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Alternative Fourier Series Expansions with Accelerated Convergence

Wenlong Li

Advanced Information Services, Fenghua, China
Email: wlee7915@yahoo.com

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Abstract

The key objective of this paper is to improve the approximation of a sufficiently smooth nonperiodic function defined on a compact interval by proposing alternative forms of Fourier series expansions. Unlike in classical Fourier series, the expansion coefficients herein are explicitly dependent not only on the function itself, but also on its derivatives at the ends of the interval. Each of these series expansions can be made to converge faster at a desired polynomial rate. These results have useful implications to Fourier or harmonic analysis, solutions to differential equations and boundary value problems, data compression, and so on.

Keywords

Fourier Series, Trigonometric Series, Fourier Approximation, Convergence Acceleration

1. Introduction

There is perhaps no better way starting the discussion than quoting directly from Iserles and Nørsett [1]: "*By any yardstick, Fourier series are one of the greatest and most influential concepts of contemporary mathematics. … It is thus with a measure of trepidation and humility that we wish to pursue a variation upon the Fourier theme in this paper.*" Since trigonometric series was first used by d'Alembert in 1747, the full formation of Fourier theories surprisingly took more than a century of endeavors highlighted by the famous d'Alembert-Euler-Bernoulli controversy and many important and/or pioneering contributions from Euler, Dirichlet, Lagrange, Lebesgue and other leading mathematicians of the time. Nevertheless, it was not long before mathematicians and scientists came to appreciate the power and far-reaching implications of Fourier's claim that any function could be expanded into a trigonometric series. Fourier's discovery is easily ranked in the "top ten" mathematical advances of all time.

Despite what has been said, the Fourier series will lose much of its luster when used to expand a sufficiently smooth nonperiodic function defined on a compact interval. It is well known that a continuous function can always be expanded into a Fourier series *inside* the interval (the word "inside" is highlighted to emphasize the fact that the two end points shall not be automatically included). This is actually the primary reason for the inefficiency of the Fourier series in approximating a nonperiodic function, and, understandably, in solving various boundary value problems. This work is aimed at overcoming the said difficulties associated with the conventional Fourier series.

It is known that a continuous function $f(x)$ defined on the interval $[-\pi, \pi]$ can always be expanded into a Fourier series

$$f(x) = a_0/2 + \sum_{m=1}^{\infty} a_m \cos mx + b_m \sin mx, \quad -\pi < x < \pi, \tag{1.1}$$

where the expansion coefficients are calculated from

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,\mathrm{d}x, \tag{1.2}$$

and

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,\mathrm{d}x. \tag{1.3}$$

The Fourier series, (1.1), reduces to

$$f(x) = \sum_{m=1}^{\infty} b_m \sin mx \tag{1.4}$$

if $f(x)$ is an odd function;
and to

$$f(x) = a_0/2 + \sum_{m=1}^{\infty} a_m \cos mx \tag{1.5}$$

if $f(x)$ is an even function.

The convergence of the Fourier series, (1.1), is well understood through the following theorems.

THEOREM 1. *If* $f(x)$ *is an absolutely integrable piecewise smooth function of period of* 2π, *then the Fourier series of* $f(x)$ *converges to* $f(x)$ *at points of continuity and to* $\frac{1}{2}\left[f(x+0)+f(x-0)\right]$ *at points of discontinuity. If* $f(x)$ *is continuous everywhere, then the series converges absolutely and uniformly.*

Proof. Pages 75-78 of Ref. [2].

THEOREM 2. *For any absolutely integrable function* $f(x)$, *its Fourier coefficients satisfy*

$$\lim_{m\to\infty} a_m = \lim_{m\to\infty} b_m = 0. \tag{1.6}$$

Proof. Pages 70-71 of Ref. [2].

THEOREM 3. *Let* $f(x)$ *be a continuous function of period* 2π, *which has n deriva-*

tives, where n − 1 derivatives are continuous and the n-th derivative is absolutely integrable (the n-th derivative may not exist at certain points). Then, the Fourier series of all n derivatives can be obtained by term-by-term differentiation of the Fourier series of $f(x)$, *where all the series, except possibly the last, converge uniformly to the corresponding derivatives. Moreover, the Fourier coefficients of the function* $f(x)$ *satisfy the relations*

$$\lim_{m\to\infty} a_m m^n = \lim_{m\to\infty} b_m m^n = 0. \tag{1.7}$$

Proof. Pages 84, 130, and 131 of Ref. [2].

As a matter of fact, (1.7) can be replaced by more explicit expressions [3] [4]

$$a_m = b_m = \mathcal{O}\left(m^{-n-1}\right) \tag{1.8}$$

and

$$\max_{-\pi\le x\le\pi}\left|f(x)-S_M(x)\right| = \mathcal{O}\left(M^{-n}\right) \tag{1.9}$$

where $S_M(x)$ is the partial sum of the Fourier series defined as

$$S_M(x) = a_0/2 + \sum_{m=1}^{M-1} a_m \cos mx + b_m \sin mx. \tag{1.10}$$

The aforementioned convergence theorems are established based on the condition that $f(x)$ is a periodic function of period 2π. It is known that the Fourier series of an analytic 2π-periodic function can actually converge at an exponential rate [5]. However, once the periodicity condition is removed, the convergence of a series expansion can be seriously deteriorated or even there is no convergence in the maximum norm. When $f(x)$ is defined only on a compact interval [−π, π], it can be viewed as the part of the 2π-periodic function which is the periodic extension of $f(x)$ onto the whole x-axis. Thus, even $f(x)$ is sufficiently smooth on [−π, π], the extended periodic function may only be piece-wise smooth due to the potential discontinuities at $x = \pi \pm 2m\pi$ ($m = 0,1,2,\cdots$). As a consequence, the series expansion of $f(x)$ converges to $f(x)$ for every $x \in (-\pi, \pi)$, and to $\left[f(-\pi)+f(\pi)\right]/2$ at $x = \pm\pi$. Understandably, such a Fourier expansion converges very slowly.

Assume, for example, that $f(x)$ is continuous on [−π, π] with an absolutely integrable derivative (which may not exist at certain points). Then we have

$$f'(x) \sim \frac{c}{2} + \sum_{m=1}^{\infty}\left[\left(mb_m + (-1)^m c\right)\cos mx - ma_m \sin mx\right] \tag{1.11}$$

where a_m and b_m are the Fourier coefficients of $f(x)$ and $c = \frac{1}{\pi}\left[f(\pi) - f(-\pi)\right]$. Since the Fourier coefficients of an absolutely integrable function tend to zero as $m \to \infty$ (Theorem 2), it is obvious that

$$\lim_{m\to\infty} ma_m = 0, \tag{1.12}$$

and

$$\lim_{m\to\infty}\left[(-1)^{m+1} mb_m\right] = c. \tag{1.13}$$

If $f(\pi)=f(-\pi)$, then we have

$$\lim_{m\to\infty} mb_m = 0 \tag{1.14}$$

which recovers the convergence rate for a continuous 2π-periodic function. Unfortunately, the condition, $f(\pi)=f(-\pi)$, is generally not true for an arbitrary function.

In recognizing this slow convergence problem, the subtraction of polynomials has been developed to remove the Gibbs phenomenon with $f(x)$ (or its related derivatives) and to thus accelerate the convergence of resulting Fourier expansions [6]-[12]. In polynomial subtraction schemes, a new (or corrected) function $F(x)$ will be created with a desired smoothness through removing the potential jumps, such as, at the end points

$$f(x)=F(x)+h(x) \tag{1.15}$$

where $h(x)$ is a polynomial of degree $2K+1$, satisfying

$$h^{(k)}(\pm\pi)=f^{(k)}(\pm\pi),\ (k=0,1,2,\cdots,K). \tag{1.16}$$

The polynomials can be easily constructed, for example, using the Lanczos's system of polynomials:

$$p_1(x)=x/\pi, \tag{1.17}$$

$$p_k'(x)=p_{k-1}(x), \quad k=2,3,\cdots \tag{1.18}$$

and

$$p_{2k+1}(0)=p_{2k+1}(\pi)=0, \quad k=1,2,\cdots. \tag{1.19}$$

Lanczos polynomials of even (odd) degrees are obviously even (odd) functions. It should be noted that Lanczos polynomials are closely related to Bernoulli polynomials which are also widely used in the methods of polynomial subtraction.

