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Numerical Experiments Using MATLAB: Superconvergence of Nonconforming Finite Element Approximation for Second-Order Elliptic Problems

Anna Harris[1*], Stephen Harris[2], Danielle Rauls[1]

[1]Department of Mathematics and Computer Science, University of Arkansas at Pine Bluff, Pine Bluff, Arkansas, USA
[2]US Food and Drug Administration, National Center for Toxicology Research, Jefferson, Arkansas, USA
Email: *harrisa@uapb.edu, stephen.harris@fda.hhs.gov

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Abstract

The superconvergence in the finite element method is a phenomenon in which the finite element approximation converges to the exact solution at a rate higher than the optimal order error estimate. Wang proposed and analyzed superconvergence of the conforming finite element method by L^2-projections. However, since the conforming finite element method (CFEM) requires a strong continuity, it is not easy to construct such finite elements for the complex partial differential equations. Thus, the nonconforming finite element method (NCFEM) is more appealing computationally due to better stability and flexibility properties compared to CFEM. The objective of this paper is to establish a general superconvergence result for the nonconforming finite element approximations for second-order elliptic problems by L^2-projection methods by applying the idea presented in Wang. MATLAB codes are published at https://github.com/annaleeharris/Superconvergence-NCFEM for anyone to use and to study. The results of numerical experiments show great promise for the robustness, reliability, flexibility and accuracy of superconvergence in NCFEM by L^2-projections.

Keywords

Nonconforming Finite Element Methods, Superconvergence, L^2-Projection, Second-Order Elliptic Equation

1. Introduction

The conforming finite element method (CFEM) requires a strong continuity; hence it is not easy to construct such finite elements for the complex partial differential equations.

The nonconforming finite element method (NCFEM) is more appealing computationally due to better stability and flexibility properties compared to CFEM [1] [2] [3]. The superconvergence in the finite element method is a phenomenon in which the finite element approximation converges to the exact solution at a rate higher than the optimal order error estimate. Wang proposed and analyzed superconvergence of the conforming finite element method by L^2-projections. The main idea behind the L^2-projections is to project the finite element solution to another finite element space with a coarse mesh and a higher order of polynomials.

The objective of this paper is to establish a general superconvergence result for the nonconforming finite element approximations for second-order elliptic problems by L^2-projection methods by applying the idea presented in Wang [4].

This paper is organized as follows. In Section 2, we present a review for the nonconforming finite element method for the second-order elliptic problem. In Section 3, we develop a general theory of superconvergence by following the idea presented in Wang [4]. In Section 4, we perform numerical experiments to support the theoretical results. Numerical experiements of superconvergence of NCFEM are performed in MATLAB and its codes are posted at https://github.com/annaleeharris/Superconvergence-NCFEM for anyone to use and to study.

2. NCFEM for the Second-Order Elliptic Problem

Consider the second-order elliptic problem with the Dirichlet boundary condition which seeks $u \in H^1(\Omega)$ satisfying

$$\begin{aligned} \Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where Δ is the Laplacian operator, Ω is a bounded, connected, and open subset of R^2, $\partial\Omega$ is a Lipschitz continuous boundary, and a given function f is the external force.

A variational formulation of (1) seeks $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v), \quad \forall v \in H_0^1(\Omega),$$

where

$$a(u,v) = (\nabla u, \nabla v) = \int_\Omega \nabla u \cdot \nabla v \mathrm{d}\Omega.$$

Let $\mathcal{T}_h$ be a quasi-uniform, *i.e.*, it is regular and satisfies the inverse assumption [5], triangulation of Ω with $diam(K) \le h, K \in \mathcal{T}_h$. Let $P_k(K)$ be the space of polynomials of degree at most k with $k \ge 0$ on K. Let $\mathcal{E}_h$ denote the union of the boundaries of all elements $K \in \mathcal{T}_h$ and let $\varepsilon_h^0 = \varepsilon_h \setminus \partial\Omega$ be the collection of all interior edges. Assume that the polynomial space in the construction of V_h contains $P_k(K)$, $k \ge 1$. Define the finite element space V_h associated with $\mathcal{T}_h$ as

$$V_h = \{ v \in L^2(\Omega) : v|_K \in P_k(K), \forall K \in \mathcal{T}_h, v \text{ is continuous at the middle point of } e \in \varepsilon_h^0, \text{ and } v \text{ is zero at the middle point of boundary edge } e \text{ on } \partial\Omega \}.$$

