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Spectral Density Estimation of Continuous Time Series

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Abstract

This paper studies spectral density estimation of a strictly stationary r-vector valued continuous time series including missing observations. The finite Fourier transform is constructed in L-joint segments of observations. The modified periodogram is defined and smoothed to estimate the spectral density matrix. We explore the properties of the proposed estimator. Asymptotic distribution is discussed.

Keywords

Joint Segments of Observations, Modified Periodograms, Spectral Density Matrix, Wishart Matrix

1. Introduction

Although spectral analysis is one of the oldest tools for time series analysis, it is still one of the most widely used analysis techniques in many branches of sciences, [1]-[6]. For zero mean r-vector valued strictly stationary time series, the spectral estimation has been studied, [7]-[17]. Time series with missing observations frequantly appear in paractice. If a block of observations is periodically unobtainable, Jones [18] provides a development for spectral estimation of a stationary time series. The theory of amplitude-modulated stationary processes has been developed by Parzen [19] and applied to periodic missing observations problems [20]. The case where an observation is made or not according to the out come of a Bernoulli trial has been discussed by Scheinok [21]. Bloomfield [22] considered the case where a more general random mechanism is involved. Broersen *et al.* [23] and [24] developed models for time series with missing observation and discussed their use for spectral estimation. Unbiased spectral estimators have been formulated assuming wavelet models of stationary time series by [25]. Their asymptotic properties have been also investigated.

In this paper, we will discuss the spectral analysis of a strictly stationary *r*-vector valued continuous time series with randomly missing observations in joint segments of observations. The paper is organized as follows. Section 2 introduces the basic definitions and assumptions. The modified series is defined in Section 3. Section 4 considers the expanded finite Fourier transform and its properties. The modified periodogram, the spectral density estimator and its properties are given in Section 5.

2. Observed Series

Let $X(t)(t \in R)$ be a zero mean *r*-vector valued strictly stationary time series with

$$E\left\{X(t+u)\bar{X}'(t)\right\} = C_{XX}(u),(t,u \in R), \tag{2.1}$$

and

$$\int_{-\infty}^{\infty} \left|C_{XX}(u)\right| \mathrm{d}u < \infty, \tag{2.2}$$

where $\left|C_{XX}(u)\right|$ denotes the matrix of absolute values, the bar denotes the complex conjugate and '' ' denotes the matrix transpose. We may then define $f_{XX}(\lambda)$ the $r \times r$ matrix of second order spectral densities by

$$f_{XX}(\lambda) = (2\pi)^{-1} \int_{-\infty}^{\infty} C_{XX}(u)\exp(-i\lambda u)\mathrm{d}u,(\lambda \in R). \tag{2.3}$$

Using the assumed stationary, we then set down

Assumption I. $X(t)$ is a strictly stationary continuous series all of whose moments exist. For each $j = 1,2,\cdots,k-1$ and any *k*-tuple $a_1, a_2, \cdots, a_k$ we have

$$\int_{R^{k-1}} \left|u_j C_{a_1,\cdots,a_k}(u_1,\cdots,u_{k-1})\right| < \infty,\ k = 2,3,\cdots \tag{2.4}$$

where

$$C_{a_1,\cdots,a_k}(u_1,u_2,\cdots,u_{k-1}) = cum\left\{X_{a_1}(t+u_1), X_{a_2}(t+u_2),\cdots, X_{a_k}(t)\right\}, \tag{2.5}$$

($a_1, a_2, \cdots, a_k = 1,2,\cdots,r$; $u_1,u_2,\cdots,u_{k-1}, t \in R; k = 2,\cdots$).

Because cumulants are measures of the joint dependence of random variables, (2.4) is seen to be a form of mixing or asymptotic independence requirement for values of $X(t)$ well separated in time. If $X(t)$ satisfies Assumption I we may define its cumulant spectral densities by

$$\begin{aligned} &f_{a_1,\cdots,a_k}(\lambda_1,\cdots,\lambda_{k-1}) \\ &= (2\pi)^{-k+1} \int_{R^{k-1}} C_{a_1,\cdots,a_k}(u_1,\cdots,u_{k-1}) \times \exp\left(-i\sum_{j=1}^{k-1}\lambda_j u_j\right)\mathrm{d}u_1\cdots\mathrm{d}u_{k-1}, \end{aligned} \tag{2.6}$$

($-\infty < \lambda_j < \infty, a_1,a_2,\cdots,a_k = 1,2,\cdots,r; k = 2,\cdots$). If $k = 2$ the cross-spectra $f_{a_1a_2}(\lambda)$ are collected together in the matrix $f_{XX}(\lambda)$ of (2.3).