The first few Lanczos polynomials can be explicitly expressed as

$$p_1(x)=x/\pi \tag{1.20}$$

$$p_2(x)=\left(3x^2-\pi^2\right)/6\pi \tag{1.21}$$

$$p_3(x)=\left(x^3-\pi^2 x\right)/6\pi \tag{1.22}$$

$$p_4(x)=\left(15x^4-30\pi^2x^2+7\pi^4\right)/360\pi. \tag{1.23}$$

For complete Fourier expansion of $F(x)$,

$$h(x)=\frac{1}{2}\sum_{k=0}^{K}\left[f^{(k)}(\pi)-f^{(k)}(-\pi)\right]p_{k+1}(x). \tag{1.24}$$

For the sine expansion of $F(x)$,

$$h(x)=\sum_{k=0}^{K}\left[f^{(2k)}(0)\,p_{2k+1}(\pi-x)+f^{(2k)}(\pi)\,p_{2k+1}(x)\right]. \tag{1.25}$$

For the cosine expansion of $F(x)$:

$$h(x)=\sum_{k=0}^{K}\left[f^{(2k+1)}(\pi)\,p_{2k+2}(x)-f^{(2k+1)}(0)\,p_{2k+2}(\pi-x)\right]. \tag{1.26}$$

Assume that $f(x)$ is C^{n-1} continuous on [−π, π] and its n-th derivative is absolutely integrable. Then the corrected function $F(x)$ can be periodically extended into a 2π-periodic function of: a) C^K continuity for $K \le n-1$ in (1.12); b) C^{2K+1} continuity for $2K+1 < n$ in (1.13) and c) C^{2K+2} continuity for $2K+2 < n$ in (1.14).

By recognizing the slower convergence of sine series than its cosine counterpart, a modified Fourier series was proposed as [1]

$$f(x) = a_0/2 + \sum_{m=1}^{\infty} a_m \cos mx + b_m \sin\left(m - \frac{1}{2}\right)x. \tag{1.27}$$

If $f(x)$ is differentiable and its derivative has bounded variation, the expansion coefficients, a_m and b_m, in (1.15) will both decay like $\mathcal{O}(m^{-2})$ [13] [14], which is still considered relatively slow in many cases.

2. An Alternative Form of Fourier Cosine Series

For a sufficiently smooth function $f(x)$ defined on a compact interval [0, π], it can always be expanded into

$$f(x) = a_0/2 + \sum_{m=1}^{\infty} a_m \cos mx, \qquad 0 \le x \le \pi \tag{2.1}$$

where

$$a_m = \frac{2}{\pi}\int_0^{\pi} f(x)\cos mx\mathrm{d}x\,. \tag{2.2}$$

It is known that expansion coefficients a_m decay like $\mathcal{O}(m^{-2})$.

To accelerate the convergence and maintain a close similarity to classical Fourier series, an alternative trigonometric expansion of $f(x)$ is here sought in the form of

$$f(x) = \mathcal{F}_{\infty,2P}[f](x) = a_0/2 + \sum_{m=1}^{\infty} a_m \cos mx + \sum_{p=1}^{2P} b_p \sin px, \quad 0 \le x \le \pi \tag{2.3}$$

where

$$a_m = \frac{2}{\pi}\int_0^{\pi}\left[f(x) - \sum_{p=1}^{2P} b_p \sin px\right]\cos mx\mathrm{d}x \tag{2.4}$$

and coefficients b_p are to be determined as described below.

THEOREM 4. *Let* $f(x)$ *have* C^{n-1} *continuity on the interval* [0, π] *and its n-th derivative is absolutely integrable* (*the n-th derivative may not exist at certain points*). *If n* ≥ 2, *then the Fourier coefficient* a_m, *as defined in* (2.4), *decays at a polynomial rate as*

$$\lim_{m\to\infty} a_m m^{2P} = 0 \quad (2P \le n) \tag{2.5}$$

provided that

$$\sum_{p=1}^{P} b_{2p}(2p)^{2q-1} = (-1)^{q-1}\left[f^{(2q-1)}(\pi) + f^{(2q-1)}(0)\right]/2 \tag{2.6}$$

and

$$\sum_{p=1}^{P} b_{2p-1}(2p-1)^{2q-1} = (-1)^{q}\left[f^{(2q-1)}(\pi) - f^{(2q-1)}(0)\right]/2, \quad (q = 1, 2, \cdots, P). \tag{2.7}$$

Proof. By integrating by part, we have

$$\begin{aligned}(\pi/2) f_m &= \int_0^{\pi} f(x)\cos mx\mathrm{d}x \\ &= \frac{\sin mx}{m} f(x)\Big|_0^{\pi} - \frac{1}{m}\int_0^{\pi} f'(x)\sin mx\mathrm{d}x \\ &= \frac{\cos mx}{m^2} f'(x)\Big|_0^{\pi} - \frac{1}{m^2}\int_0^{\pi} f''(x)\cos mx\mathrm{d}x \\ &= \sum_{q=1}^{Q}\frac{(-1)^{q-1}\cos mx}{m^{2q}} f^{(2q-1)}(x)\Big|_0^{\pi} + \frac{(-1)^Q}{m^{2Q}}\int_0^{\pi} f^{(2Q)}(x)\cos mx\mathrm{d}x \\ &= \sum_{q=1}^{Q}\frac{(-1)^{q-1}}{m^{2q}}\left[(-1)^m f^{(2q-1)}(\pi) - f^{(2q-1)}(0)\right] + \frac{(-1)^Q}{m^{2Q}}\int_0^{\pi} f^{(2Q)}(x)\cos mx\mathrm{d}x.\end{aligned} \tag{2.8}$$

Denote $h(x)=\sum_{p=1}^{2P} b_p \sin px$, then

$$\begin{aligned}(\pi/2) h_m &= \sum_{p=1}^{2P} b_p \int_0^{\pi}\sin px\cos mx\mathrm{d}x \\ &= \sum_{p=1}^{2P} b_p \int_0^{\pi}\frac{1}{2}\left[\sin(p+m)x + \sin(p-m)x\right]\mathrm{d}x \\ &= -\frac{1}{2}\sum_{p=1}^{2P} b_p \left(\frac{\cos(p+m)x}{p+m} + \frac{\cos(p-m)x}{p-m}\right)\Bigg|_0^{\pi} \\ &= \sum_{p=1}^{2P} b_p \frac{p}{m^2-p^2}\left[(-1)^{p+m}-1\right] \\ &= \sum_{p=1}^{2P}\left[(-1)^{p+m}-1\right] b_p\left[\frac{p}{m^2}+\frac{p^3}{m^4}+\frac{p^5}{m^6}+\cdots\right] \\ &= \sum_{q=1}^{Q}\frac{1}{m^{2q}}\left(\sum_{p=1}^{2P}\left[(-1)^{p+m}-1\right]p^{2q-1}b_p\right) + \mathcal{O}\left(m^{-2Q-2}\right)\end{aligned} \tag{2.9}$$

for sufficiently large *m*.

Substracting (2.9) from (2.8) leads to

$$\begin{aligned}(\pi/2) a_m = (\pi/2)(f_m - h_m) &= \sum_{q=1}^{Q}\frac{(-1)^{m+q-1}}{m^{2q}}\left[f^{(2q-1)}(\pi) - \sum_{p=1}^{2P} b_p p^{(2q-1)}(-1)^{q+p-1}\right] \\ &\quad - \sum_{q=1}^{Q}\frac{(-1)^{q-1}}{m^{2q}}\left[f^{(2q-1)}(0) - \sum_{p=1}^{2P}(-1)^{q-1} b_p p^{(2q-1)}\right] \\ &\quad + \frac{(-1)^Q}{m^{2Q}}\int_0^{\pi} f^{(2Q)}(x)\cos mx\mathrm{d}x + \mathcal{O}\left(m^{-2Q-2}\right).\end{aligned} \tag{2.10}$$

The first two terms in (2.10) vanish if

$$\sum_{p=1}^{2P} b_p p^{2q-1}(-1)^{q+p-1} = f^{(2q-1)}(\pi) \tag{2.11}$$

and

$$\sum_{p=1}^{2P} b_p p^{2q-1}(-1)^{q-1} = f^{(2q-1)}(0), \quad q=1,2,\cdots,Q. \tag{2.12}$$

In order to have a unique and smallest set of coefficients, b_p, we set $Q = P$ in (2.11) and (2.12), or equivalently, in (2.6) and (2.7). The convergence estimate, (2.5), becomes evident from (2.10) according to Theorem 2. □