The finite element space V_h is assumed to satisfy the following approximation property for any $u \in H^{m+1}(\Omega)$ [6]:

$$\inf_{v\in V_h}\left(\|u-v\|+h\|u-v\|_1\right)\le Ch^{m+1}\|u\|_{m+1},\quad 0\le m\le k. \tag{2}$$

The nonconforming finite element approximation problem (2) seeks $u_h \in V_h$ such that

$$a_h(u_h,v)=(f,v),\quad \forall\, v\in V_h, \tag{3}$$

where

$$a_h(u_h,v)=\sum_{K\in\mathcal{T}_h}(\nabla u_h,\nabla v)_K=\sum_{K\in\mathcal{T}_h}\int_K \nabla u_h\cdot\nabla v\mathrm{d}x.$$

A well known error estimate for the finite element approximation solution u_h is the following [7]:

$$\|u-u_h\|+h\|u-u_h\|_{1,h}\le Ch^{k+1}\|u\|_{k+1}, \tag{4}$$

where C is a constant independent of the mesh size h.

To apply the superconvergence of finite element approximation, we assume that domain Ω is so regular that it ensures a $H^s, s\ge 1$, regularity for the solution of (2). In other words, for any $f\in H^{s-2}(\Omega)$ the problem (2) has a unique solution $u\in H_0^1(\Omega)$ satisfying the following a priori estimate

$$\|u\|_s\le C\|f\|_{s-2},\quad \forall\, f\in H^{s-2}(\Omega), s\ge 1, \tag{5}$$

where C is a constant independent of data f.

3. Superconvergence of NCFEM

Let $\mathcal{T}_\tau$ be another finite element partition with coarse mesh size τ where $h\ll\tau$. Assume that τ and h have the following relation:

$$\tau=h^\alpha,\quad \alpha\in(0,1). \tag{6}$$

Let V_τ be any finite element space consisting of piecewise polynomial of degree r associated with the partition $\mathcal{T}_\tau$. Define Q_τ to be the L^2-projection from $L^2(\Omega)$ onto the finite element space V_τ. The finite element space V_τ is defined as follows:

$$V_\tau=\left\{v\in L^2(\Omega): v|_K\in P_r(K), \forall\, K\in\mathcal{T}_\tau\right\}.$$

The following lemma will provide an error estimate for $Q_\tau u-Q_\tau u_h$.

Lemma 1 *Assume that the second-order elliptic problem* (2) *holds* (5) *with* $1\le s\le k+1$ *and* $V_\tau\subset H^{s-2}(\Omega)$. *Then there exists a constant C independent of h and* τ *such that*

$$\|Q_\tau u-Q_\tau u_h\|\le Ch^{k+s-1+\alpha\min(0,2-s)}\|u\|_{k+1}, \tag{7}$$

where $\alpha\in(0,1)$ *and* $\tau\gg h$.

Proof. Using the definition of $\|\cdot\|$ and Q_τ, we have

$$\|Q_\tau u - Q_\tau u_h\| = \sup_{\phi \in L^2(\Omega), \|\phi\|=1} \left|\left(Q_\tau u - Q_\tau u_h, \phi\right)\right|$$

and

$$\left(Q_\tau u - Q_\tau u_h, \phi\right) = \left(u - u_h, Q_\tau \phi\right).$$

Then

$$\|Q_\tau u - Q_\tau u_h\| = \sup_{\phi \in L^2(\Omega), \|\phi\|=1} \left|\left(u - u_h, Q_\tau \phi\right)\right|. \tag{8}$$

Consider the following problem:

$$\begin{aligned} -\Delta w &= Q_\tau \phi \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{9}$$

Multiplying the second-order elliptic Equation (1) by v and integrating it over Ω give

$$a_h(u,v) - \sum_{K \in \mathcal{T}_h} \langle \nabla u \cdot \boldsymbol{n}, v \rangle_{\partial K} = (f, v), \tag{10}$$

where n is the unit outward normal.

