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On AP-Henstock Integrals of Interval-Valued Functions and Fuzzy-Number-Valued Functions

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Abstract

In 2000, Wu and Gong [1] introduced the thought of the Henstock integrals of interval-valued functions and fuzzy-number-valued functions and obtained a number of their properties. The aim of this paper is to introduce the thought of the AP-Henstock integrals of interval-valued functions and fuzzy-number-valued functions which are extensions of [1] and investigate a number of their properties.

Keywords

Fuzzy Numbers, AP-Henstock Integrals of Interval-Valued Functions, AP-Henstock Integrals of Fuzzy-Number-Valued Functions

1. Introduction

As it is well known, the Henstock integral for a real function was first defined by Henstock [2] in 1963. The Henstock integral is a lot of powerful and easier than the Lebesgue, Wiener and Richard Phillips Feynman integrals. Furthermore, it is also equal to the Denjoy and the Perron integrals [2] [3]. In 2016, Hamid and Elmuiz [4] introduced the concept of the Henstock-Stieltjes (HS) integrals of interval-valued functions and fuzzy-number-valued functions and discussed a number of their properties.

In this paper, we introduce the concept of the AP-Henstock integrals of interval-valued functions and fuzzy-number-valued functions and discuss some of their properties.

The paper is organized as follows. In Section 2, we have a tendency to provide the preliminary terminology used in this paper. Section 3 is dedicated to discussing the AP-Henstock integral of interval-valued functions. In Section 4, we introduce the AP-Henstock integral of fuzzy-number-valued functions. The last section provides conclusions.

2 Preliminaries

Let E be a measurable set and let c be a real number. The density of E at c is defined by

$$d_c E = \lim_{h\to 0^+} \frac{\mu\left(E\bigcap(c-h,c+h)\right)}{2h}, \tag{2.1}$$

provided the limit exists. The point c is called a point of density of E if $d_c E = 1$. The set E^d represents the set of all points $x \in E$ such that x is a point of density of E.

A measurable set $S_x \subseteq [a,b]$ is called an approximate neighborhood (br.ap-nbd) of $x \in [a,b]$ if it containing x as a point of density. We choose an ap-nbd $S_x \subseteq [a,b]$ for each $x \in E \subseteq [a,b]$ and denote a choice on E by $S = \{S_x : x \in E\}$. A tagged interval-point pair $([u,v],\xi)$ is said to be S-fine if $\xi \in [u,v]$ and $u,v \in S_\xi$.

A division P is a finite collection of interval-point pairs $\left\{([u_i,v_i];\xi_i)\right\}_{i=1}^n$, where $\left\{[u_i,v_i]\right\}_{i=1}^n$ are non-overlapping subintervals of $[a,b]$. We say that $P = \left\{([u_i,v_i];\xi_i)\right\}_{i=1}^n$ is

1) a division of $[a,b]$ if $\bigcup_{i=1}^n [u_i,v_i] = [a,b]$;

2) S-fine division of $[a,b]$ if $\xi_i \in [u_i,v_i]$ and $([u_i,v_i],\xi_i)$ is S-fine for all $i = 1,2,\cdots,n$.

Definition 2.1. [2] [3] *A real-valued function* $f:[a,b]\to R$ *is said to be Henstock integrable to* A *on* $[a,b]$ *if for every* $\varepsilon > 0$, *there is a function* $\delta(t) > 0$ *such that for any* δ*-fine division* $P = \{[u_i,v_i];\xi_i\}_{i=1}^n$ *of* $[a,b]$, *we have*

$$\left|\sum_{i=1}^n f(\xi_i)(v_i - u_i) - A\right| < \varepsilon, \tag{2.2}$$

where the sum $\sum$ is understood to be over P and we write $(H)\int_a^b f(t)\mathrm{d}t = A$, and $f \in H[a,b]$.

Definition 2.2. [5] *A function* $f:[a,b]\to R$ *is AP-Henstock integrable if there exists a real number* $A \in R$ *such that for each* $\varepsilon > 0$ *there is a choice* S *such that*

$$\left|\sum_{i=1}^n f(\xi_i)(v_i - u_i) - A\right| < \varepsilon \tag{2.3}$$

for each S-fine division $P = \left\{([u_i,v_i];\xi_i)\right\}_{i=1}^n$ of $[a,b]$. A is called AP-Henstock integral of f on $[a,b]$, and we write $A = (APH)\int_a^b f$.

