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Exact Solutions of Gardner Equations through tanh-coth Method

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Abstract

In this paper, we apply the tanh-coth method and traveling wave transformation method for solving Gardner equations, including (1 + 1)-Gardner and (2 + 1)-Gardner equations. The tanh-coth method proved to be reliable and effective in handling a large number of nonlinear dispersive and disperse equations. Through tanh-coth method, we get analytical expressions of soliton solutions of Gardner equations. The one-soliton solution is characterized by an infinite wing or infinite tail.

Keywords

tanh-coth Method, Gardner Equations, Soliton Solutions

1. Introduction

In the study of nonlinear science, finding the exact solution of nonlinear evolution equations is an important subject. Different methods have their different types of specific applications for nonlinear evolution equations. In recent years many scholars put forward and developed several new methods for solving PDEs which based on the original method, such as Hirota's bilinear method [1], homogeneous balance method [2] [3], projective Riccati equation method [4] [5], Jacobi elliptic functions method [6], auxiliary equation method [7], and separation of variables [8] [9] [10]. Among them, the tanh-coth method and the sine-cosine method are powerful and widely used in several research works. For single soliton solution, the tanh-coth method is easy to use and has been applied for a wide variety of nonlinear problems.

In the plasma physics, solid physics, fluid mechanics, etc., the Gardner equation is written as

$$u_t + \alpha u u_x + \beta u^2 u_x + \gamma u_{xxx} = 0, \tag{1}$$

which is also called the KdV-mKdV equation. The model can be well described the wave propagation in a one-dimensional nonlinear lattice with a non harmonic bound particle. Gardner equations have very important application in mathematics, physics, engineering and other fields. Different types of equations can be obtained by changing the value of α, β, γ.

With $\beta = 0$, $\gamma = 1$, the KdV equation is written as

$$u_t + \alpha u u_x + u_{xxx} = 0, \tag{2}$$

where the parameter α can be scaled to any real number, usually taking $\alpha = \pm 1$ or $\alpha = \pm 6$. KdV equation simulates a variety of nonlinear phenomena, including the ion acoustic waves and diving waves in the plasma.

With $\alpha = 0, \gamma = 1$, we get the mKdV equation which is written as

$$u_t + \beta u^2 u_x + u_{xxx} = 0. \tag{3}$$

It is completely integrable [11] and can be obtained by Miura transformation of the KdV equation.

The Gardner equations are used to describe many physical models, which are closely related to the study of physics. So it is very important to study it deeply.

With $\alpha = -6, \beta = -6\epsilon^2, \gamma = 1$, the (1 + 1)-Gardner equation turns out to be

$$\upsilon_t - 6\upsilon\upsilon_x - 6\epsilon^2\upsilon^2\upsilon_x + \upsilon_{xxx} = 0. \tag{4}$$

Further, the (2 + 1)-dimensional Gardner Equation [12] [13] [14] [15] is written as

$$\begin{aligned} &\omega_t + \omega_{xxx} - \frac{3}{2}\alpha^2\omega^2\omega_x + 6\beta\omega\omega_x + 3\gamma^2\upsilon_y - 3\alpha\gamma\omega_x\upsilon = 0 \\ &\upsilon_x = \omega_y, \end{aligned} \tag{5}$$

which reduces to the (1 + 1)-Gardner equation with $\omega_y = 0$.

For $\alpha = 0$, Equation (5) is transformed into the KP equation as

$$\omega_t + \omega_{xxx} + 6\beta\omega\omega_x + 3\gamma^2\partial_x^{-1}\upsilon_{yy} = 0, \tag{6}$$

while it is the modified KP equation with $\beta = 0$. Therefore, the (2 + 1)-Gardner equation combines KP equation and modified KP equation.

With $\alpha = 2, \gamma = -1, \beta = \tau$, the (2 + 1)-Gardner equation turns out to be

$$\omega_t + \omega_{xxx} - 6\omega^2\omega_x + 6\omega_x\partial_x^{-1}\omega_y + 6\tau\omega\omega_x + 3\partial_x^{-1}\omega_{yy} = 0. \tag{7}$$

We had found soliton solutions, travelling wave solutions and plane periodic solutions of KdV and mKdV equations through tanh-coth method. In order to prove superiority of the tanh-coth method, we apply it on Gardner equations which are more complex and have higher dimensions.