Subtract (3) from the above Equation (10) gives

$$a_h(u - u_h, v) = \sum_{K \in \mathcal{T}_h} \langle \nabla u \cdot \boldsymbol{n}, v \rangle_{\partial K}, \quad \forall v \in V_h. \tag{11}$$

Multiplying (9) by $u - u_h$, integrating it over Ω, adding and subtracting $w_I \in V_h$, and using the result (11) we have

$$\begin{aligned} (Q_\tau \phi, u - u_h) &= (-\Delta w, u - u_h) \\ &= a_h(w, u - u_h) - \sum_{K \in \mathcal{T}_h} \langle \nabla w \cdot \boldsymbol{n}, u - u_h \rangle_{\partial K} \\ &= a_h(w - w_I + w_I, u - u_h) - \sum_{K \in \mathcal{T}_h} \langle \nabla w \cdot \boldsymbol{n}, u - u_h \rangle_{\partial K} \\ &= a_h(w - w_I, u - u_h) + a_h(w_I, u - u_h) - \sum_{K \in \mathcal{T}_h} \langle \nabla w \cdot \boldsymbol{n}, u - u_h \rangle_{\partial K} \\ &= a_h(w - w_I, u - u_h) + \sum_{K \in \mathcal{T}_h} \langle \nabla u \cdot \boldsymbol{n}, w_I \rangle_{\partial K} - \sum_{K \in \mathcal{T}_h} \langle \nabla w \cdot \boldsymbol{n}, u - u_h \rangle_{\partial K}. \end{aligned}$$

The line integrals of the above equations are approximated in [6] as follows:

$$\sum_{K \in \mathcal{T}_h} \langle \nabla u \cdot \boldsymbol{n}, w_I \rangle_{\partial K} \le C h^{k+s-1} \|u\|_{k+1} \|w\|_s, \tag{12}$$

$$\sum_{K \in \mathcal{T}_h} \langle \nabla w \cdot \boldsymbol{n}, u - u_h \rangle_{\partial K} \le C h^{k+s-1} \|u\|_{k+1} \|w\|_s. \tag{13}$$

Using the Cauchy-Schwartz inequality, the approximation property (2), and line integral approximations (12) and (13) we have

$$\begin{aligned} \left|(Q_\tau \phi, u - u_h)\right| &= a_h(w - w_I, u - u_h) + \sum_{K \in \mathcal{T}_h} \langle \nabla u \cdot \boldsymbol{n}, w_I \rangle_{\partial K} - \sum_{K \in \mathcal{T}_h} \langle \nabla w \cdot \boldsymbol{n}, u - u_h \rangle_{\partial K} \\ &\le \|w - w_I\| \|u - u_h\| + C h^{k+s-1} \|u\|_{k+1} \|w\|_s \\ &\le C h^{k+s-1} \|w\|_s \|u\|_{k+1}. \end{aligned}$$

Substituting $\|w\|_s$ as $\|Q_\tau\phi\|_{s-2}$ by the H^s regularity, applying the inverse inequality to the term $\|Q_\tau\phi\|_{s-2}$ and using the definition of $\tau = h^\alpha$ we have

$$\begin{aligned}\left|\left(Q_\tau\phi, u-u_h\right)\right| &\le Ch^{k+s-1}\|Q_\tau\phi\|_{s-2}\|u\|_{k+1} \\ &\le Ch^{k+s-1}\tau^{\min(0,2-s)}\|Q_\tau\phi\|\|u\|_{k+1} \\ &\le Ch^{k+s-1}\tau^{\min(0,2-s)}\|\phi\|\|u\|_{k+1} \\ &\le Ch^{k+s-1+\alpha\min(0,2-s)}\|u\|_{k+1}.\end{aligned}$$

Combining the above equation with the Equation (8) we have

$$\|Q_\tau u - Q_\tau u_h\| \le Ch^{k+s-1+\alpha\min(0,2-s)}\|u\|_{k+1}, \tag{14}$$

which completes the proof of the lemma.

The following theorem provides an error estimate for $u - Q_\tau u_h$.