Assumption II. Let $h_a^{(T)}(t) = h_a\left(\frac{t}{T}\right), t \in [0,T)$ is bounded, is of bounded variation and vanishes for all t outside the interval $[0,T)$, that is called data window.

3. Modified Series

Let $D(t)=\{D_a(t), t\in R\}_{a=1,2,\cdots,r}$ be a process independent of $X(t)$ such that, for every t

$$P\{D_a(t)=1\}=p, P\{D_a(t)=0\}=q,$$

note that

$$E\{D_a(t)\}=p.$$

The success of recording an observation not depend on the fail of another and so it is independent. We may then define the modified series

$$Y(t)=D(t)X(t),$$

with components,

$$Y_a(t)=D_a(t)X_a(t),$$

where

$$D_a(t)=\begin{cases}1 & \text{if } X_a(t) \text{ is observed}\\ 0 & \text{if } X_a(t) \text{ is missed}\end{cases}$$

4. Expanded Finite Fourier Transform in *L*-Joint Segments of Observations

In the case when there are some randomly missing observations, Elhassanein [17] constructed the expanded finite Fourier transform on disjoint segments of observations. In this section the expanded finite Fourier transform is constructed in *L*-joint segments of observations for a strictly stationary *r*-vector valued time series. Expression for its mean, variance and cumulant will be derived. The results introduced here may be regarded as a generalization to [13] and [17]. Let $X(t)(t\in(0,T))$ be an observed stretch of data with some randomly missing observations. Let $T=L(N-M)+M$, where L is the number of joint segments and N is the length of each segment and M is the length of joint parts, $0\le M<N$, where $M=0$ we get the results in [17]. The expanded finite Fourier transform of a given stretch of data, is defined by

$$\begin{aligned} d_Y^{l(N-M)}(\lambda)=&\left(2\pi\int_{l(N-M)}^{(l+1)(N-M)+M}\left[h^{(N)}(t-l(N-M))\right]^2 \mathrm{d}t\right)^{-\frac{1}{2}}\\ &\times\int_{l(N-M)}^{(l+1)(N-M)+M} h^{(l(N-M))}(t-l(N-M))\exp(-i\lambda t)Y(t)\mathrm{d}t, \end{aligned} \quad (4.1)$$

where $-\infty<\lambda<\infty, l=0,1,\cdots,L-1,$ and $h(t)$ is the data window satisfies *Assumption II*.

Theorem 4.1. Let $X(t)(t\in(0,T))$ be a strictly stationary *r*-vector valued time series with mean zero, and satisfy Assumption I. Let $d_a^{l(N-M)}(\lambda)$ be defined as (3.1), and $h_a(t)$ satisfy *Assumption II*, for $a=1,2,\cdots,r,$ then

$$E\left\{d_a^{l(N-M)}(\lambda)\right\}=0 \tag{4.2}$$

$$\begin{aligned}&Cov\left\{d_a^{l(N-M)}(\lambda_1),d_b^{l(N-M)}(-\lambda_2)\right\}\\&=p^2\exp\left(-i(\lambda_1-\lambda_2)l(N-M)\right)\int_{-N}^{N}C_{ab}(u)\exp(-i\lambda_1 u)H_{ab}^{(N)}(u,\lambda_1-\lambda_2)\mathrm{d}u\\&=p^2\exp\left(-i(\lambda_1-\lambda_2)l(N-M)\right)\int_R f_{ab}(\nu)\Phi_{ab}^{(N)}(\lambda_1-\nu,\lambda_2-\nu)\mathrm{d}\nu\end{aligned} \tag{4.3}$$