Remark. If $2P < n$, the relationship, (2.5), in Theorem 4 can be modified to

$$\lim_{m\to\infty} a_m m^{2P+1} = 0 \tag{2.13}$$

or, more explicitly,

$$a_m \sim \mathcal{O}\left(m^{-2P-2}\right). \tag{2.14}$$

Alternatively, (2.4) can be expressed as

$$a_m = \frac{2}{\pi}\int_0^\pi f(x)\cos mx\mathrm{d}x - \sum_{p=1}^{2P} \alpha_{mp} b_p \tag{2.15}$$

where

$$\alpha_{mp} = \begin{cases} \dfrac{2p\left[(-1)^{m+p} - 1\right]}{\pi\left(m^2 - p^2\right)} & \text{for } m \neq p, \\ 0 & \text{for } m = p. \end{cases} \tag{2.16}$$

Equations (2.6) and (2.7) can be rewritten in matrix form as

$$\begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_2 \end{bmatrix} \begin{Bmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{F}_1 \\ \boldsymbol{F}_2 \end{Bmatrix} \tag{2.17}$$

where

$$\left[\boldsymbol{A}_1\right]_{p,q} = (2p-1)^{2q-1} \tag{2.18}$$

$$\left[\boldsymbol{A}_2\right]_{p,q} = (2p)^{2q-1} \tag{2.19}$$

$$\boldsymbol{B}_1 = \left\{b_1 \quad b_3 \quad \cdots \quad b_{2p-1} \quad \cdots \quad b_{2P-1}\right\}^{\mathrm{T}} \tag{2.20}$$

$$\boldsymbol{B}_2 = \left\{b_2 \quad b_4 \quad \cdots \quad b_{2p} \quad \cdots \quad b_{2P}\right\}^{\mathrm{T}} \tag{2.21}$$

$$\boldsymbol{F}_1 = \left\{-f_-^{(1)} \quad f_-^{(3)} \quad \cdots \quad (-1)^q f_-^{(2q-1)} \quad \cdots \quad (-1)^P f_-^{(2P-1)}\right\}^{\mathrm{T}} \tag{2.22}$$

and

$$\boldsymbol{F}_2 = \left\{f_+^{(1)} \quad -f_+^{(3)} \quad \cdots \quad (-1)^{q-1} f_+^{(2q-1)} \quad \cdots \quad (-1)^{P-1} f_+^{(2P-1)}\right\}^{\mathrm{T}} \tag{2.23}$$

in which

$$f_+^{(2q-1)} = \left[f^{(2q-1)}(\pi) + f^{(2q-1)}(0)\right]\big/2 \tag{2.24}$$

and

$$f_-^{(2q-1)} = \left[f^{(2q-1)}(\pi) - f^{(2q-1)}(0)\right]\big/2\,. \tag{2.25}$$

Determination of the coefficients, $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$, involves the inversion of a Vandermonde-like matrix

$$\boldsymbol{X} = \begin{bmatrix} x_1 & x_2 & \cdots & x_j & \cdots & x_P \\ x_1^3 & x_2^3 & \cdots & x_j^3 & \cdots & x_P^3 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ x_1^{2i-1} & x_2^{2i-1} & \cdots & x_j^{2i-1} & \cdots & x_P^{2i-1} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ x_1^{2P-1} & x_2^{2P-1} & \cdots & x_j^{2P-1} & \cdots & x_P^{2P-1} \end{bmatrix} \qquad (i, j = 1, 2, 3, \cdots, P) \tag{2.26}$$

which is always invertable if $x_k \neq x_j$ for $j \neq k$.

Consider a polynomial of degree $2P-1$

$$\phi_i(x) = \frac{x}{x_i} \prod_{\substack{k=1 \\ k\neq i}}^{P} \frac{x^2 - x_k^2}{x_i^2 - x_k^2} = \sum_{k=1}^{P} c_{ik} x^{2k-1} . \tag{2.27}$$

Then it is obvious that

$$\phi_i(x_j) = \sum_{k=1}^{P} c_{ik} x_j^{2k-1} = \delta_{ij} \tag{2.28}$$

where δ_{ij} is Kronecker's symbol.

According to (2.28), matrix $\boldsymbol{C} = [c_{ik}]$ is actually the inverse of matrix $\boldsymbol{X}$.

To find an explicit expression for matrix $\boldsymbol{C}$, let

$$\prod_{\substack{q=1 \\ q\neq i}}^{P} \left(x^2 - x_q^2\right) = \sum_{q=0}^{P-1} \beta_q x^{2q} = \frac{1}{x^2 - x_i^2} \prod_{q=1}^{P} \left(x^2 - x_q^2\right) = \frac{1}{x^2 - x_i^2} \sum_{q=0}^{P} \alpha_q x^{2q} \tag{2.29}$$

where

$$\alpha_q = \sum_{1 \le j_1 < \cdots < j_{P-q} \le P} (-1)^q x_{j_1}^2 x_{j_2}^2 \cdots x_{j_{P-q}}^2 \tag{2.30}$$

and

$$\beta_q = -\frac{1}{x_i^{2q+2}} \sum_{s=0}^{q} \alpha_s x_i^{2s} . \tag{2.31}$$

Thus, we have

$$\phi_i(x) = -\frac{1}{\prod_{\substack{j=1 \\ j\neq i}}^{P} \left(x_i^2 - x_j^2\right)} \sum_{q=1}^{P} \left(\frac{1}{x_i^{2q+1}} \sum_{s=0}^{q-1} \alpha_s x_i^{2s} \right) x^{2q-1} . \tag{2.32}$$

Comparing (2.32) with (2.27) leads to

$$c_{ik} = -\frac{\sum_{s=0}^{k-1} \alpha_s x_i^{2s}}{x_i^{2k+1} \prod_{\substack{j=1 \\ j\neq i}}^{P} \left(x_i^2 - x_j^2\right)} \tag{2.33}$$

or

$$c_{ik} = \frac{\sum_{\substack{1 \le j_1 < \cdots < j_{P-k} \le P \\ j_1, \cdots, j_{P-k} \neq i}} (-1)^{k+1} x_{j_1}^2 \cdots x_{j_{P-k}}^2}{x_i \prod_{\substack{j=1 \\ j\neq i}}^{P} \left(x_j^2 - x_i^2\right)} . \tag{2.34}$$

In light of (2.34), the coefficients b_p ($p=1,2,\cdots,2P$) can be obtained as

$$b_p=\sum_{k=1}^{P}(-1)^p\left[f^{(2k-1)}(\pi)+(-1)^p f^{(2k-1)}(0)\right]\frac{\sum\limits_{\substack{1\le j_1<\cdots<j_{P-k}\le P\\ j_1,\cdots,j_{P-k}\ne i}} x_{j_1}^2\cdots x_{j_{P-k}}^2}{x_i\prod\limits_{\substack{j=1\\ j\ne i}}^{P}\left(x_j^2-x_i^2\right)},$$

$$p=x_i=\begin{cases}2i-1 & \text{if } p \text{ is odd}\\ 2i & \text{if } p \text{ is even}\end{cases},\qquad (i=1,2,\cdots,P). \tag{2.35}$$

By making use of (2.21), the first few coefficients, for example, are readily found as:

$$b_1=-f_-^{(1)} \tag{2.36}$$

and

$$b_2=f_+^{(1)}/2 \tag{2.37}$$

for P=1;

$$b_1=-9f_-^{(1)}/8-f_-^{(3)}/8 \tag{2.38}$$

$$b_2=2f_+^{(1)}/3+f_+^{(3)}/24 \tag{2.39}$$

$$b_3=f_-^{(1)}/24+f_-^{(3)}/24 \tag{2.40}$$

and

$$b_4=-f_+^{(1)}/12-f_+^{(3)}/48 \tag{2.41}$$

for $P=2$;

$$b_1=-75f_-^{(1)}/64-17f_-^{(3)}/96-f_-^{(5)}/192 \tag{2.42}$$

$$b_2=3f_+^{(1)}/4+13f_+^{(3)}/192+f_+^{(5)}/768 \tag{2.43}$$

$$b_3=25f_-^{(1)}/384+13f_-^{(3)}/192+f_-^{(5)}/384 \tag{2.44}$$

$$b_4=-3f_+^{(1)}/20-f_+^{(3)}/24-f_+^{(5)}/960 \tag{2.45}$$

$$b_5=-3f_-^{(1)}/640-f_-^{(3)}/192-f_-^{(5)}/1920 \tag{2.46}$$

and

$$b_6=f_+^{(1)}/60+f_+^{(3)}/192+f_+^{(5)}/3840 \tag{2.47}$$

for $P=3$.