Theorem 1 *Assume that* (5) *holds true with* $1 \le s \le k+1$ *and* $V_\tau \subset H^{s-2}(\Omega)$. *If* u_h *is the finite element approximation of the exact solution* $u \in H^{k+1}(\Omega) \cap H^{r+1}(\Omega) \cap H_0^1(\Omega)$ *of* (2), *then there exists a constant C independent of h and* τ *such that*

$$\begin{aligned}&\|u - Q_\tau u_h\| + h^\alpha\|\nabla_\tau(u - Q_\tau u_h)\| \\ &\le Ch^{\alpha(r+1)}\|u\|_{r+1} + Ch^{k+s-1+\alpha\min(0,2-s)}\|u\|_{k+1}.\end{aligned} \tag{15}$$

Proof. Since we assume the exact solution u is sufficiently smooth and by the definitions of Q_τ and τ, we have

$$\|u - Q_\tau u_h\| \le C\tau^{r+1}\|u\|_{r+1} = Ch^{\alpha(r+1)}\|u\|_{r+1}. \tag{16}$$

Using the triangle inequality and combining (16) and Lemma 1 we obtain

$$\begin{aligned}\|u - Q_\tau u_h\| &\le \|u - Q_\tau u\| + \|Q_\tau u - Q_\tau u_h\| \\ &\le Ch^{\alpha(r+1)}\|u\|_{r+1} + Ch^{k+s-1+\alpha\min(0,2-s)}\|u\|_{k+1},\end{aligned}$$

which completes the error estimate of $\|u - Q_\tau u_h\|$.

Similarly, we estimate $h^\alpha\|\nabla_\tau(u - Q_\tau u_h)\|$.

Using the inverse inequality and the definitions of Q_τ and τ we have

$$\|\nabla_\tau(u - Q_\tau u_h)\| \le C\tau^r\|u\|_{r+1} = Ch^{\alpha r}\|u\|_{r+1}. \tag{17}$$

Using the triangle inequality and combining (17) and Lemma 1 we have

$$\begin{aligned}&h^\alpha\|\nabla_\tau(u - Q_\tau u_h)\| \le h^\alpha\|\nabla_\tau u - \nabla_\tau Q_\tau u\| + h^\alpha\|\nabla_\tau Q_\tau u - \nabla_\tau Q_\tau u_h\| \\ &\le Ch^{\alpha(r+1)}\|u\|_{r+1} + Ch^{k+s-1+\alpha\min(0,2-s)}\|u\|_{k+1}.\end{aligned}$$

Hence the theorem has been proved.

The optimal α is selected using Theorem 1 for the error estimates:

$$\alpha(r+1) = k+s-1+\alpha\min(0,2-s),$$

$$\alpha = \frac{k+s-1}{r+1-\min(0,2-s)}. \tag{18}$$

4. Numerical Experiments of Superconvergence of NCFEM by L^2-Projection Methods

In this section, we present numerical experiments for second-order elliptic problems to support our theoretical results. Assume that the exact solution of the second-order elliptic problem has the H^s regularity for some $1 \le s \le 2$ and for simplicity, assume $k = 1, s = 2,$ and $r = 2$ which gives $\alpha = \frac{2}{3}$ using the α formula (18).

From the theoretical result (15) we have the following optimal error estimates:

$$\left\| u - Q_\tau u_h \right\| \le Ch^{\alpha(r+1)} \left\| u \right\|_{r+1} + Ch^{k+s-1+\alpha\min(0,2-s)} \left\| u \right\|_{k+1} \le Ch^2 \left\| u \right\|_3 \tag{19}$$

and

$$\left\| \nabla_\tau \left(u - Q_\tau u_h \right) \right\| \le Ch^{\alpha r} \left\| u \right\|_{r+1} + Ch^{k+s-1-\alpha+\alpha\min(0,2-s)-\alpha} \left\| u \right\|_{k+1} \le Ch^{\frac{4}{3}} \left\| u \right\|_3 . \tag{20}$$

From the results (19) and (20), theoretically, in L^2 norm the L^2-projection to the existing numerical approximation does not improve the convergence rate but in H_1 norm the L^2-projection to the existing numerical solution provides some superconvergence.