Theorem 2.1. *If* f *and* g *are AP-Henstock integrable on* $[a,b]$ *and* $f \le g$ *almost everywhere on* $[a,b]$, *then*

$$(APH)\int_a^b f \le (APH)\int_a^b g. \tag{2.4}$$

Proof. The proof is similar to the Theorem 3.6 in [3]. □

3. The AP-Henstock Integral of Interval-Valued Functions

In this section, we shall give the definition of the AP-Henstock integrals of interval-valued functions and discuss some of their properties.

Definition 3.1. [1] Let

$I_R = \{ I = [I^-, I^+] : \text{I is the closed bounded interval on the real line } R \}$.

For $A, B \in I_R$, we define $A \le B$ iff $A^- \le B^-$ and $A^+ \le B^+$, $A + B = C$ iff $C^- = A^- + B^-$ and $C^+ = A^+ + B^+$, and $A \cdot B = \{ a \cdot b : a \in A, b \in B \}$, where

$$(A \cdot B)^- = \min \{ A^- \cdot B^-, A^- \cdot B^+, A^+ \cdot B^-, A^+ \cdot B^+ \} \tag{3.1}$$

and

$$(A \cdot B)^+ = \max \{ A^- \cdot B^-, A^- \cdot B^+, A^+ \cdot B^-, A^+ \cdot B^+ \}. \tag{3.2}$$

Define $\mathrm{d}(A, B) = \max(|A^- - B^-|, |A^+ - B^+|)$ as the distance between intervals A and B.

Definition 3.2. [1] *Let* $F : [a, b] \to I_R$ *be an interval-valued function.* $I_0 \in I_R$, *for every* $\varepsilon > 0$ *there is a* $\delta(t) > 0$ *such that for any* δ*-fine division* $P = \{ [u_i, v_i], \xi_i \}_{i=1}^n$ *we have*

$$\mathrm{d}\left(\sum_{i=1}^n F(\xi_i)(v_i - u_i), I_0 \right) < \varepsilon, \tag{3.3}$$

then $F(t)$ is said to be Henstock integrable over $[a, b]$ and write

$(IH)\int_a^b F(t)\mathrm{d}t = I_0$. For brevity, we write $F(t) \in IH[a, b]$.

Definition 3.3. *A interval-valued function* $F : [a, b] \to I_R$ *is AP-Henstock integrable to* $I_0 \in I_R$, *if for every* $\varepsilon > 0$ *there exists a choice* S *on* $[a, b]$ *such that*

$$\mathrm{d}\left(\sum_{i=1}^n F(\xi_i)(v_i - u_i), I_0 \right) < \varepsilon, \tag{3.4}$$

whenever $P = \{ ([u_i, v_i]; \xi_i) \}_{i=1}^n$ is a S-fine division of $[a, b]$, we write

$(APIH)\int_a^b F = I_0$ and $F \in APIH[a, b]$.

Theorem 3.1. *If* $F \in APIH[a, b]$, *then the integral value is unique.*

Proof. Let integral value is not unique and let $B_1 = (APIH)\int_a^b F$ and

$B_2 = (APIH)\int_a^b F$. Let $\varepsilon > 0$ be given. Then there exists a choice S on $[a, b]$ such that

$$\mathrm{d}\left(\sum_{i=1}^n F(\xi_i)(v_i - u_i), B_1 \right) < \frac{\varepsilon}{2}, \tag{3.5}$$

$$\mathrm{d}\left(\sum_{i=1}^{n}F(\xi_i)(v_i-u_i),B_2\right)<\frac{\varepsilon}{2} \tag{3.6}$$

whenever $P=\{([u_i,v_i];\xi_i)\}_{i=1}^{n}$ is a S-fine division of $[a,b]$.