where

$$H_{ab}^{(N)}(u,\lambda_1-\lambda_2)=(2\pi)^{-1}\left[\int_0^N\int_0^N\left(h_a^{(N)}(t_1)\right)^2\left(h_b^{(N)}(t_2)\right)^2\mathrm{d}t_1\mathrm{d}t_2\right]^{-\frac{1}{2}}\times\int_0^N h_a^{(N)}(u+t)h_b^{(N)}(t)\exp\left(-it(\lambda_1-\lambda_2)\right)\mathrm{d}t,$$

and

$$\Phi_{ab}^{(N)}(\lambda_1,\lambda_2)=(2\pi)^{-1}\left[\int_0^N\int_0^N\left(h_a^{(N)}(t_1)\right)^2\left(h_b^{(N)}(t_2)\right)^2\mathrm{d}t_1\mathrm{d}t_2\right]^{-\frac{1}{2}}\times H_a^{(N-M)}(\lambda_1)\overline{H_b^{(N-M)}(\lambda_2)}$$

where

$$H_a^{(N)}(\lambda)=\int_0^N h_a^{(N)}(t)\exp(-i\lambda t)\mathrm{d}t$$

for $\lambda_1=\lambda_2=\lambda, a=b$ then

$$Var\left\{d_a^{(l(N-M))}(\lambda)\right\}=p\int_{-\infty}^{\infty}f_{aa}(\lambda-\nu)\Phi_{aa}^{(N)}(\nu)\mathrm{d}\nu, \tag{4.4}$$

$$\begin{aligned}&Cum\left\{d_{a_1}^{l_1(N-M)}(\lambda_1),\cdots,d_{a_k}^{l_k(N-M)}(\lambda_k)\right\}\\&=(2\pi)^{\frac{k}{2}-1}p^k\left(\prod_{j=1}^{k}\int_0^N\left(h_{a_j}^{(N)}(t_j)\right)^2\mathrm{d}t_j\right)^{-\frac{1}{2}}f_{a_1a_2\cdots a_k}(\lambda_1,\lambda_2,\cdots,\lambda_{k-1})G_{a_1\cdots a_k}^{(N)}\left(\sum_{j=1}^{k}\lambda_j\right)+O\left(N^{-\frac{k}{2}}\right)\end{aligned} \tag{4.5}$$

where $O\left(N^{-\frac{k}{2}}\right)$ is uniform in $\lambda_1,\lambda_2,\cdots,\lambda_{k-1}$ as $N\to\infty$, $k=2,\cdots$ and

$$G_{a_1\cdots a_k}^{(N)}=\int_0^N\left(\prod_{j=1}^{k}h_{aj}^{(N)}(t_j)\right)\exp(-i\lambda t)\mathrm{d}t,\ \lambda\neq 0,\lambda,t\in R,$$

Proof. We will prove (4.5), by (4.1) we get

$$\begin{aligned}&Cum\left\{d_{a_1}^{(l(N-M))}(\lambda_1),\cdots,d_{a_k}^{(l(N-M))}(\lambda_k)\right\}\\&=(2\pi)^{\frac{k}{2}}p^k\left(\prod_{j=1}^{k}\int_0^N\left(h_{a_j}^{(N)}(t_j)\right)^2\mathrm{d}t_j\right)^{-\frac{1}{2}}\int_0^N\cdots\int_0^N\left(\prod_{j=1}^{k}h_{a_j}^{(N)}(t_j)\right)\exp\left(-i\sum_{j=1}^{k}\lambda_j t_j\right)\\&\times C_{a_1,a_2,\cdots,a_k}(t_1-t_k,\cdots,t_{k-1}-t_k)\mathrm{d}t_1\cdots\mathrm{d}t_k,\end{aligned}$$

let $t_j - t_k = u_j, t_k = t, j = 1,2,\cdots,k-1,$ and since

$$\left|\int_0^N\left(\prod_{j=1}^{k-1}h_{aj}^{(N)}(u_j+t)\right)h_{a_k}^{(N)}(t)\exp(-i\lambda t)\mathrm{d}t-\int_0^N\left(\prod_{j=1}^{k}h_{aj}^{(N)}(t_j)\right)\exp(-i\lambda t)\mathrm{d}t\right|\le A^{k-1}C\left(\sum_{j=1}^{k-1}|u_j|\right)$$