This paper is organized as follows. In Section 2, we introduce the tanh-coth method. In Section 3, we first substitute the wave variable $\xi = x - ct$ into the (1 + 1)-Gardner equation, and then integrate once. Based on the tanh-coth method, the soliton and kink solutions of the (1 + 1)-Gardner are given. In Section 4, we would like to search for solutions to the dimensionally reduced (2 + 1)-Gardner equation from substituting the wave variable $\xi = x + dy - ct$. The solutions are obtained by tanh-coth method and the soliton solution is graphically revealed. A conclusion is given in Section 5.

2. The tanh-coth Method

A wave variable $\xi = x - ct$ converts any PDE

$$P\left(u, u_t, u_x, u_{xx}, u_{xxx}, \cdots\right) = 0 \tag{8}$$

to an ODE

$$Q\left(u, u', u'', u''', \cdots\right) = 0. \tag{9}$$

Equation (9) is then integrated as long as all terms contain derivatives where integration constants are considered zeros.

Introducing an independent variable

$$Y = \tanh\left(\mu\xi\right), \xi = x - ct, \tag{10}$$

where μ is the wave number. The tanh-coth method admits the use of the finite expansion

$$u\left(\mu\xi\right) = S\left(Y\right) = \sum_{k=0}^{M} a_k Y^k + \sum_{k=1}^{M} b_k Y^{-k}, \tag{11}$$

where M is a positive integer, in most cases, that will be determined by balance method. And we usually balance the highest order nonlinear terms with the linear terms of highest order by using the scheme given as follows:

$$u \to M,$$

$$u^n \to nM,$$

$$u' \to M + 1,$$

$$u^{(r)} \to M + r. \tag{12}$$

Substituting (11) into the reduced ODE results. We then collect all coefficients of each power of $Y^k, 0 \le k \le nM$ in the resulting equation where these coefficients have to vanish. This will give a system of algebraic equations involving the parameters a_k, b_k, μ and *c*. Finally, we obtain an analytic solution $u\left(x, t\right)$ in a closed form.

3. The Solutions of (1 + 1)-Gardner Equation

We first substitute the wave variable $\xi = x - ct$ into the (1 + 1)-Gardner equation

$$u_t - 6\left(u + \epsilon^2 u^2\right)u_x + u_{xxx} = 0, \tag{13}$$

that gives

$$-cu' - 6\left(u + \epsilon^2 u^2\right)u' + u''' = 0. \tag{14}$$

Integrating once to obtain

$$-cu - 3u^2 - 2\epsilon^2 u^3 + u'' = 0. \tag{15}$$

We then balance the nonlinear term $-2\epsilon^2 u^3$, that has the exponent $3M$, with the highest order derivative u'', that has the exponent $M+2$. Using the balance process leads to

$$3M = M + 2, \tag{16}$$

that gives

$$M = 1. \tag{17}$$

The tanh-coth method allows us to use the substitution

$$u(x,t) = S(Y) = a_0 + a_1 Y + b_1 Y^{-1}. \tag{18}$$

Substituting (18) into (15), collecting the coefficients of each power of $Y^i, 0 \le i \le 4$, setting each coefficient to zero, we find

$$Y^0 := -ca_0 - 3a_0^2 - 6a_1 b_1 - 2\epsilon^2 a_0^3 - 12\epsilon^2 a_0 a_1 b_1 = 0,$$

$$Y^1 := -ca_1 - 6a_0 a_1 - 6\epsilon^2 a_0^2 a_1 - 6\epsilon^2 a_1^2 b_1 - 2a_1\mu^2 = 0,$$

$$Y^3 := -3a_1^2 - 6\epsilon^2 a_1^2 a_0 = 0,$$

$$Y^4 := -2\epsilon^2 a_1^3 + 2a_1\mu^2 = 0.$$

We find the following sets of solutions:

$$(i)\quad a_0 = a_1 = 0, b_1 = \sqrt{-\frac{c}{2}}, \mu = \sqrt{-\frac{c}{2}}, c < 0, \tag{19}$$

$$(ii)\quad b_1 = 0, a_0 = -\frac{1}{2\epsilon^2}, a_1 = \frac{1}{\epsilon^2}\sqrt{\frac{1}{2}}, \mu = \frac{1}{2\epsilon}, c = \frac{1}{\epsilon^2} > 0, \tag{20}$$

$$(iii)\quad a_1 = b_1 = \sqrt{-\frac{1}{4\epsilon^2}}, a_0 = -\frac{1}{2\epsilon^2}, \mu = \frac{1}{4\epsilon}, c = \frac{1}{\epsilon^2} > 0. \tag{21}$$