EXAMPLE 1. Consider function $f(x)=Ax^2+Bx+C$ $(0\le x\le\pi)$. Its conventional Fourier expansions are easily obtained as

$$f(x)=\frac{A\pi^2}{3}+\frac{B\pi}{2}+C+\sum_{m=1}^{\infty}\frac{A}{m^2}\cos 2mx-\sum_{m=1}^{\infty}\frac{(B+\pi A)}{m}\sin 2mx,\qquad 0<x<\pi \tag{2.48}$$

or

$$f(x)=\frac{A\pi^2}{3}+\frac{B\pi}{2}+C+\sum_{m=1}^{\infty}a_m\cos mx,\quad 0\le x\le\pi \tag{2.49}$$

where

$$a_m = \frac{-2B + (-1)^m (2B + 4A\pi)}{m^2 \pi}. \tag{2.50}$$

Under the current framework, this function can be expanded as:

$$\begin{aligned} f(x) = A + \frac{\pi B}{2} + C + \sum_{m=1}^{\infty} a_m \cos mx \\ - A\pi \sin x + \frac{1}{2}(\pi A + B)\sin 2x, \quad 0 \le x \le \pi \end{aligned} \tag{2.51}$$

where

$$a_m = \begin{cases} \dfrac{-4A}{m^2 (m^2 - 1)} & \text{if } m \text{ is even} \\ \dfrac{16(\pi A + B)}{\pi m^2 (m^2 - 4)} & \text{if } m \text{ is odd} \end{cases} \tag{2.52}$$

for $P = 1$;

$$\begin{aligned} f(x) = \frac{1}{9}(20 + 3\pi^2)A + \frac{\pi B}{2} + C + \sum_{m=1}^{\infty} a_m \cos mx - \frac{9A\pi}{8}\sin x \\ + \frac{2}{3}(B + A\pi)\sin 2x + \frac{A\pi}{24}\sin 3x - \frac{1}{12}(B + A\pi)\sin 4x, \quad 0 \le x \le \pi \end{aligned} \tag{2.53}$$

where

$$a_m = \begin{cases} \dfrac{36A}{m^2 (m^2 - 1)(m^2 - 9)} & \text{if } m \text{ is even} \\ \dfrac{-256(\pi A + B)}{\pi m^2 (m^2 - 4)(m^2 - 16)} & \text{if } m \text{ is odd} \end{cases} \tag{2.54}$$

for $P = 2$;

$$\begin{aligned} f(x) = \frac{A(518 + 75\pi^2)}{225} + \frac{B\pi}{2} + C + \sum_{m=1}^{\infty} a_m \cos mx - \frac{75A\pi}{64}\sin x \\ + \frac{3}{4}(B + A\pi)\sin 2x + \frac{25A\pi}{384}\sin 3x - \frac{3}{20}(B + A\pi)\sin 4x \\ - \frac{3A\pi}{640}\sin 5x + \frac{1}{60}(B + A\pi)\sin 6x, \quad 0 \le x \le \pi \end{aligned} \tag{2.55}$$

where

$$a_m = \begin{cases} \dfrac{-900A}{m^2 (m^2 - 1)(m^2 - 9)(m^2 - 25)} & \text{if } m \text{ is even} \\ \dfrac{9216(\pi A + B)}{\pi m^2 (m^2 - 4)(m^2 - 16)(m^2 - 36)} & \text{if } m \text{ is odd} \end{cases} \tag{2.56}$$

for $P = 3$.

A graphic display of the results, (2.48), (2.49), (2.51), (2.53) and (2.55), is given in Figure 1 for $A = 4$, $B = 2$, and $C = 1$. The corresponding truncation errors are plotted in Figure 2 and Figure 3.

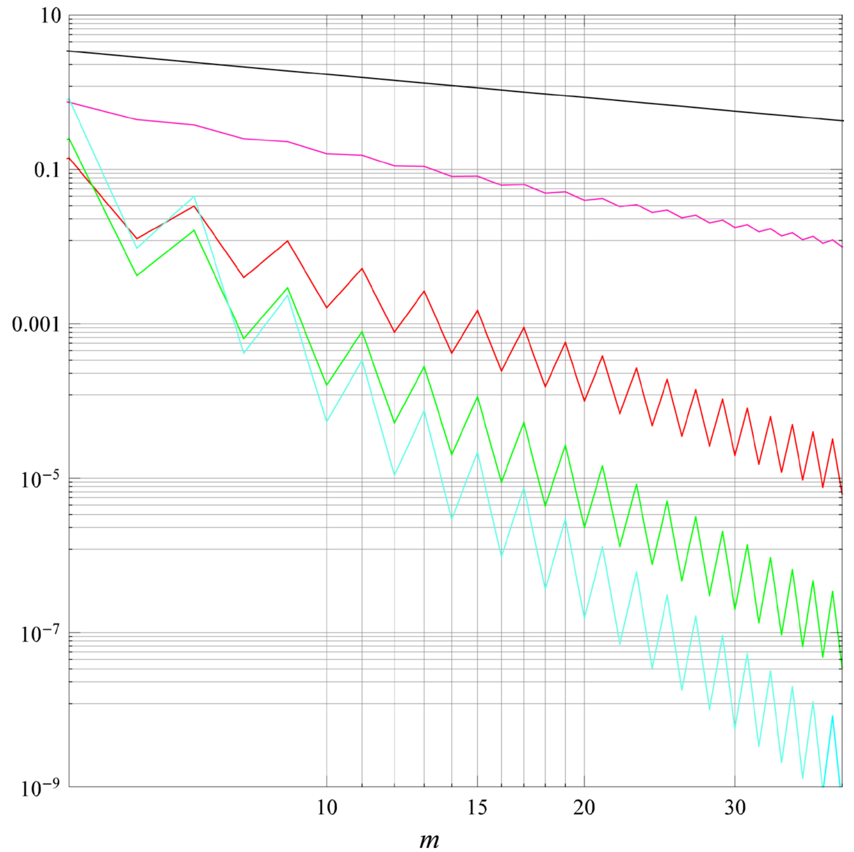

Figure 1. Decays of expansion coefficients: ——— b_m in (2.48), ——— a_m in (2.49), ——— a_m in (2.51), ——— a_m in (2.53), and ——— a_m in (2.55).

3. An Alternative Form of Fourier Sine Series

Similarly, $f(x)$ can also be expanded into sine series:

$$f(x) = \mathcal{F}_{\infty,2P}[f](x) = \sum_{m=1}^{\infty} a_m \sin mx + \sum_{p=1}^{2P} b_p \cos px, \quad 0 \le x \le \pi \tag{3.1}$$

where b_p are the expansion coefficients to be determined, and

$$a_m = \frac{2}{\pi}\int_0^{\pi}\left[f(x) - \sum_{p=1}^{2P} b_p \cos px\right]\sin mx dx . \tag{3.2}$$

THEOREM 5. *Let* $f(x)$ *have* C^{n-1} *continuity on the interval* $[0, \pi]$ *and the n-th derivative is absolutely integrable* (*the n-th derivative may not exist at certain points*). *Then for* $2P \le n+1$ *the Fourier coefficients of* $f(x)$ *defined in* (3.1) *decay at a polynomial rate as*

$$\lim_{m\to\infty} a_m m^{2P-1} = 0 \quad (2P \le n+1) \tag{3.3}$$

provided that

$$\sum_{p=1}^{P} b_{2p}(2p)^{2q-2} = (-1)^{q-1}\left[f^{(2q-2)}(\pi) + f^{(2q-2)}(0)\right]/2 \tag{3.4}$$

and

$$\sum_{p=1}^{P} b_{2p-1}(2p-1)^{2q-2} = (-1)\left[f^{(2q-2)}(\pi)+f^{(2q-2)}(0)\right]\Big/2, \quad (q=1,2,\cdots,P). \tag{3.5}$$

Proof. By integrating by part, we have

$$\begin{aligned}(\pi/2)a_m &= \int_0^\pi \left[f(x)-\sum_{p=1}^{2P} b_p \cos px\right]\sin mx\mathrm{d}x \\ &= -\left[f(x)-\sum_{p=1}^{2P} b_p \cos px\right]\frac{\cos mx}{m}\Bigg|_0^\pi + \frac{1}{m}\int_0^\pi\left[f'(x)-\sum_{p=1}^{2P} b_p \cos' px\right]\cos mx\mathrm{d}x \\ &= \sum_{q=1}^{Q}\frac{(-1)^{q+m}}{m^{2q-1}}\left[f^{(2q-2)}(\pi)-\sum_{p=1}^{2P} b_p p^{2q-2}(-1)^{p+q-1}\right]-\frac{(-1)^q}{m^{2q-1}}\left[f^{(2q-2)}(0)-\sum_{p=1}^{2P} b_p p^{2q-2}(-1)^{q-1}\right] \\ &\quad -\frac{(-1)^Q}{m^{2Q-1}}\int_0^\pi\left[f^{(2Q-1)}(x)-\sum_{p=1}^{2P} b_p \cos^{(2Q-1)} px\right]\Bigg|\cos mx\mathrm{d}x.\end{aligned} \tag{3.6}$$