for some constants A,C and $(t_j,u_j,\lambda\in R, j=1,\cdots,k)$, we get

$$\begin{aligned}&Cum\left\{d_{a_1}^{(l(N-M))}(\lambda_1),\cdots,d_{a_k}^{(l(N-M))}(\lambda_k)\right\}\\&=(2\pi)^{\frac{k}{2}}p^k\left(\prod_{j=1}^{k}\int_0^N\left(h_{a_j}^{(N)}(t_j)\right)^2\mathrm{d}t_j\right)^{-\frac{1}{2}}\int_0^N\left(\prod_{j=1}^{k}h_{a_j}^{(N)}(t_j)\right)\exp\left(-it\sum_{j=1}^{k}\lambda_j\right)\mathrm{d}t\\&\times\int_{-N}^{N}\cdots\int_{-N}^{N}C_{a_1,a_2,\cdots,a_k}(u_1,\cdots,u_{k-1})\exp\left(-i\sum_{j=1}^{k-1}\lambda_ju_j\right)\mathrm{d}u_1\cdots\mathrm{d}u_{k-1}+\varepsilon_T,\end{aligned}$$

where

$$\begin{aligned}|\varepsilon_T|&\le(2\pi)^{\frac{k}{2}}p^k\left(\prod_{j=1}^{k}\int_0^{N-1}\left(h_{a_j}^{(N)}(t_j)\right)^2\mathrm{d}t_j\right)^{-\frac{1}{2}}\\&\times\int_{-(N-1)}^{N-1}\cdots\int_{-(N-1)}^{N-1}A^{k-1}C\left(\sum_{j=1}^{k-1}|u_j|\right)\left|C_{a_1,a_2,\cdots,a_k}(u_1,\cdots,u_{k-1})\right|\mathrm{d}u_1\cdots\mathrm{d}u_{k-1}\end{aligned}$$

since $h_{a_j}^{(N)}(t_j)$ satisfy Assumption II for $j=1,\cdots,k$ then

$$\prod_{j=1}^{k}\int_0^N\left(h_{a_j}^{(N)}(t_j)\right)^2\mathrm{d}t_j\sim T^k\prod_{j=1}^{k}\int_0^1\left(h(u_j)\right)^2\mathrm{d}u_j$$

which implies to $\varepsilon_T = O\left(T^{-\frac{k}{2}}\right)$, using (2.6) the proof is completed. □

5. Estimation

Using expanded finite Fourier transform (4.1), we construct the modified periodogram as

$$I_{ab}^{l(N-M)}(\lambda)=\left(2\pi p^2\int_{l(N-M)}^{(l+1)(N-M)+M}h_a^{(N)}(t)h_b^{(N)}(t)\mathrm{d}t\right)^{-1}\alpha_a^{l(N-M)}(\lambda)\overline{\alpha_b'^{l(N-M)}(\lambda)},\tag{5.1}$$

such that

$$\alpha_a^{(N-M)}(\lambda)=\sqrt{2\pi\int_{l(N-M)}^{(l+1)(N-M)+M}\left[h_a^{(N)}(t)\right]^2\mathrm{d}t}\,d_a^{(l(N-M))}(\lambda),$$

where the bar denotes the complex conjugate. The smoothed spectral density estimate is constructed as

$$f_{ab}^{(T)}(\lambda)=\frac{1}{L}\int_0^L I_{ab}^{l(N-M)}(\lambda)\mathrm{d}u, a,b=1,2,\cdots,r\tag{5.2}$$

Theorem 5.1. Let $X(t)(t\in R)$ be a strictly stationary *r*-vector valued continuous time series with mean zero, and satisfy Assumption I. Let $I_{YY}^{(T)}(\lambda)=\left\{I_{ab}^{(T)}(\lambda)\right\}_{a,b=1,2,\cdots,r}$ be given by (3.6), and $\Phi_a(t)$ satisfy *Assumption II* for $a=1,2,\cdots,r,$ then

$$E\left\{I_{ab}^{l(N-M)}(\lambda)\right\} = f_{ab}(\lambda) + O\left(N^{-1}\right), p \to 1 \tag{5.3}$$