Consequently, we obtain the following solutions:

$$u_1(x,t) = \frac{\sqrt{2}}{2\epsilon^2}\tanh\left[\frac{1}{2\epsilon}(x-ct)\right], c > 0, \tag{22}$$

$$u_2(x,t) = -\frac{1}{2\epsilon^2} + \frac{1}{2\epsilon^2}\coth\left[\frac{1}{2\epsilon}(x-ct)\right], c > 0, \tag{23}$$

$$u_3(x,t) = -\frac{1}{2\epsilon^2} - \sqrt{\frac{3}{2\epsilon^4} - \frac{c}{\epsilon^2}}\,\mathrm{sech}\left[\frac{2c\epsilon^2 - 3}{2\epsilon^2}(x-ct)\right], c > 0. \tag{24}$$

Following immediately. Figure 1 shows a single soliton solution $u_3(x,t)$ for $c = 1, \epsilon = 1$. In the graph, the X axis is t, the Y axis is x, and the Z axis is $u_3(x,t)$. It can be see that the one-soliton solution is characterized by an infinite wing. This shows that $u \to -\frac{1}{2\epsilon^2}$ as x and t one of them tends to infinity, that is, ξ tends to infinity.

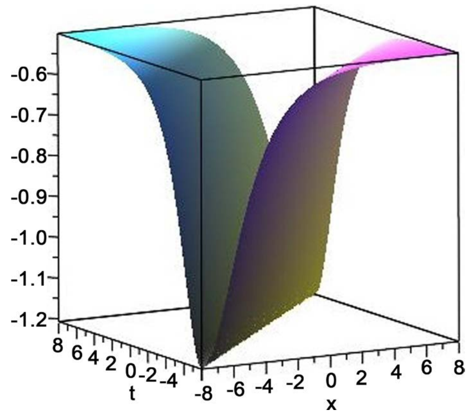

Figure 1. Graph of the one-solution solution $u_3(x,t)$ for $c=1,\epsilon=1$ characterized by an infinite wing.

4. The Solutions of (2 + 1)-Gardner Equation

We first substitute the wave variable $\xi = x+dy-ct$ into the (2 + 1)-Gardner equation

$$\omega_t+\omega_{xxx}+6\omega^2\omega_x+6\omega_x\partial_x^{-1}\omega_y+6\tau\omega\omega_x-3\partial_x^{-1}\omega_{yy}=0, \tag{25}$$

that gives

$$\begin{cases}-c\omega'+\omega'''+6\omega^2\omega'+6d\omega'\upsilon+6\tau\omega\omega'-3d^2\upsilon'=0\\ \upsilon_x=\omega'\end{cases}$$

Based on $\dfrac{\partial\upsilon}{\partial x}=\dfrac{\partial\upsilon}{\partial\xi}\cdot\dfrac{\partial\xi}{\partial x}=\upsilon'$, we get

$$-c\omega'+\omega'''+6\omega^2\omega'+6d\omega\omega'+6\tau\omega\omega'-3d^2\omega'=0. \tag{26}$$

Integrating once to obtain

$$\left(-c-3d^2\right)\omega+\left(3d+3\tau\right)\omega^2+2\omega^3+\omega''=0. \tag{27}$$

We then balance the nonlinear term $2\omega^3$, that has the exponent $3M$, with the highest order derivative ω'', that has the exponent $M+2$. Using the balance process leads to

$$3M=M+2, \tag{28}$$

that gives

$$M=1. \tag{29}$$

The tanh-coth method allows us to use the substitution

$$\omega(x,t)=S(Y)=a_0+a_1Y+b_1Y^{-1}. \tag{30}$$

Substituting (30) into (27), that gives

$$\left(-3d^2-c\right)\left(a_1Y+a_0+b_1Y^{-1}\right)-2a_1\mu^2Y+2a_1\mu^2Y^3+2b_1\mu^2Y^{-3}$$
$$-2b_1\mu^2Y^{-1}+\left(3d+3\tau\right)\left(a_1Y+a_0+b_1Y^{-1}\right)^2+2\left(a_1Y+a_0+b_1Y^{-1}\right)^3=0.$$