The first two terms in (3.6) will both vanish if

$$\sum_{p=1}^{2P} b_p p^{2q-2}(-1)^{p+q-1} = f^{(2q-2)}(\pi) \tag{3.7}$$

and

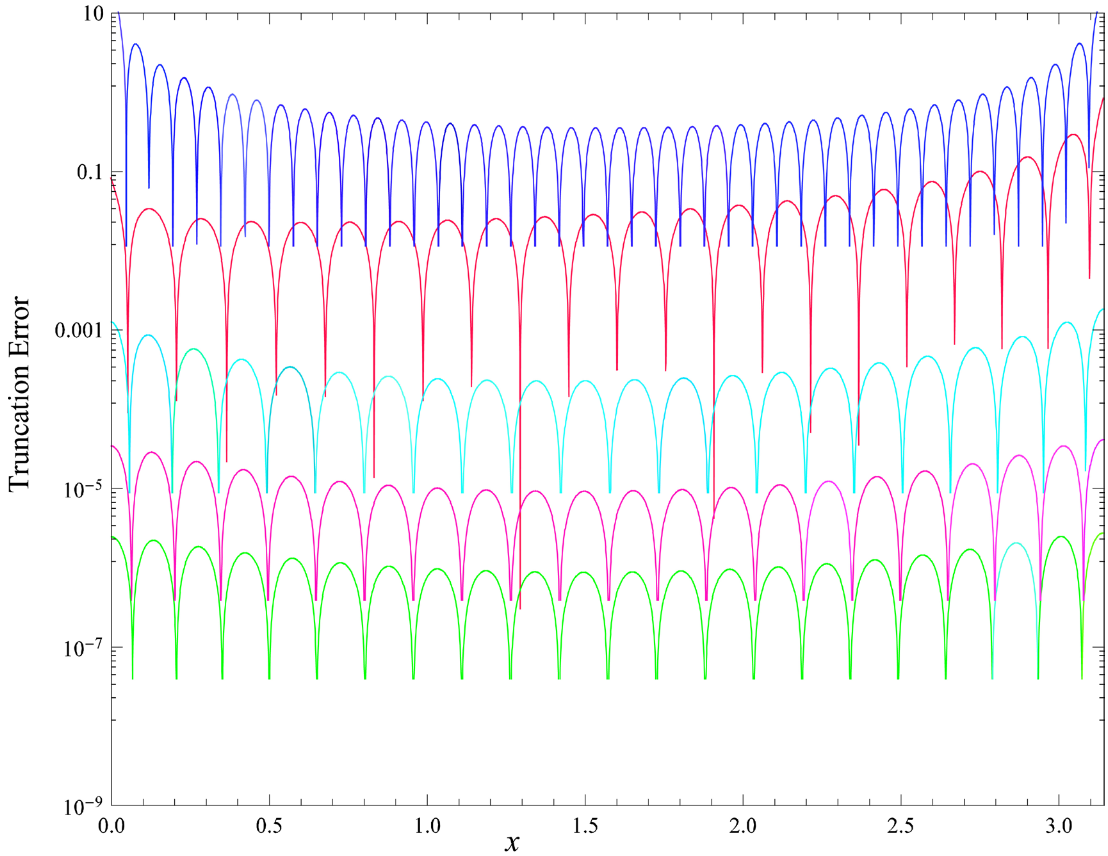

Figure 2. Truncation errors, $\left|f(x)-\mathcal{F}_{M,2P}[f](x)\right|$, for the series expansions: (2.48), (2.49), (2.51), (2.53), and (2.55). $M = 20$.

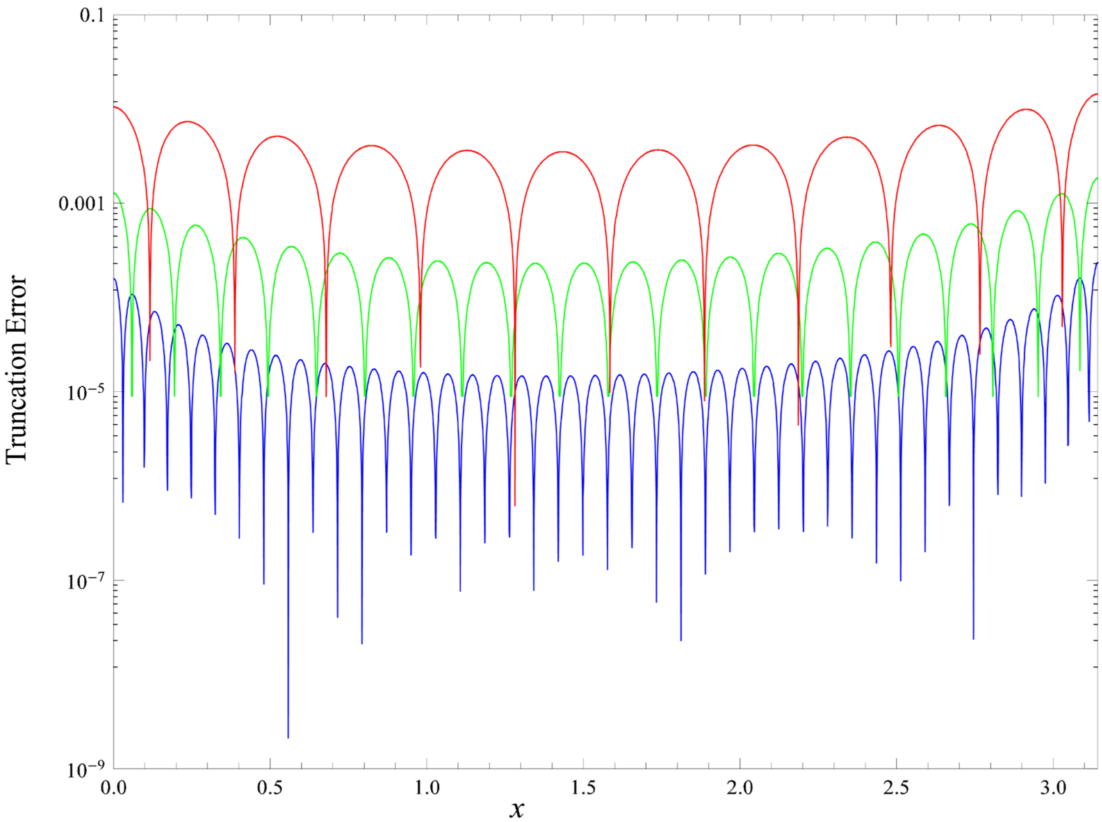

Figure 3. Errors, $\left|f(x)-\mathcal{F}_{M,2}[f](x)\right|$, for series expansion (2.51): ——— M = 10, ——— M = 20 and ——— M = 40.

$$\sum_{p=1}^{2P} b_p p^{2q-2}(-1)^{q-1} = f^{(2q-2)}(0), \quad q = 1, 2, \cdots, Q. \tag{3.8}$$

In order to have a unique and smallest set of coefficients, b_p, we set $Q = P$ in (3.7) and (3.8), or equivalently, in (3.4) and (3.5). The convergence estimate, (3.3), then becomes evident according to Theorem 2. □

Remark. If $2P < n+1$, the relationship (3.3) can be modified to

$$\lim_{m\to\infty} a_m m^{2P} = 0 \tag{3.9}$$

which can be further written in a shaper form as

$$a_m \sim \mathcal{O}\left(m^{-2P-1}\right), \quad \text{for sufficiently large } m. \tag{3.10}$$

The expansion coefficients, a_m, can be alternatively expressed as

$$a_m = \frac{2}{\pi}\int_0^{\pi} f(x)\sin mx\,dx - \sum_{p=1}^{2P} \beta_{mp} b_p \tag{3.11}$$

where

$$\beta_{mp} = \begin{cases} \dfrac{2m\left[(-1)^{m+p}-1\right]}{\pi\left(p^2-m^2\right)} & \text{for } m \neq p, \\ 0 & \text{for } m = p. \end{cases} \tag{3.12}$$