$$\begin{aligned}
&Cov\left\{I_{a_1b_1}^{l(N-M)}(\lambda_1), I_{a_2b_2}^{l(N-M)}(\lambda_2)\right\} = \left(G_{a_1b_1}G_{a_2b_2}\Phi_{a_1b_1}^{(N)}(0)\Phi_{a_2b_2}^{(N)}(0)\right)^{-1} \\
&\times\Big[G_{a_1a_2}G_{b_1b_2}\Phi_{a_1a_2}^{(N)}(\lambda_1-\lambda_2)\overline{\Phi_{b_1b_2}^{(N)}(\lambda_1-\lambda_2)}f_{a_1a_2}(\lambda_1)f_{b_1b_2}(-\lambda_1) \\
&+G_{a_1b_2}G_{b_1a_2}\Phi_{a_1b_2}^{(N)}(\lambda_1-\lambda_2)\overline{\Phi_{b_1a_2}^{(N)}(\lambda_1+\lambda_2)}f_{a_1b_2}(\lambda_1)f_{b_1a_2}(-\lambda_1) \\
&+(2\pi)G_{a_1b_1a_2b_2}\Phi_{a_1b_1a_2b_2}^{(N)}(0)f_{a_1b_1a_2b_2}(\lambda_1,-\lambda_1,\lambda_2)\Big] + O\left(N^{-1}\right)
\end{aligned} \tag{5.4}$$

$$\begin{aligned}
&Cum\left\{I_{a_1b_1}^{l_1(N-M)}(\lambda_1),\cdots,I_{a_kb_k}^{l_k(N-M)}(\lambda_k)\right\} \\
&=\left(\prod_{i=1}^{k}G_{a_ib_i}\Phi_{a_ib_i}^{(N)}(0)\right)^{-1}\sum\left\{\prod_{j=1}^{k}G_{a_jb_j}\exp\left(-il_k(N-M)\sum_{j=1}^{k}(\mu_j+\gamma_j)\right)\right. \\
&\left.\times\Phi_{c_jd_j}^{(N)}(\mu_j+\gamma_j)\right\}\left\{\prod_{j=1}^{k}f_{c_jd_j}(\mu_j)\right\}+O\left(N^{-1}\right)
\end{aligned} \tag{5.5}$$

where the summation extends over all partitions $\{(c_1,\mu_1),(d_1,\gamma_1)\},\cdots,\{(c_k,\mu_k),(d_k,\gamma_k)\}$, into pairs of the quantities $(a_1,\lambda_1),(b_1,-\lambda_1),\cdots,(a_k,\lambda_k),(b_k,-\lambda_k)$ excluding the case with $\mu_j=-\gamma_j=\lambda_m$ for some j,m, where $O\left(N^{-1}\right)$ is uniform in $\lambda_1,\cdots,\lambda_k$.

Proof. By (5.1), we have

$$\begin{aligned}
E\left\{I_{ab}^{l(N-M)}(\lambda)\right\} &= \left(p^2G_{ab}\Phi_{ab}^{(N)}(0)\right)^{-1}E\left\{d_a^{l(N-M)}(\lambda)\overline{d_b'^{l(N-M)}(\lambda)}\right\} \\
&= Cov\left\{d_a^{l(N-M)}(\lambda), d_b^{l(N-M)}(\lambda)\right\}
\end{aligned}$$

then by (4.3) the proof of (5.3) is completed. From (5.1), and by *Theorem* (2.3.2) in [10] p. 21, we have

$$\begin{aligned}
&Cov\left\{I_{a_1b_1}^{l(N-M)}(\lambda_1), I_{a_2b_2}^{l(N-M)}(\lambda_2)\right\} \\
&= Cov\left\{d_{a_1}^{l(N-M)}(\lambda_1)d_{b_1}'^{l(N-M)}(-\lambda_1), d_{a_2}^{l(N-M)}(\lambda_2)d_{b_2}'^{l(N-M)}(-\lambda_2)\right\} \\
&= Cum\left\{d_{a_1}^{l(N-M)}(\lambda_1), d_{b_1}^{l(N-M)}(-\lambda_1), d_{a_2}^{l(N-M)}(\lambda_2), d_{b_2}^{l(N-M)}(-\lambda_2)\right\} \\
&\quad + Cov\left\{d_{a_1}^{l(N-M)}(\lambda_1), d_{a_2}^{l(N-M)}(\lambda_2)\right\}Cov\left\{d_{b_1}^{l(N-M)}(-\lambda_1), d_{b_2}^{l(N-M)}(-\lambda_2)\right\} \\
&\quad + Cov\left\{d_{a_1}^{l(N-M)}(\lambda_1), d_{b_2}^{l(N-M)}(-\lambda_2)\right\}Cov\left\{d_{b_1}^{l(N-M)}(-\lambda_1), d_{a_2}^{l(N-M)}(\lambda_2)\right\}.
\end{aligned}$$