The finite element partition $\mathcal{T}_h$ is constructed by dividing the domain into an $n^3 \times n^3$ rectangular mesh then dividing the rectangular mesh with the positive slope to form two triangles. The coarse finite element partition $\mathcal{T}_\tau$ is also constructed by dividing the domain into an $n^2 \times n^2$ rectangular mesh then dividing the rectangular mesh with the positive slope to form two triangles. The finite element space V_h consists of the space of the linear polynomials $P_1(K)$ associated with the partition $\mathcal{T}_h$ and the dual finite element space V_τ consists of the space of the quadratic polynomials $P_2(K)$ associated with the partition $\mathcal{T}_\tau$. The finite element spaces V_h and V_τ are defined as follows:

$$V_h = \big\{ v \in L^2(\Omega) : v|_K \in P_k(K), \forall K \in \mathcal{T}_h, v \text{ is continuous at the middle point of } e \in \varepsilon_h^0, \text{ and } v \text{ is zero at the middle point of boundary edge } e \text{ on } \partial\Omega \big\}.$$

and

$$V_\tau = \left\{ v \in L^2(\Omega) : v|_K \in P_2(K), \forall\, K \in \mathcal{T}_\tau \right\}.$$

The numerical approximation is refined as $h = n^{-3}$ where $n = 2, \cdots, 6$. The length of $\tau = n \cdot h, n = 2, \cdots, 6$ and each τ element contains $n^2 \cdot h$ elements.

Using the α Equation (18) and our choice of $k = 1, s = 2,$ and $r = 2$ we have

$$\alpha = \frac{k + s - 2}{r + 1 - \min(0, 2 - s)} = \frac{2}{3}.$$

Using the difference in mesh size and a higher degree of polynomials we shall produce some superconvergence of NCFEM for the second-order elliptic problems.

Example 1. Let the domain $\Omega = [0,1] \times [0,1]$ and the exact solution is assumed to be as follows:

$$u = x(1-x)y(1-y).$$

From **Table 1** we observe that the L^2-projection to the existing numerical approximation u_h reduced the error estimates in L^2 norm and in H_1 norm. In L^2 norm the convergence rate of $\|u - Q_\tau u_h\|$ is similar to the convergence rate of $\|u - u_h\|$ which is the same as the theoretical result (19). The convergence rate of $|u - Q_\tau u_h|_1$ is about 33% faster than the convergence rate of $|u - u_h|_1$ in H_1 norm (see **Figure 2**). The surface plots of $Q_\tau u_h$ in coarse meshes and u_h in fine meshes are shown in **Figure 1**. The numerical example 1 clearly supports the theoretical result and confirms the superconvergence of NCFEM for the second-order elliptic problem.

Example 2. Let the domain $\Omega = [0,1] \times [0,1]$ and let the analytical solution be given as

$$u = x(1-x)y\cos(1.5\pi y).$$

From **Table 2**, we can see that the numerical example 2 supports the theoretical result (15). See **Figure 3**, when $h = 3^{-3}$ and $\tau = 3^{-2}$, we can project 3^2 fine triangle elements onto one coarse triangle element. Thus, as n increases, we can project n^2 more fine triangle elements to one coarse triangle element in which the process of refining elements produces better error estimates. The L^2-projection to the existing numerical approximation u_h produced some superconvergence in H_1 norm and did not affect the convergence rate in L^2 norm (see **Figure 4**). The numerical example 2 also

Table 1. Numerical error approximation results using NCFEM in Example 1, $u = x(1-x)y(1-y)$.

iter	h	$\|\nabla_h(u-u_h)\|$	$\|u-u_h\|$	$\|\nabla_\tau(u-Q_\tau u_h)\|$	$\|u-Q_\tau u_h\|$
1	2^{-3}	0.1388e−1	0.3909e−3	0.8184e−2	0.3920e−3
2	3^{-3}	0.4138e−2	0.3443e−4	0.1635e−2	0.3431e−4
3	4^{-3}	0.1747e−2	0.6104e−5	0.5190e−3	0.6104e−5
4	5^{-3}	0.8944e−3	0.1600e−5	0.2127e−3	0.1600e−5
5	6^{-3}	0.5176e−3	0.5358e−6	0.1026e−3	0.5359e−6
	$O(h^r)$	0.9981	2.000	1.3287	2.0010