Whence it follows from the Triangle Inequality that:

$$\mathrm{d}(B_1,B_2)=\mathrm{d}\left(\sum_{i=1}^{n}F(\xi_i)(v_i-u_i),B_1\right)+\mathrm{d}\left(\sum_{i=1}^{n}F(\xi_i)(v_i-u_i),B_2\right)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon. \tag{3.7}$$

Since for $\forall\varepsilon>0$, there exists a choice S on $[a,b]$ as above so $B_1=B_2$. □

Theorem 3.2. *An interval-valued function* $F\in APIH[a,b]$ *if and only if* $F^-,F^+\in APH[a,b]$ *and*

$$(APIH)\int_a^b F=\left[(APH)\int_a^b F^-,(APH)\int_a^b F^+\right]. \tag{3.8}$$

Proof. Let $F\in APIH[a,b]$, from Definition 3.3 there is a unique interval number $I_0=[I_0^-,I_0^+]$ with the property that for any $\varepsilon>0$ there exists a choice S on $[a,b]$ such that

$$\mathrm{d}\left(\sum_{i=1}^{n}F(\xi_i)(v_i-u_i),I_0\right)<\varepsilon, \tag{3.9}$$

whenever $P=\{([u_i,v_i];\xi_i)\}_{i=1}^{n}$ is a S-fine division of $[a,b]$. Since $v_i-u_i\ge 0$ for $1\le i\le n$, we have

$$\begin{aligned}&\mathrm{d}\left(\sum_{i=1}^{n}F(\xi_i)(v_i-u_i),I_0\right)<\varepsilon\\&=\max\left(\left|\left[\sum_{i=1}^{n}F(\xi_i)(v_i-u_i)\right]^- -I_0^-\right|,\left|\left[\sum_{i=1}^{n}F(\xi_i)(v_i-u_i)\right]^+ -I_0^+\right|\right)<\varepsilon.\\&=\max\left(\left|\sum_{i=1}^{n}F^-(\xi_i)(v_i-u_i)-I_0^-\right|,\left|\sum_{i=1}^{n}F^+(\xi_i)(v_i-u_i)-I_0^+\right|\right)<\varepsilon.\end{aligned} \tag{3.10}$$

Hence $\left|\sum_{i=1}^{n}F^-(\xi_i)(v_i-u_i)-I_0^-\right|<\varepsilon$, $\left|\sum_{i=1}^{n}F^+(\xi_i)(v_i-u_i)-I_0^+\right|<\varepsilon$ whenever $P=\{([u_i,v_i];\xi_i)\}_{i=1}^{n}$ is a S-fine division of $[a,b]$. Thus $F^-,F^+\in APH[a,b]$ and

$$(APIH)\int_a^b F=\left[(APH)\int_a^b F^-,(APH)\int_a^b F^+\right]. \tag{3.11}$$

Conversely, let $F^-,F^+\in APH[a,b]$. Then there exists $H_1,H_2\in R$ with the property that given $\varepsilon>0$ there exists a choice S on $[a,b]$ such that

$$\left|\sum_{i=1}^{n}F^-(\xi_i)(v_i-u_i)-H_1\right|<\varepsilon,\quad\left|\sum_{i=1}^{n}F^+(\xi_i)(v_i-u_i)-H_2\right|<\varepsilon$$

whenever $P=\{([u_i,v_i];\xi_i)\}_{i=1}^{n}$ is a S-fine division of $[a,b]$. We define $I_0=[H_1,H_2]$, then if $P=\{[u_i,v_i],\xi_i\}_{i=1}^{n}$ is a S-fine division of $[a,b]$, we have

$$d\left(\sum_{i=1}^{n} F(\xi_i)(v_i - u_i), I_0\right) < \varepsilon. \tag{3.12}$$

Hence $F:[a,b] \to I_R$ is AP-Henstock integrable on $[a,b]$. □

Theorem 3.3. *If* $F, G \in APIH[a,b]$ *and* $\beta, \gamma \in R$. *Then* $\beta F + \gamma G \in APIH[a,b]$ *and*

$$(APIH)\int_a^b (\beta F + \gamma G) = \beta (APIH)\int_a^b F + \gamma (APIH)\int_a^b G. \tag{3.13}$$

Proof. If $F, G \in APIH[a,b]$, then $F^-, F^+, G^-, G^+ \in APH[a,b]$ by Theorem 3.2. Hence $\beta F^- + \gamma G^-, \beta F^- + \gamma G^+, \beta F^+ + \gamma G^-, \beta F^+ + \gamma G^+ \in APH[a,b]$.