Actually,

$$\beta_{mp} = \alpha_{pm} \quad \text{(see (2.16))}. \tag{3.13}$$

We can rewrite (3.4) and (3.5) in matrix form as

$$\begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_2 \end{bmatrix} \begin{Bmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{F}_1 \\ \boldsymbol{F}_2 \end{Bmatrix} \tag{3.14}$$

where

$$[\boldsymbol{A}_1]_{p,q} = (2p-1)^{2q-2} \tag{3.15}$$

$$[\boldsymbol{A}_2]_{p,q} = (2p)^{2q-2} \tag{3.16}$$

$$\boldsymbol{F}_1 = \left\{ -f_-^{(0)} \quad f_-^{(2)} \quad \cdots \quad (-1)^{q+1} f_-^{(2q-2)} \quad \cdots \quad (-1)^{P+1} f_-^{(2P-2)} \right\}^{\mathrm{T}} \tag{3.17}$$

and

$$\boldsymbol{F}_2 = \left\{ f_+^{(0)} \quad -f_+^{(2)} \quad \cdots \quad (-1)^{q} f_+^{(2q-2)} \quad \cdots \quad (-1)^{P} f_+^{(2P-2)} \right\}^{\mathrm{T}}. \tag{3.18}$$

Following the same procedures as described earlier, coefficients b_p can be obtained from

$$b_p = \sum_{k=1}^{P} (-1)^p \left[f^{(2k-2)}(\pi) + (-1)^p f^{(2k-2)}(0) \right] \frac{\sum\limits_{\substack{1\le j_1<\cdots<j_{P-k}\le P \\ j_1,\cdots,j_{P-k}\neq i}} x_{j_1}^2 \cdots x_{j_{P-k}}^2}{\prod\limits_{\substack{j=1 \\ j\neq i}}^{P} \left(x_j^2 - x_i^2 \right)},$$

$$p = x_i = \begin{cases} 2i-1 & \text{if } p \text{ is odd} \\ 2i & \text{if } p \text{ is even} \end{cases}, \quad (i = 1, 2, \cdots, P). \tag{3.19}$$

Using this formula, the first several coefficients are easily determined as:

$$b_1 = -f_-^{(0)} \tag{3.20}$$

and

$$b_2 = f_+^{(0)} \tag{3.21}$$

for $P = 1$;

$$b_1 = -9 f_-^{(0)}/8 - f_-^{(2)}/8 \tag{3.22}$$

$$b_2 = 4 f_+^{(0)}/3 + f_+^{(2)}/12 \tag{3.23}$$

$$b_3 = f_-^{(0)}/8 + f_-^{(2)}/8 \tag{3.24}$$

and

$$b_4 = -f_+^{(0)}/3 - f_+^{(2)}/12 \tag{3.25}$$

for $P = 2$;

$$b_1 = -75 f_-^{(0)}/64 - 17 f_-^{(2)}/96 - f_-^{(4)}/192 \tag{3.26}$$

$$b_2 = 3 f_+^{(0)}/2 + 13 f_+^{(2)}/96 + f_+^{(4)}/384 \tag{3.27}$$

$$b_3 = 25 f_-^{(0)}/128 + 13 f_-^{(2)}/64 + f_-^{(4)}/128 \tag{3.28}$$

$$b_4 = -3 f_+^{(0)}/5 - f_+^{(2)}/6 - f_+^{(4)}/240 \tag{3.29}$$

$$b_5 = -3 f_-^{(0)}/128 - 5 f_-^{(2)}/192 - f_-^{(4)}/384 \tag{3.30}$$

and

$$b_6 = f_+^{(0)}/10 + f_+^{(2)}/32 + f_+^{(4)}/640 \tag{3.31}$$

for $P = 3$.

EXAMPLE 2. Consider function $f(x) = Ax^2 + Bx + C$ $(0 \le x \le \pi)$. The classical sine series expansion of this function is easily found as

$$f(x) = \sum_{m=1}^{\infty} a_m \sin mx \tag{3.32}$$

where

$$a_m = \begin{cases} \dfrac{2m^2\left(2c + B\pi + \pi^2 A\right) - 8A}{m^3\pi} & \text{if } m \text{ is odd,} \\ -\dfrac{2(B + A\pi)}{m} & \text{if } m \text{ is even.} \end{cases} \tag{3.33}$$

In the context of the current framework, this function can be expressed as

$$\begin{aligned} f(x) = {} & \sum_{m=1}^{\infty} a_m \sin mx - \frac{1}{2}\left(B\pi + A\pi^2\right)\cos x \\ & + \frac{1}{2}\left(2C + B\pi + A\pi^2\right)\cos 2x, \quad 0 \le x \le \pi \end{aligned} \tag{3.34}$$

where

$$a_m = \begin{cases} \dfrac{32A - 8m^2\left[2C + B\pi + A\left(1 + \pi^2\right)\right]}{m^3\left(m^2 - 4\right)\pi} & \text{if } m \text{ is odd} \\ \dfrac{2(B + A\pi)}{m\left(m^2 - 1\right)} & \text{if } m \text{ is even} \end{cases} \tag{3.35}$$

for $P = 1$;

$$\begin{aligned} f(x) = {} & \sum_{m=1}^{\infty} a_m \sin mx - \frac{9}{16}\pi(B + A\pi)\cos x \\ & + \frac{1}{6}\left(A + 8C + 4B\pi + 4A\pi^2\right)\cos 2x + \frac{1}{16}\pi(B + A\pi)\cos 3x \\ & - \frac{1}{6}\left(A + 2C + B\pi + A\pi^2\right)\cos 4x, \quad 0 \le x \le \pi \end{aligned} \tag{3.36}$$

where

$$a_m = \begin{cases} -\dfrac{18(B + A\pi)}{m\left(m^2 - 1\right)\left(m^2 - 9\right)} & \text{if } m \text{ is even} \\ \dfrac{128m^2(2C + B\pi) + 32A\left[m^2\left(5 + 4\pi^2\right) - 16\right]}{\pi m^3\left(m^2 - 4\right)\left(m^2 - 16\right)} & \text{if } m \text{ is odd} \end{cases} \tag{3.37}$$

for $P = 2$.

4. An Alternative Form of Fourier Series Expansion

Let $f(x)$ be defined on the interval $[-\pi, \pi]$. It can also be expanded into a complete trigonometric series as

$$f(x) = \mathcal{F}_{\infty,2P,2Q}[f](x)$$
$$= a_0/2 + \sum_{m=1}^{\infty}\left(a_m + \bar{a}_{2Q,m}\operatorname{sgn}(x)\right)\cos mx + \left(b_m + \bar{b}_{2P,m}\operatorname{sgn}(x)\right)\sin mx \quad (4.1)$$
$$\left(\bar{a}_{2Q,m} \equiv 0, \text{ for } m > 2Q;\ \bar{b}_{2P,m} \equiv 0, \text{ for } m > 2P\right), \quad -\pi \le x \le \pi$$

where a_m and b_m are the expansion coefficients to be calculated from

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(f(x) - \sum_{p=1}^{2P}\bar{b}_{2P,p}\operatorname{sgn}(x)\sin px\right)\cos mx\mathrm{d}x \quad (4.2)$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(f(x) - \sum_{q=1}^{2Q}\bar{a}_{2Q,q}\operatorname{sgn}(x)\cos qx\right)\sin mx\mathrm{d}x \quad (4.3)$$

and $\bar{a}_{2Q,m}$ and $\bar{b}_{2P,m}$ are to be determined as follows.