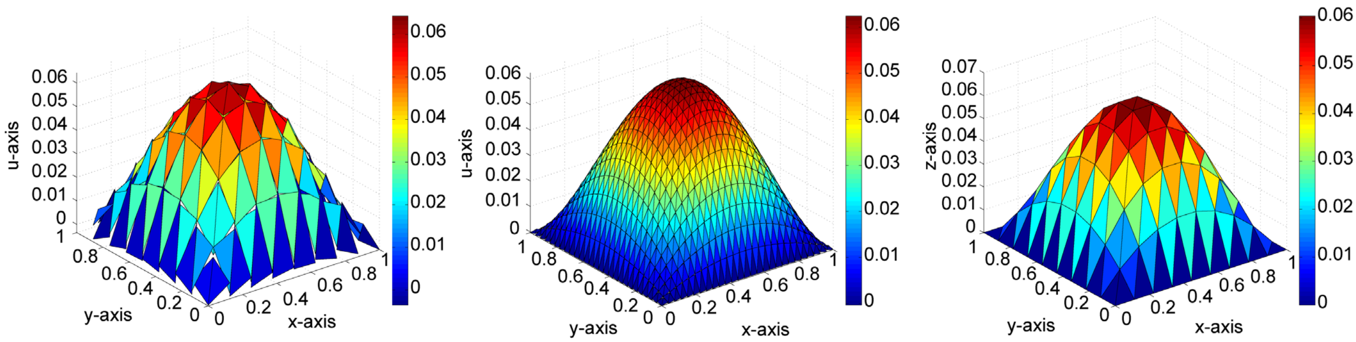

Figure 1. Surface plots of approximation using NCFEM in Example 1, $u = x(1-x)y(1-y)$. (L): Surface plot of u_h when $h = 2^{-3}$. (M): Surface of plot of u_h when $h = 3^{-3}$. (R): Surface plot of $Q_\tau u_h$ when $\tau = 3^{-2}$.

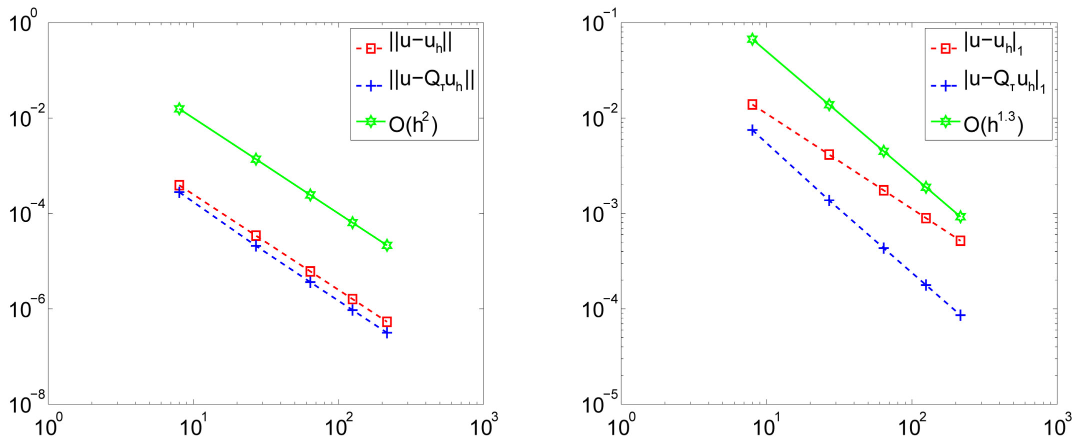

Figure 2. Error convergence rates using NCFEM in Example 1, $u = x(1-x)y(1-y)$. (L): L^2 norm error; (R): H_1 norm error.

Table 2. Numerical error approximation results using NCFEM in Example 2, $u = x(1-x)y\cos(1.5\pi y)$.

iter	h	$\|\nabla_h(u-u_h)\|$	$\|u-u_h\|$	$\|\nabla_\tau(u-Q_\tau u_h)\|$	$\|u-Q_\tau u_h\|$
1	2^{-3}	0.3933e−1	0.8429e−3	0.2214e−1	0.8453e−3
2	3^{-3}	0.1189e−1	0.7404e−4	0.4387e−2	0.7408e−4
3	4^{-3}	0.5019e−2	0.1317e−4	0.1392e−2	0.1318e−4
4	5^{-3}	0.2570e−2	0.3454e−5	0.5708e−3	0.3455e−5
5	6^{-3}	0.1487e−2	0.1156e−5	0.2754e−3	0.1157e−5
	$O(h^r)$	0.9983	1.9998	1.3311	2.0006