(1) If $\beta > 0$ and $\gamma > 0$, then

$$\begin{aligned}(APH)\int_a^b (\beta F + \gamma G)^- &= (APH)\int_a^b (\beta F^- + \gamma G^-) \\ &= \beta (APH)\int_a^b F^- + \gamma (APH)\int_a^b G^- \\ &= \beta\left((APIH)\int_a^b F\right)^- + \gamma\left((APIH)\int_a^b G\right)^- \\ &= \left(\beta (APIH)\int_a^b F + \gamma (APIH)\int_a^b G\right)^-.\end{aligned}$$

(2) If $\beta < 0$ and $\gamma < 0$, then

$$\begin{aligned}(APH)\int_a^b (\beta F + \gamma G)^- &= (APH)\int_a^b (\beta F^+ + \gamma G^+) \\ &= \beta (APH)\int_a^b F^+ + \gamma (APH)\int_a^b G^+ \\ &= \beta\left((APIH)\int_a^b F\right)^+ + \gamma\left((APIH)\int_a^b G\right)^+ \\ &= \left(\beta (APIH)\int_a^b F + \gamma (APIH)\int_a^b G\right)^-.\end{aligned}$$

(3) If $\beta > 0$ and $\gamma < 0$, (or $\beta < 0$ and $\gamma < 0$,), then

$$\begin{aligned}(APH)\int_a^b (\beta F + \gamma G)^- &= (APH)\int_a^b (\beta F^- + \gamma G^+) \\ &= \beta (APH)\int_a^b F^- + \gamma (APH)\int_a^b G^+ \\ &= \beta\left(((APIH))\int_a^b F\right)^- + \gamma\left((APIH)\int_a^b G\right)^+ \\ &= \left(\beta (APIH)\int_a^b F + \gamma (APIH)\int_a^b G\right)^-.\end{aligned}$$

Similarly, for four cases above we have

$$(APH)\int_a^b(\beta F+\gamma G)^+=\left(\beta(APIH)\int_a^b F+\gamma(APIH)\int_a^b G\right)^+. \quad (3.14)$$

Hence by Theorem 3.2 $\beta F+\gamma G\in APIH[a,b]$ and

$$(APIH)\int_a^b(\beta F+\gamma G)=\beta(APIH)\int_a^b F+\gamma(APIH)\int_a^b G. \quad (3.15)$$

□

Theorem 3.4. *If* $F\in APIH[a,c]$ *and* $F\in APIH[c,b]$, *then* $F\in APIH[a,b]$ *and*

$$(APIH)\int_a^b F=(APIH)\int_a^c F+(APIH)\int_c^b F. \quad (3.16)$$

Proof. If $F\in APIH[a,c]$ and $F\in APIH[c,b]$, then by Theorem 3.2 $F^-,F^+\in APH[a,c]$ and $F^-,F^+\in APH[c,b]$. Hence $F^-,F^+\in APH[a,b]$ and

$$(APH)\int_a^b F^-=(APH)\int_a^c F^-+(APH)\int_c^b F^-$$

$$=\left((APIH)\int_a^c F+(APIH)\int_c^b F\right)^-.$$

Similarly, $(APH)\int_a^b F^+=\left((APIH)\int_a^c F+(APIH)\int_c^b F\right)^+$. Hence by Theorem 3.2 $F\in APIH[a,b]$ and

$$(APIH)\int_a^b F=(APIH)\int_a^c F+(APIH)\int_c^b F. \quad (3.17)$$

□

Theorem 3.5. *If* $F\le G$ *nearly everywhere on* $[a,b]$ *and* $F,G\in APIH[a,b]$, *then*

$$(APIH)\int_a^b F\le(APIH)\int_a^b G. \quad (3.18)$$

Proof. Let $F\le G$ nearly everywhere on $[a,b]$ and $F,G\in APIH[a,b]$ Then $F^-,F^+,G^-,G^+\in APH[a,b]$ and $F^-\le G^-$, $F^+\le G^+$ nearly everywhere on $[a,b]$ By Theorem 2.1 $(APH)\int_a^b F^-\le(APH)\int_a^b G^-$ and $(APH)\int_a^b F^+\le(APH)\int_a^b G^+$. Hence