THEOREM 6. *Let* $f(x)$ *have* C^{n-1} *continuity on the interval* $[-\pi, \pi]$ *and the n-th derivative is absolutely integrable* (*the n-th derivative may not exist at certain points*). *Then the Fourier coefficients of* $f(x)$, *as defined in* (4.2) *and* (4.3), *decay at a polynomial rate as*

$$\lim_{m\to\infty} a_m m^{2P} = 0 \quad (2P \le n) \quad (4.4)$$

and

$$\lim_{m\to\infty} b_m m^{2Q-1} = 0 \quad (2Q \le n+1) \quad (4.5)$$

provided that

$$\sum_{p=1}^{P}\bar{b}_{2P,2p}(2p)^{2r-1} = (-1)^{r-1}\left[f^{(2r-1)}(\pi) - f^{(2r-1)}(-\pi)\right]/4 \quad (4.6)$$

$$\sum_{p=1}^{P}\bar{b}_{2P,2p-1}(2p-1)^{2r-1} = (-1)^{r}\left[f^{(2r-1)}(\pi) - f^{(2r-1)}(-\pi)\right]/4, \quad (r = 1,2,\cdots,P) \quad (4.7)$$

$$\sum_{q=1}^{Q}\bar{a}_{2Q,2q}(2q)^{2r-2} = (-1)^{r-1}\left[f^{(2r-2)}(\pi) - f^{(2r-2)}(-\pi)\right]/4 \quad (4.8)$$

and

$$\sum_{q=1}^{Q}\bar{a}_{2Q,2q-1}(2q-1)^{2r-2} = (-1)^{r}\left[f^{(2r-2)}(\pi) - f^{(2r-2)}(-\pi)\right]/4, \quad (r = 1,2,\cdots,Q). \quad (4.9)$$

Proof. Function $f(x)$ can be considered as the superposition of an even function $g(x) = [f(x) + f(-x)]/2$ and an odd function $h(x) = [f(x) - f(-x)]/2$. Theorems 4 is then directly applicable to $g(x)$ on $[0, \pi]$. Thus, (4.4) holds if

$$\sum_{p=1}^{P}\bar{b}_{2P,2p}(2p)^{2r-1} = (-1)^{r-1}\left[g^{(2r-1)}(\pi) + g^{(2r-1)}(0)\right]/2 \quad (4.10)$$

and

$$\sum_{p=1}^{P} \bar{b}_{2P,2p-1} (2p-1)^{2r-1} = (-1)^r \left[g^{(2r-1)}(\pi) + g^{(2r-1)}(0) \right] \Big/ 2, \quad (r = 1, 2, \cdots, P). \tag{4.11}$$

Since

$$g^{(2r-1)}(0) = \left[f^{(2r-1)}(x) - f^{(2r-1)}(-x) \right] \Big/ 2 \Big|_{x=0} \equiv 0 \tag{4.12}$$

and

$$g^{(2r-1)}(\pi) = \left[f^{(2r-1)}(\pi) - f^{(2r-1)}(-\pi) \right] \Big/ 2, \tag{4.13}$$

(4.10) and (4.11) can be rewritten as (4.6) and (4.7), respectively.

Similarly, relationship (4.5) is readily obtained from applying Theorem 5 to the odd function $h(x)$ on interval $[0, \pi]$ by recognizing that

$$h^{(2r-2)}(0) = \left[f^{(2r-2)}(x) - f^{(2r-2)}(-x) \right] \Big/ 2 \Big|_{x=0} \equiv 0 \tag{4.14}$$

and

$$h^{(2r-2)}(\pi) = \left[f^{(2r-2)}(\pi) - f^{(2r-2)}(-\pi) \right] \Big/ 2. \tag{4.15}$$

The expansion coefficients of $g(x)$ are determined from

$$\begin{aligned} a_m &= \frac{2}{\pi} \int_0^{\pi} \left[\frac{f(x) + f(-x)}{2} - \sum_{p=1}^{2P} \bar{b}_{2P,p} \sin px \right] \cos mx \mathrm{d}x \\ &= \frac{1}{\pi} \int_0^{\pi} f(x) \cos mx \mathrm{d}x + \frac{1}{\pi} \int_{-\pi}^{0} f(x) \cos mx \mathrm{d}x \\ &\quad - \frac{1}{\pi} \int_0^{\pi} \sum_{p=1}^{2P} \bar{b}_{2P,p} \sin px \cos mx \mathrm{d}x + \frac{1}{\pi} \int_{-\pi}^{0} \sum_{p=1}^{2P} \bar{b}_{2P,p} \sin px \cos mx \mathrm{d}x \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx \mathrm{d}x - \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\sum_{p=1}^{2P} \bar{b}_{2P,p} \operatorname{sgn}(x) \sin px \right) \cos mx \mathrm{d}x. \end{aligned} \tag{4.16}$$

Similarly, the expansion coefficients of $h(x)$ are determined from

$$\begin{aligned} b_m &= \frac{2}{\pi} \int_0^{\pi} \left[\frac{f(x) - f(-x)}{2} - \sum_{q=1}^{2Q} \bar{a}_{2Q,q} \cos qx \right] \sin mx \mathrm{d}x \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \mathrm{d}x - \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\sum_{q=1}^{2Q} \bar{a}_{2Q,q} \operatorname{sgn}(x) \cos qx \right) \sin mx \mathrm{d}x. \end{aligned} \tag{4.17}$$

The even (odd) extension of $g(x) - \sum_{p=1}^{2P} \bar{b}_{2P,p} \sin px$ ($h(x) - \sum_{q=1}^{2Q} \bar{a}_{2Q,q} \cos qx$) onto $[-\pi, 0)$ will lead to an even (odd) function $g(x) - \sum_{p=1}^{2P} \bar{b}_{2P,p} \operatorname{sgn}(x) \sin px$ ($h(x) - \sum_{q=1}^{2Q} \bar{a}_{2Q,q} \operatorname{sgn}(x) \cos qx$) on $[-\pi, \pi]$. Expression (4.1) will then become evident. □

Alternatively, (4.2) and (4.3) can be expressed as

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx \mathrm{d}x - \sum_{p=1}^{2P} \alpha_{mp} \bar{b}_{2P,p} \tag{4.18}$$

and

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\mathrm{d}x - \sum_{q=1}^{2Q}\alpha_{qm}\bar{a}_{2Q,q} \tag{4.19}$$

where α_{mp} is given in (2.16).

The coefficients $\bar{b}_{2P,p}$ and $\bar{a}_{2Q,q}$ are readily calculated from (2.35) and (3.19), respectively, by letting

$$f_+^{(2r-1)} = f_-^{(2r-1)} = \left[f^{(2r-1)}(\pi) - f^{(2r-1)}(-\pi)\right]/4, \quad r = 1, 2, \cdots, P \tag{4.20}$$

and

$$f_+^{(2r-2)} = f_-^{(2r-2)} = \left[f^{(2r-2)}(\pi) - f^{(2r-2)}(-\pi)\right]/4, \quad r = 1, 2, \cdots, Q. \tag{4.21}$$

EXAMPLE 3. Consider function $f(x) = Ax^2 + Bx + C$ $(-\pi \le x \le \pi)$. Its classical Fourier expansion is easily found as

$$f(x) = \frac{A\pi^2}{3} + C + 4A\sum_{m=1}^{\infty}\frac{(-1)^m}{m^2}\cos mx - 2B\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\sin mx, \quad -\pi < x < \pi. \tag{4.22}$$

By setting $P = 0$ and $Q = 1$ in (4.1), we have

$$\bar{a}_{2Q,1} = -\bar{a}_{2Q,2} = -\frac{f(\pi) - f(-\pi)}{4} = -\frac{B\pi}{2} \tag{4.23}$$

and

$$\begin{aligned} f(x) = {} & \frac{A\pi^2}{3} + C + 4A\sum_{m=1}^{\infty}\frac{(-1)^m}{m^2}\cos mx - \frac{\pi B}{2}\operatorname{sgn}(x)\cos x \\ & + \frac{\pi B}{2}\operatorname{sgn}(x)\cos 2x + \sum_{m=1}^{\infty} b_m \sin mx, \qquad -\pi \le x \le \pi \end{aligned} \tag{4.24}$$

where

$$\begin{aligned} b_m &= \frac{(-1)^{m+1}2B}{m} - \bar{a}_{2Q,1}\alpha_{1m} - \bar{a}_{2Q,2}\alpha_{2m} \\ &= \begin{cases} \dfrac{2B}{m(m^2-1)} & \text{if } m \text{ is even,} \\ \dfrac{-8B}{m(m^2-4)} & \text{if } m \text{ is odd.} \end{cases} \end{aligned} \tag{4.25}$$

It is seen from (4.25) that the sine series now converges at a rate of m^{-3} which is faster than m^2 for its cosine counterpart. If desired, the convergence of the series expansion in the form of (4.1) can be further accelerated by setting $P = Q = 1$. Accordingly, in addition to (4.23), we have

$$\begin{aligned} \bar{b}_{2P,1} &= -2\bar{b}_{2P,2} \\ &= -\frac{f'(\pi) - f'(-\pi)}{4} = -A\pi \end{aligned} \tag{4.26}$$

and

$$f(x)=\frac{A\pi^2}{3}+C-2A-\frac{\pi B}{2}\operatorname{sgn}(x)\cos x+\frac{\pi B}{2}\operatorname{sgn}(x)\cos 2x$$
$$-\pi A\operatorname{sgn}(x)\sin x+\frac{\pi A}{2}\operatorname{sgn}(x)\sin 2x \tag{4.27}$$
$$+\sum_{m=1}^{\infty}(a_m\cos mx+b_m\sin mx),\quad -\pi\le x\le\pi$$

where

$$a_m=\frac{(-1)^m 4A}{m}-\bar{b}_{2P,1}\alpha_{m1}-\bar{b}_{2P,2}\alpha_{m2}=\begin{cases}\dfrac{-4A}{m^2(m^2-1)} & \text{if } m \text{ is even,}\\[2ex] \dfrac{16A}{m^2(m^2-4)} & \text{if } m \text{ is odd.}\end{cases} \tag{4.28}$$

The series expansion given in (4.27) will converge at a rate of m^{-3} in comparison with m^{-2} for that in (4.24).