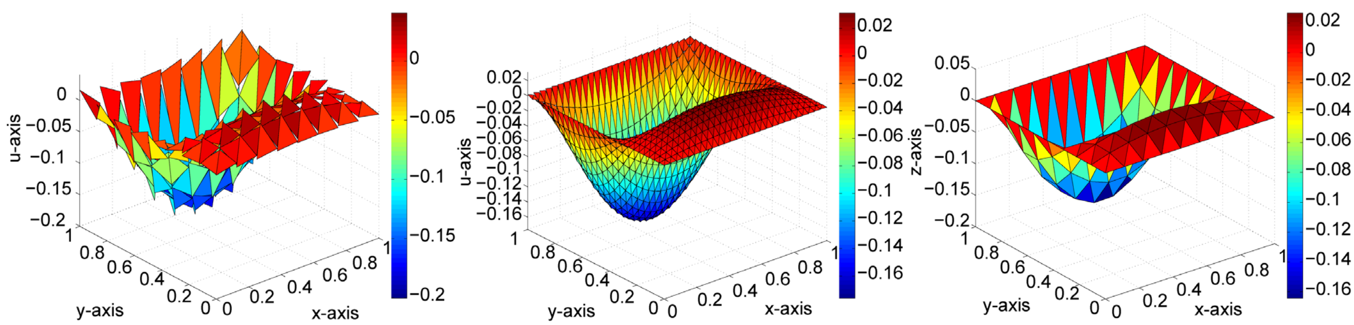

Figure 3. Surface plots of approximation using NCFEM in Example 2, $u = x(1-x)y\cos(1.5\pi y)$. (L): Surface plot of u_h when $h = 2^{-3}$. (M): Surface of plot of u_h when $h = 3^{-3}$. (R): Surface plot of $Q_\tau u_h$ when $\tau = 3^{-2}$.

supports the theoretical result and confirms the superconvergence of NCFEM for the second-order elliptic problem.

5. Conclusion

The L^2-projection to the existing numerical approximation u_h produced some superconvergence in H_1 norm, convergence rate ≥ 1.3, but did not affect the convergence

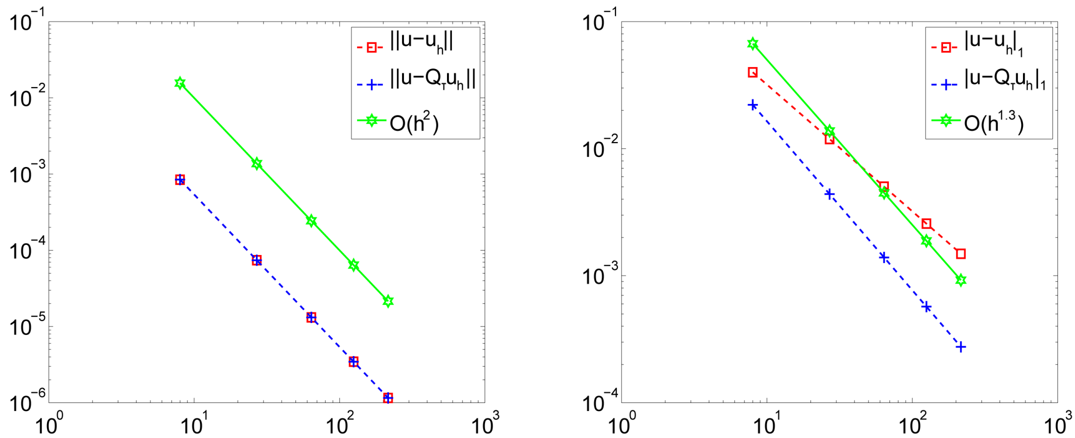

Figure 4. Error convergence rates using NCFEM in Example 2, $u = x(1-x)y\cos(1.5\pi y)$. (L): L^2 norm error; (R): H_1 norm error.

rate in L^2 norm. With the numerical experiments we can conclusively support the theoretical result and confirm the superconvergence of NCFEM for second-order elliptic problems by L^2-projection method.

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