$$(APIH)\int_a^b F\le(APIH)\int_a^b G, \quad (3.19)$$

by Theorem 3.2. □

Theorem 3.6. *Let* $F,G\in APIH[a,b]$ *and* $\mathrm{d}(F,G)$ *is Lebesgue integrable on* $[a,b]$. *Then*

$$\mathrm{d}\left((APIH)\int_a^b F,(APIH)\int_a^b G\right)\le(L)\int_a^b \mathrm{d}(F,G). \quad (3.20)$$

Proof. By definition of distance,

$$
\begin{aligned}
&\mathrm{d}\left((APIH)\int_a^b F,(APIH)\int_a^b G\right)\\
&=\max\left(\left|\left((APIH)\int_a^b F\right)^- -\left((APIH)\int_a^b G\right)^-\right|,\left|\left((APIH)\int_a^b F\right)^+ -\left((APIH)\int_a^b G\right)^+\right|\right)\\
&=\max\left(\left|(APH)\int_a^b\left(F^- -G^-\right)\right|,\left|(APH)\int_a^b\left(F^+ -G^+\right)\right|\right)\\
&\le\max\left((L)\int_a^b\left|F^- -G^-\right|,(L)\int_a^b\left|F^+ -G^+\right|\right)\\
&\le(L)\int_a^b\max\left(\left|F^- -G^-\right|,\left|F^+ -G^+\right|\right)\\
&=(L)\int_a^b\mathrm{d}(F,G).
\end{aligned}
\tag{3.12}
$$

□

4. The AP-Henstock Integral of Fuzzy-Number-Valued Functions

This section introduces the concept of the AP-Henstock integral of fuzzy-number-valued functions and investigates some of their properties.

Definition 4.1. [6] [7] [8] *Let* $\tilde{A}\in F(R)$ *be a fuzzy subset on* R*. If for any* $\lambda\in[0,1]$, $A_\lambda=\left[A_\lambda^-,A_\lambda^+\right]$ *and* $A_1\neq\phi$, *where* $A_\lambda=\left\{t:\tilde{A}(t)\ge\lambda\right\}$, *then* $\tilde{A}$ *is called a fuzzy number. If* $\tilde{A}$ *is convex, normal, upper semi-continuous and has the compact support, we say that* $\tilde{A}$ *is a compact fuzzy number.*

Let $\tilde{R}$ denote the set of all fuzzy numbers.

Definition 4.2. [6] *Let* $\tilde{A},\tilde{B}\in\tilde{R}$, *we define* $\tilde{A}\le\tilde{B}$ *iff* $A_\lambda\le B_\lambda$ *for all* $\lambda\in(0,1]$, $\tilde{A}+\tilde{B}=\tilde{C}$ *iff* $A_\lambda+B_\lambda=C_\lambda$ *for any* $\lambda\in(0,1]$, $\tilde{A}\cdot\tilde{B}=\tilde{D}$ *iff* $A_\lambda\cdot B_\lambda=D_\lambda$ *for any* $\lambda\in(0,1]$.

For $\tilde{A},\tilde{B}\in\tilde{R}^C$, $D\left(\tilde{A},\tilde{B}\right)=\sup_{\lambda\in[0,1]}\mathrm{d}\left(A_\lambda,B_\lambda\right)$ is called the distance between $\tilde{A}$ and $\tilde{B}$.