5. Corollaries

COROLLARY 1. *Let* $f(x)$ *have* C^{n-1} *continuity on the interval* $[0, \pi]$ *and the n-th derivative is absolutely integrable* (*the n-th derivative may not exist at certain points*). *Assume* $n\ge 2$. *Then* $f(x)$ *can be expanded as*

$$f(x)=\mathcal{F}_{\infty,2P}[f](x)=a_0/2+\sum_{m=1}^{\infty}A_m\cos(mx-\theta_m),\quad(\theta_m\equiv 0 \text{ if } m>2P),\quad 0\le x\le\pi \tag{5.1}$$

and

$$\lim_{m\to\infty}A_m m^{2P}=0\quad(2P\le n). \tag{5.2}$$

Provided that

$$A_m=\begin{cases}\sqrt{a_m^2+b_m^2} & \text{for } 1\le m\le 2P\\ a_m & \text{otherwise}\end{cases} \tag{5.3}$$

and

$$\theta_m=\tan^{-1}(b_m/a_m) \tag{5.4}$$

where a_m *and* b_m *are calculated from* (2.4) *and* (2.35), *respectively.*

Proof. For $1\le m\le 2P$, denote

$$a_m=A_m\cos\theta_m \tag{5.5}$$

and

$$b_m=A_m\sin\theta_m \tag{5.6}$$

or, alternatively, (5.3) and (5.4).

Then expansion (5.1) follows immediately from (2.3) in view that

$$a_m\cos mx+b_m\sin mx=A_m\cos\theta_m\cos mx+A_m\sin\theta_m\sin mx=A_m\cos(mx-\theta_m).$$

Since $A_m=a_m$ for $m>2P$, (5.2) is evident from (2.5). □

COROLLARY 2. *Let* $f(x)$ *have* C^{n-1} *continuity on the interval* $[0, \pi]$ *and the n-th derivative is absolutely integrable* (*the n-th derivative may not exist at certain points*). *Then for* $2P \le n+1$, $f(x)$ *can be expanded as*

$$f(x) = \mathcal{F}_{\infty,2P}[f](x) = \sum_{m=1}^{\infty} A_m \sin(mx + \theta_m), \quad (\theta_m \equiv 0 \text{ if } m > 2P), \quad 0 \le x \le \pi \tag{5.7}$$

and

$$\lim_{m\to\infty} A_m m^{2P-1} = 0 \tag{5.8}$$

provided that A_m *and* θ_m *satisfy* (5.3) *and* (5.4), *and* a_m *and* b_m *are calculated from* (3.2) *and* (3.19), *respectively*.

Proof. By (5.5) and (5.6), we have

$$\begin{aligned} a_m \sin mx + b_m \cos mx &= A_m \cos\theta_m \sin mx + A_m \sin\theta_m \cos mx \\ &= A_m \sin(mx + \theta_m), \quad \text{for } 1 \le m \le 2P. \end{aligned} \tag{5.9}$$

Thus, (5.7) and (5.8) become obvious from (3.1) and (3.3), respectively. □

COROLLARY 3. *Let* $f(x)$ *have* C^{n-1} *continuity on the interval* $[-\pi, \pi]$ *and the n-th derivative is absolutely integrable* (*the n-th derivative may not exist at certain points. Then for* $2P \le n$ *and* $2Q \le n+1$, $f(x)$ *can be expanded as*

$$\begin{aligned} f(x) &= \mathcal{F}_{\infty,2P,2Q}[f](x) \\ &= a_0/2 + \sum_{m=1}^{\infty}\left[A_m \cos(mx - \text{sgn}(x)\theta_m) + B_m \sin(mx + \text{sgn}(x)\phi_m)\right], \\ &(\theta_m \equiv 0 \text{ if } m > 2P, \phi_m \equiv 0 \text{ if } m > 2Q) \quad -\pi \le x \le \pi \end{aligned} \tag{5.10}$$

and

$$\lim_{m\to\infty} A_m m^{2P} = 0 \tag{5.11}$$

and

$$\lim_{m\to\infty} B_m m^{2Q-1} = 0 \tag{5.12}$$

provided that

$$A_m = \begin{cases} \sqrt{a_m^2 + \bar{b}_{2P,m}^2} & \text{for } 1 \le m \le 2P \\ a_m & \text{otherwise} \end{cases} \tag{5.13}$$

$$B_m = \begin{cases} \sqrt{b_m^2 + \bar{a}_{2Q,m}^2} & \text{for } 1 \le m \le 2Q \\ b_m & \text{otherwise} \end{cases} \tag{5.14}$$

$$\theta_m = \tan^{-1}\left(\bar{b}_{2P,m}/a_m\right) \tag{5.15}$$

and

$$\phi_m = \tan^{-1}\left(\bar{a}_{2Q,m}/b_m\right) \tag{5.16}$$

where a_m, b_m, $\bar{a}_{2Q,m}$ and $\bar{b}_{2P,m}$ are defined in the same way as those in (4.1).

Since Corollary 3 is obvious from Theorem 6 and Corollaries 1 and 2, its proof will not be given here.

Notice that in (5.1)

$$A_m \cos(mx-\theta_m) = \frac{A_m}{2}\left(e^{i(mx-\theta_m)} + e^{-i(mx-\theta_m)}\right) = C_m e^{imx} + C_{-m} e^{-imx} \tag{5.17}$$

where

$$C_m = \frac{A_m e^{-i\theta_m}}{2} = \frac{1}{2}(A_m \cos\theta_m - iA_m \sin\theta_m) = \begin{cases} (a_m - ib_m)/2 & 1 \le m \le 2P \\ a_m/2 & \text{otherwise} \end{cases} \tag{5.18}$$

and $C_{-m} = C_m^*$ (superposed * indicates taking complex conjugate).

Remark. In Corollary 1, (5.1) can be alternatively written as

$$f(x) = \mathcal{F}_{\infty,2P}[f](x) = \sum_{m=-\infty}^{\infty} A_m e^{imx}, \quad 0 \le x \le \pi \tag{5.19}$$

where

$$A_m = \begin{cases} (a_m - ib_m)/2 & 1 \le m \le 2P \\ a_m/2 & \text{otherwise} \end{cases} \tag{5.20}$$

and $A_{-m} = A_m^*$.

Similarly, (5.7) in Corollary 2 can be written as

$$f(x) = \mathcal{F}_{\infty,2P}[f](x) = \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} A_m e^{imx}, \quad 0 \le x \le \pi \tag{5.21}$$

where

$$A_m = \begin{cases} (a_m + ib_m)/2i & 1 \le m \le 2P \\ a_m/2i & \text{otherwise} \end{cases} \tag{5.22}$$

and $A_{-m} = A_m^*$.

And (5.9) in Corollary 3 as

$$f(x) = \mathcal{F}_{\infty,2P,2Q}[f](x) = \sum_{m=-\infty}^{\infty} A_m e^{imx}, \quad -\pi \le x \le \pi \tag{5.24}$$

where

$$A_m = \left[\left(a_m + \bar{a}_m \operatorname{sgn}(x)\right) - i\left(b_m + \bar{b}_m \operatorname{sgn}(x)\right)\right]/2 \tag{5.25}$$

and $A_{-m} = A_m^*$.

6. Conclusion

Alternative Fourier series expansions have been presented in an effort of better representing a sufficiently smooth function in a compact interval. The series expansions can take various forms, resulting in different rates of convergence. When one of the series expansions, for example, is used to solve a boundary value problem, its convergence rate needs to be compatible with the smoothness of the solution "physically" dictated by the problem. Thus, there may exist the best form for any given problem. Among other important applications, the new Fourier series will potentially lead to a new path for solving differential equations and boundary value problems.

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