By *Theorem* (4.1) the proof of (5.4) is completed. From (5.1), we have

$$\begin{aligned}
&Cum\left\{I_{a_1b_1}^{l_1(N-M)}(\lambda_1),\cdots,I_{a_kb_k}^{l_k(N-M)}(\lambda_k)\right\} = p^{-2k}\left(\prod_{i=1}^{k}G_{a_ib_i}\Phi_{a_ib_i}^{(N)}(0)\right)^{-1} \\
&\times Cum\left\{d_{a_1}^{l_1(N-M)}(\lambda_1)d_{b_1}'^{l_1(N-M)}(-\lambda_1),\cdots,d_{a_k}^{l_k(N-M)}(\lambda_k)d_{b_k}'^{l_k(N-M)}(-\lambda_k)\right\}
\end{aligned}$$

By *Theorem* (2.3.2) in [10] p. 21, we get

$$\begin{aligned}
&Cum\left\{d_{a_1}^{l_1(N-M)}(\lambda_1)d_{b_1}'^{l_1(N-M)}(-\lambda_1),\cdots,d_{a_k}^{l_k(N-M)}(\lambda_k)d_{b_k}'^{l_k(N-M)}(-\lambda_k)\right\} \\
&= \sum_{\nu}Cum\left\{d^{l_i(N-M)}(\lambda_i); i\in\nu_1\right\}\cdots Cum\left\{d^{l_i(N-M)}(\lambda_i); i\in\nu_s\right\},
\end{aligned}$$

where the summation extends over all indecomposable partitions $\nu = \left[\bigcup_{j=1}^{s} \nu_j\right] \in I$, $I = (a_1, \cdots, a_k; b_1, \cdots, b_k)$, $1 \le s \le k$ of the transformed table

$$\begin{matrix} (a_1,\lambda_1),(b_1,-\lambda_1) & & \{(c_1,\mu_1),(d_1,\gamma_1)\} \\ (a_2,\lambda_2),(b_2,-\lambda_2) & \rightarrow & \{(c_2,\mu_2),(d_2,\gamma_2)\} \\ \vdots & & \vdots \\ (a_k,\lambda_k),(b_k,-\lambda_k) & & \{(c_k,\mu_k),(d_k,\gamma_k)\}. \end{matrix}$$

Then, by *Theorem* (4.1), we get the proof of (5.5). □

Theorem 5.2. Let $X(t)(t \in R)$ be a strictly stationary *r*-vector valued time series with mean zero, and satisfy Assumption I. Let $I_{YY}^{l(N-M)}(\lambda) = \left\{I_{ab}^{l(N-M)}(\lambda)\right\}_{a,b=1,2,\cdots,r}$ be given by (3.6), $2\lambda_j, \lambda_j \pm \lambda_k \not\equiv 0 (\bmod 2\pi)$ for $1 \le j < k \le J$ and $\Phi_a(t)$ satisfy *Assumption II* for $a = 1, 2, \cdots, r$. Then $I_{YY}^{l(N-M)}(\lambda_j), j = 1, 2, \cdots, J$ are asymptotically independent $W_r^c(1, f_{XX}(\lambda_j))$ variates. Also if $\lambda = \pm\pi, \pm 3\pi, \cdots$. then $I_{YY}^{l(N-M)}(\lambda)$ is asymptotically $W_r(1, f_{XX}(\lambda))$ independent of the previous variates. Where, $W_r(\gamma, \Sigma)$ denotes an $r \times r$ symmetric matrix-valued Wishart variate with covariance matrix Σ and γ degree of freedom and $W_r^c(\gamma, \Sigma)$ denotes an $r \times r$ Hermitian matrix-valued complex Wishart variate with covariance matrix Σ and γ degree of freedom.