Lemma 4.1. [9] *If a mapping* $H:[0,1]\to I_R$, $\lambda\to H(\lambda)=\left[m_\lambda,n_\lambda\right]$, *satisfies* $\left[m_{\lambda_1},n_{\lambda_1}\right]\supset\left[m_{\lambda_2},n_{\lambda_2}\right]$ *when* $\lambda_1<\lambda_2$, *then*

$$\tilde{A}:=\bigcup_{\lambda\in(0,1]}\lambda H(\lambda)\in\tilde{R} \tag{4.1}$$

and

$$A_\lambda=\bigcap_{n=1}^{\infty}H\left(\lambda_n\right), \tag{4.2}$$

where $\lambda_n=\left[1-\dfrac{1}{(n+1)}\right]\lambda.$

Definition 4.3. [1] *Let* $\tilde{F}:[a,b]\to\tilde{R}$*. If the interval-valued function* $F_\lambda(t)=\left[F_\lambda^-(t),F_\lambda^+(t)\right]$ *is Henstock integrable on* $[a,b]$ *for any* $\lambda\in(0,1]$, *then we say that* $\tilde{F}(t)$ *is Henstock integrable on* $[a,b]$ *and the integral value is defined by*

$$(FH)\int_a^b \tilde{F}(t)\mathrm{d}t := \bigcup_{\lambda\in(0,1]} \lambda (IH)\int_a^b F_\lambda(t)\mathrm{d}t$$
$$= \bigcup_{\lambda\in(0,1]} \lambda \left[(H)\int_a^b F_\lambda^- \mathrm{d}t, (H)\int_a^b F_\lambda^+ \mathrm{d}t\right].$$

For brevity, we write $\tilde{F}(t)\in FH[a,b]$.

Definition 4.4. *Let* $\tilde{F}:[a,b]\to\tilde{R}$. *If the interval-valued function* $F_\lambda(t)=\left[F_\lambda^-(t),F_\lambda^+(t)\right]$ *is AP-Henstock integrable on* $[a,b]$ *for any* $\lambda\in(0,1]$, *then* $\tilde{F}(t)$ *is called AP-Henstock integrable on* $[a,b]$ *and the integral value is defined by*

$$(APFH)\int_a^b \tilde{F}(t)\mathrm{d}t := \bigcup_{\lambda\in(0,1]} \lambda (APIH)\int_a^b F_\lambda(t)\mathrm{d}t$$
$$= \bigcup_{\lambda\in(0,1]} \lambda \left[(APH)\int_a^b F_\lambda^- dt, (APH)\int_a^b F_\lambda^+ \mathrm{d}t\right].$$

We write $\tilde{F}(t)\in APFH[a,b]$.

Theorem 4.1. $\tilde{F}\in APFH[a,b]$, *then* $(APFH)\int_a^b \tilde{F}(t)\mathrm{d}t\in\tilde{R}$ *and*

$$\left((APFH)\int_a^b \tilde{F}(t)\mathrm{d}t\right)_\lambda = \bigcap_{n=1}^{\infty}(APIH)\int_a^b F_{\lambda_n}(t)\mathrm{d}t, \tag{4.3}$$

where $\lambda_n=\left[1-\frac{1}{(n+1)}\right]\lambda.$

Proof. Let $H:(0,1]\to I_R$, be defined by

$$H(\lambda)=\left[(APH)\int_a^b F_\lambda^-(t)\mathrm{d}t,(APH)\int_a^b F_\lambda^+(t)\mathrm{d}t\right].$$

Since $F_\lambda^-(t)$ and $F_\lambda^+(t)$ are increasing and decreasing on λ respectively, therefore, when $0<\lambda_1\le\lambda_2\le 1$, we have $F_{\lambda_1}^-(t)\le F_{\lambda_2}^-(t)$, $F_{\lambda_1}^+(t)\ge F_{\lambda_2}^+(t)$, on $[a,b]$. From Theorem 3.5 we have

$$\left[(APH)\int_a^b F_{\lambda_1}^-(t)\mathrm{d}t,(APH)\int_a^b F_{\lambda_1}^+(t)\mathrm{d}t\right]\supset\left[(APH)\int_a^b F_{\lambda_2}^-(t)\mathrm{d}t,(APH)\int_a^b F_{\lambda_2}^+(t)\mathrm{d}t\right]. \tag{4.4}$$

From Theorem 3.2 and Lemma 4.1 we have

$$(APFH)\int_a^b \tilde{F}(t)\mathrm{d}t := \bigcup_{\lambda\in(0,1]}\lambda\left[(APH)\int_a^b F_\lambda^-\mathrm{d}t,(APH)\int_a^b F_\lambda^+\mathrm{d}t\right]\in\tilde{R} \tag{4.5}$$

and for all $\lambda\in(0,1]$, $\left[(APFH)\int_a^b \tilde{F}(t)\mathrm{d}t\right]_\lambda = \bigcap_{n=1}^{\infty}(APIH)\int_a^b F_{\lambda_n}(t)\mathrm{d}t$, where