Collecting the coefficients of each power of $Y^i, 0\le i\le 4$ and then setting each coefficient to zero that leads to the following set of constraining equations for the parameters:

$$Y^0 := \left(-3d^2 - c\right)a_0 + \left(3d + 3\tau\right)\left(a_0^2 + a_1 b_1\right) + 2a_0^3 + 12a_0 a_1 b_1 = 0,$$

$$Y^1 := \left(-3d^2 - c\right)a_1 + \left(3d + 3\tau\right)2a_0 a_1 + 6a_0^2 a_1 + 6a_1^2 b_1 - 2a_1\mu^2 = 0,$$

$$Y^3 := \left(3d + 3\tau + 6a_0\right)a_1^2 = 0,$$

$$Y^4 := 2a_1\left(\mu^2 + a_1^2\right) = 0.$$

We find the following sets of solutions:

$$(i)\quad a_0 = a_1 = 0, b_1 = \sqrt{c}, \mu = \sqrt{c}, c > 0, \tag{31}$$

$$(ii)\quad a_1 = 0, a_0 = \frac{-\left(3d + 3\tau\right) + \sqrt{18d\tau + 9\tau^2 + 33d^2 + 8c}}{4}, b_1 = 1, \mu = 1, \tag{32}$$

$$(iii)\quad a_1 = 0, a_0 = b_1 = \frac{-\left(3d + 3\tau\right) + \sqrt{18d\tau + 9\tau^2 + 33d^2 + 8c}}{4}, \mu = 1. \tag{33}$$

Consequently, we obtain the solutions as

$$\omega_1\left(x, y, t\right) = \sqrt{c}\coth\left[\sqrt{c}\left(x + dy - ct\right)\right], c > 0, \tag{34}$$

$$\omega_2\left(x, y, t\right) = -\frac{-\left(3d + 3\tau\right) + \sqrt{18d\tau + 9\tau^2 + 33d^2 + 8c}}{4}\coth\left[\left(x + dy - ct\right)\right], \tag{35}$$

$$\omega_3\left(x, y, t\right) = -\frac{d + \tau}{2} + \sqrt{\frac{9d^2 + 6d\tau + 3\tau^2 + 2c}{2}}\operatorname{sech}\left[\sqrt{\frac{9d^2 + 6d\tau + 3\tau^2 + 2c}{2}}\left(x + dy - ct\right)\right], \tag{36}$$

where ω_1 and ω_2 are kink solutions. And the kink solution's graph is changing along with b_1 taking different values.

Following immediately. Figure 2 below shows the one-soliton solution $\omega_3\left(x, y, t\right)$ for $c = 1, d = 1, \tau = 1, t = 1$. From the graph we can see, the one-soliton solution is characterized by infinite tail. This shows that $\omega \to -\frac{d + \tau}{2}$ as x and t one of them tends to infinity, that is, ξ tends to infinity.

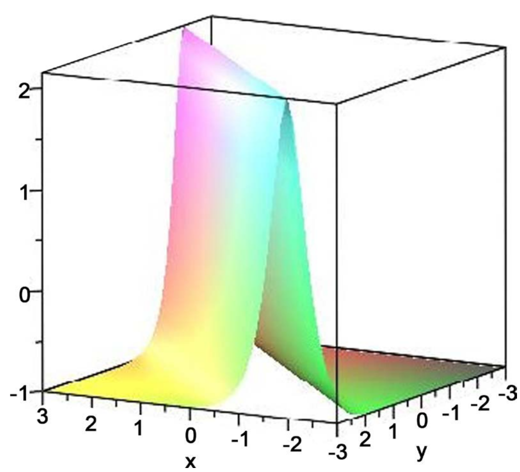

Figure 2. Graph of $\omega_3\left(x, y, t\right)$ for $c = 1, d = 1, \tau = 1, t = 1$ characterized by an infinite tail.

5. Conclusion

In this paper, we obtain the soliton and kink solutions of the (1 + 1)-Gardner equation and (2 + 1)-Gardner equation through the tanh-coth method. The biggest advantage is that by traveling wave transformation, the problem of solving nonlinear partial differential equations is transformed into the problem of solving nonlinear ordinary differential equations or nonlinear algebraic equations. The tanh-coth method is convenient to use, and can be further extended to solve other nonlinear partial differential equations.

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