Proof. The proof comes directly from *Theorem* (4.2), for more details about Wishart distribution see [26]. □

Theorem 5.3. Let $X(t)(t \in R)$ be a strictly stationary *r*-vector valued time series with mean zero, and satisfy Assumption I. Let $f_{ab}^{(T)}(\lambda)$ be given by (3.7), $a, b = 1, 2, \cdots, r$, then

$$E\left\{f_{ab}^{(T)}(\lambda)\right\} = f_{ab}(\lambda) + O(N^{-1}) \tag{5.6}$$

$$\begin{aligned} & Cov\left\{f_{a_1b_1}^{(T)}(\lambda_1), f_{a_2b_2}^{(T)}(\lambda_2)\right\} \\ & = \left(L^2 \Phi_{a_1b_1}^{(N)}(0) \Phi_{a_2b_2}^{(N)}(0)\right)^{-1} \int_0^L \int_0^L \left(G_{a_1b_1}(l_1,l_1) G_{a_2b_2}(l_2,l_2)\right)^{-1} \\ & \times \Big[G_{a_1a_2}(l_1,l_2) G_{b_1b_2}(l_1,l_2) \Phi_{a_1a_2}^{(N)}(\lambda_1 - \lambda_2) \overline{\Phi_{b_1b_2}^{(N)}(\lambda_1 - \lambda_2)} \\ & \times \exp(-il_2(N-M)(\lambda_1 - \lambda_2)) f_{a_1a_2}(\lambda_1) f_{b_1b_2}(-\lambda_1) \\ & + G_{a_1b_2}(l_1,l_2) G_{b_1a_2}(l_1,l_2) \Phi_{a_1b_2}^{(N)}(\lambda_1 + \lambda_2) \overline{\Phi_{b_1a_2}^{(N)}(\lambda_1 + \lambda_2)} \\ & \times \exp(-il_2(N-M)(\lambda_1 + \lambda_2)) f_{a_1b_2}(\lambda_1) f_{b_1a_2}(-\lambda_1) \\ & + (2\pi) G_{a_1b_1a_2b_2}(l_1,l_1,l_2,l_2) \Phi_{a_1b_1a_2b_2}^{(N)}(0) f_{a_1b_1a_2b_2}(\lambda_1, -\lambda_1, \lambda_2)\Big] \mathrm{d}u_1 \mathrm{d}u_2 + O(N^{-1}) \end{aligned} \tag{5.7}$$

Proof. By (5.2), we have

$$E\left\{f_{ab}^{(T)}(\lambda)\right\} = \frac{1}{L} \int_0^L E\left\{I_{ab}^{l(N-M)}(\lambda)\right\}$$

then by (5.3) the proof of (5.6) is completed. From (5.2), we get

$$Cov\left\{f_{a_1b_1}^{(T)}(\lambda_1), f_{a_2b_2}^{(T)}(\lambda_2)\right\} = \frac{1}{L^2} \int_0^L \int_0^L Cov\left\{I_{a_1b_1}^{l(N-M)}(\lambda_1), I_{a_2b_2}^{l(N-M)}(\lambda_2)\right\} \mathrm{d}u_1 \mathrm{d}u_2.$$

which completes the proof of (5.7). □

Theorem 5.4. Let $X(t)(t \in R)$ be a strictly stationary *r*-vector valued time series with mean zero, and satisfy Assumption I. Let $f_{ab}^{l(N-M)}(\lambda)$ be given by (5.2), $a,b=1,2,\cdots,r$, $2\lambda_j, \lambda_j \pm \lambda_k \not\equiv 0(\mathrm{mod}\, 2\pi)$ for $1 \le j < k \le J$, Then $Lf_{ab}^{l(N-M)}(\lambda_j), j=1,2,\cdots,J$ are asymptotically independent $W_r^c(L, f_{ab}(\lambda_j))$ variates. Also if $\lambda = \pm\pi, \pm 3\pi, \cdots$. then $Lf_{ab}^{l(N-M)}(\lambda)$ is asymptotically $W_r(L, f_{ab}(\lambda))$ independent of the previous variates.

Proof. The proof comes directly by *Theorem* (5.3) and *Theorem* (7.3.2) in [26] p. 162. □

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