$\lambda_n=\left[1-\frac{1}{(n+1)}\right]\lambda.$ □

Theorem 4.2. *If* $\tilde{F},\tilde{G}\in APFH[a,b]$ *and* $\beta,\gamma\in R$. *Then* $\beta\tilde{F}+\gamma\tilde{G}\in APFH[a,b]$ *and*

$$(APFH)\int_a^b\left(\beta\tilde{F}+\gamma\tilde{G}\right)\mathrm{d}t=\beta(APFH)\int_a^b\tilde{F}(t)\mathrm{d}t+\gamma(APFH)\int_a^b\tilde{G}(t)\mathrm{d}t. \tag{4.6}$$

Proof. If $\tilde{F},\tilde{G}\in APFH[a,b]$, then the interval-valued function

$F_\lambda(t)=\left[F_\lambda^-(t),F_\lambda^+(t)\right]$ and $G_\lambda(t)=\left[G_\lambda^-(t),G_\lambda^+(t)\right]$ are AP-Henstock integrable on $[a,b]$ for any $\lambda\in(0,1]$ and $(APFH)\int_a^b\tilde{F}(t)\mathrm{d}t=\bigcup_{\lambda\in(0,1]}\lambda(APIH)\int_a^b F_\lambda(t)\mathrm{d}t$ and $(APFH)\int_a^b\tilde{G}(t)\mathrm{d}t=\bigcup_{\lambda\in(0,1]}\lambda(APIH)\int_a^b G_\lambda(t)\mathrm{d}t$. From Theorem 3.3 we have $\beta F_\lambda+\gamma G_\lambda\in APIH[a,b]$ and
$(APIH)\int_a^b(\beta F_\lambda+\gamma G_\lambda)\mathrm{d}t=\beta(APIH)\int_a^b F_\lambda\mathrm{d}t+\gamma(APIH)\int_a^b G_\lambda\mathrm{d}t$ for any $\lambda\in(0,1]$.
Hence $\beta\tilde{F}+\gamma\tilde{G}\in APFH[a,b]$ and

$$
\begin{aligned}
(APFH)\int_a^b\left(\beta\tilde{F}+\gamma\tilde{G}\right)\mathrm{d}t &= \bigcup_{\lambda\in(0,1]}\lambda(APIH)\int_a^b(\beta F_\lambda+\gamma G_\lambda)\mathrm{d}t\\
&= \bigcup_{\lambda\in(0,1]}\lambda\left(\beta(APIH)\int_a^b F_\lambda\mathrm{d}t+\gamma(APIH)\int_a^b G_\lambda\mathrm{d}t\right)\\
&= \beta\bigcup_{\lambda\in(0,1]}\lambda(APIH)\int_a^b F_\lambda\mathrm{d}t+\gamma\bigcup_{\lambda\in(0,1]}\lambda(APIH)\int_a^b G_\lambda\mathrm{d}t\\
&= \beta(APFH)\int_a^b\tilde{F}(t)\mathrm{d}t+\gamma(APFH)\int_a^b\tilde{G}(t)\mathrm{d}t.
\end{aligned}
$$

□

Theorem 4.3. *If* $\tilde{F}\in APFH[a,c]$ *and* $\tilde{F}\in APFH[c,b]$, *then* $\tilde{F}\in APFH[a,b]$ *and*

$$(APFH)\int_a^b\tilde{F}(t)\mathrm{d}t=(APFH)\int_a^c\tilde{F}(t)\mathrm{d}t+(APFH)\int_c^b\tilde{F}(t)\mathrm{d}t. \tag{4.7}$$

Proof. If $\tilde{F}\in APFH[a,c]$ and $\tilde{F}\in APFH[c,b]$, then the interval-valued function $F_\lambda(t)=\left[F_\lambda^-(t),F_\lambda^+(t)\right]$ is AP-Henstock integrable on $[a,c]$ and $[c,b]$ for any $\lambda\in(0,1]$ and $(APFH)\int_a^c\tilde{F}(t)\mathrm{d}t=\bigcup_{\lambda\in(0,1]}\lambda(APIH)\int_a^c F_\lambda(t)\mathrm{d}t$ and

$(APFH)\int_c^b\tilde{F}(t)\mathrm{d}t=\bigcup_{\lambda\in(0,1]}\lambda(APIH)\int_c^b F_\lambda(t)\mathrm{d}t$. From Theorem 3.4 we have

$F_\lambda\in APIH[a,b]$ and $(APIH)\int_a^b F_\lambda\mathrm{d}t=(APIH)\int_a^c F\mathrm{d}t+(APIH)\int_c^b F_\lambda\mathrm{d}t$ for any $\lambda\in(0,1]$. Hence $\tilde{F}\in APFH[a,b]$ and

$$
\begin{aligned}
(APFH)\int_a^b\tilde{F}(t)\mathrm{d}t &= \bigcup_{\lambda\in(0,1]}\lambda(APIH)\int_a^b F_\lambda(t)\mathrm{d}t\\
&= \bigcup_{\lambda\in(0,1]}\lambda\left((APIH)\int_a^c F_\lambda(t)\mathrm{d}t+(APIH)\int_c^b F_\lambda(t)\mathrm{d}t\right)\\
&= \bigcup_{\lambda\in(0,1]}\lambda(APIH)\int_a^c F_\lambda(t)\mathrm{d}t+\bigcup_{\lambda\in(0,1]}\lambda(APIH)\int_c^b F_\lambda(t)\mathrm{d}t\\
&= (APFH)\int_a^c\tilde{F}(t)\mathrm{d}t+(APFH)\int_c^b\tilde{F}(t)\mathrm{d}t.
\end{aligned}
$$

□

Theorem 4.4. *If* $\tilde{F}(t) \le \tilde{G}(t)$ *nearly everywhere on* $[a,b]$ *and* $\tilde{F}, \tilde{G} \in APFH[a,b]$, *then*

$$(APFH)\int_a^b \tilde{F}(t)\,\mathrm{d}t \le (APFH)\int_a^b \tilde{G}(t)\,\mathrm{d}t. \tag{4.8}$$

Proof. If $\tilde{F}(t) \le \tilde{G}(t)$ nearly everywhere on $[a,b]$ and $\tilde{F}, \tilde{G} \in APFH[a,b]$, then $F_\lambda(t) \le G_\lambda(t)$ nearly everywhere on $[a,b]$ for any $\lambda \in (0,1]$ and $F_\lambda(t)$ and $G_\lambda(t)$ are AP-Henstock integrable on $[a,b]$ for any $\lambda \in (0,1]$ and

$(APFH)\int_a^b \tilde{F}(t)\,\mathrm{d}t = \bigcup_{\lambda\in(0,1]} \lambda (APIH)\int_a^b F_\lambda(t)\,\mathrm{d}t$ and

$(APFH)\int_a^b \tilde{G}(t)\,\mathrm{d}t = \bigcup_{\lambda\in(0,1]} \lambda (APIH)\int_a^b G_\lambda(t)\,\mathrm{d}t$. From Theorem 3.5 we have

$(APIH)\int_a^b F_\lambda(t)\,\mathrm{d}t \le (APIH)\int_a^b G_\lambda(t)\,\mathrm{d}t$ for any $\lambda \in (0,1]$. Hence

$$\begin{aligned}(APFH)\int_a^b \tilde{F}(t)\,\mathrm{d}t &= \bigcup_{\lambda\in(0,1]} \lambda (APIH)\int_a^b F_\lambda(t)\,\mathrm{d}t \\ &\le \bigcup_{\lambda\in(0,1]} \lambda (APIH)\int_a^b G_\lambda(t)\,\mathrm{d}t \\ &= (APFH)\int_a^b \tilde{G}(t)\,\mathrm{d}t.\end{aligned}$$

□

5. Conclusion

In this paper, we have a tendency to introduce the concept of the AP-Henstock integrals of interval-valued functions and fuzzy number-valued functions and investigate some properties of those integrals.

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