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Asymptotically Antiperiodic Solutions for a Nonlinear Differential Equation with Piecewise Constant Argument in a Banach Space*

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Abstract

In this paper, we give sufficient conditions for the existence and uniqueness of asymptotically ω-antiperiodic solutions for a nonlinear differential equation with piecewise constant argument in a Banach space when ω is an integer. This is done using the Banach fixed point theorem. An example involving the heat operator is discussed as an illustration of the theory.

Keywords

Asymptotically ω-Antiperiodic Functions, Differential Equations with Piecewise Constant Argument, Semi-Group

1. Introduction

We are concerned with the differential Cauchy problem with piecewise constant argument:

$$\begin{cases} x'(t) = Ax(t) + A_0 x([t]) + g(t, x([t])) \\ x(0) = c_0 \end{cases} \tag{1}$$

where A_0 is a bounded linear operator, $[.]$ is the largest integer function, g is a continuous function on $\mathbb{R}^+ \times \mathbb{X}$ and A is the infinitesimal generator of an exponentially semigroup $T(t), t \geq 0$ acting on the Banach space $\mathbb{X}$. The main purpose of this work is to study, for the first time, the existence and the uniqueness of asymptotically ω-antiperiodic solutions to (1) when ω is an integer.

Differential equations with piecewise constant argument (EPCA) have the structure

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of continuous dynamical systems in intervals of constant length. Therefore they combine the properties of both differential and difference equations. They are used to model problems in biology, economy and in many other fields (see [1]-[7]).

The study of the existence and uniqueness of periodic solutions of differential equations is a well-established fact. The concept of asymptotical periodicity has been introduced to handle phenomena which behave periodically as time grows (see for instance [8]-[10]). However, antiperiodicity has a great importance in the qualitative study of differential equations. For instance, many phenomena in biology, ecology, quantum physics and engineering are antiperiodic (see [10]-[17] and references therein).

Recently, the authors of [18] introduced the concept of asymptotically antiperiodic functions and studied semilinear integrodifferential equations in this framework. In [19], a new composition theorem for asymptotically antiperiodic functions is proved. This result is used to show the existence and the uniqueness of asymptotically antiperiodic mild solution to some fractional functional integro-differential equations in a Banach space. Motivated by [18] and [19], we will show the existence and uniqueness of asymptotically antiperiodic mild solution for (1).

This work is organized as follows. In Section 2, we recall some fundamental properties of asymptotically antiperiodic functions. Section 3 is devoted to our main results. We illustrate our main result in Section 4, dealing with the existence and the uniqueness of asymptotically antiperiodic solution for a partial differential equation.

2 Preliminaries

Let $\mathbb{X}$ be a Banach space. The space $BC(\mathbb{R}^+,\mathbb{X})$ of the continuous bounded functions from $\mathbb{R}^+$ into $\mathbb{X}$, endowed with the norm $\|f\|_\infty := \sup_{t\geq 0}\|f(t)\|$, is a Banach space. The Banach subspace of functions *f* such that $\lim_{t\to\infty} f(t)=0$ is denoted by $C_0(\mathbb{R}^+,\mathbb{X})$. A positive number ω being given, $P_\omega(\mathbb{X})$ will be the subset of $BC(\mathbb{R}^+,\mathbb{X})$ constituted of all ω-periodic functions; it is also a Banach space. We recall the following properties of antiperiodic and asymptotically antiperiodic functions. We refer to [18] where they are proved.

Definition 2.1. *A function* $f\in BC(\mathbb{R},\mathbb{X})$ *is said to be ω-antiperiodic* (*or simply antiperiodic*) *if there exists* $\omega>0$ *such that* $f(t+\omega)=-f(t)$ *for all* $t\in\mathbb{R}$. *The least such ω will be called the antiperiod of f.*

We will denote by $P_{\omega ap}(\mathbb{X})$, the space of all ω-antiperiodic functions $\mathbb{R}\to\mathbb{X}$.

Theorem 2.1. *Let* $f, f_1, f_2\in P_{\omega ap}(\mathbb{X})$. *Then the following are also in* $P_{\omega ap}(\mathbb{X})$.

i) f_1+f_2, cf, c is an arbitrary real number.

ii) $g(t):=\frac{1}{f}(t)$, provided $f\neq 0$ on $\mathbb{R}$. Here $\mathbb{X}=\mathbb{R}$.

iii) $f_a(t):=f(t+a)$, a is an arbitrary real number.

Theorem 2.2. $P_{\omega ap}(\mathbb{X})$ *is a Banach space equipped with the supnorm.*

Now we consider asymptotically ω-antiperiodic function.

Definition 2.2. *A function* $f\in BC(\mathbb{R},\mathbb{X})$ *is said to be asymptotically ω-antiperiodic if there exist* $u\in P_{\omega ap}(\mathbb{X})$ *and* $h\in C_0(\mathbb{R}^+,\mathbb{X})$, *such that*

$$f = u + h, \quad \forall t \in \mathbb{R}^+.$$

g and *h* are called respectively the principal and corrective terms of *f*.

We will denote by $AP_{\omega ap}(\mathbb{X})$, the space of all asymptotically ω-antiperiodic $\mathbb{X}$-valued functions.

Remark 2.1. $AP_{\omega ap}(\mathbb{X})$ *is a Banach space equipped with the supnorm and the decomposition of an asymptotically antiperiodic is unique.*

3. Main Results

We begin with the definition of a solution to (1).

Definition 3.1. *A solution of Equation* (1) *on* $\mathbb{R}^+$ *is a function* $x(t)$ *that satisfies the conditions*:

1-x(t) is continuous on $\mathbb{R}^+$.

2-The derivative $x'(t)$ exists at each point $t \in \mathbb{R}^+$, with possible exception of the points $[t] \in \mathbb{R}^+$ where one-sided derivatives exists.

3-*Equation* (1) is satisfied on each interval $[n, n+1)$ with $n \in \mathbb{N}$.

Let $T(t)$ be the C_0 semigroup generated by A and x a solution of (1). Then the function m defined by $m(s) = T(t-s)x(s)$ is differentiable for $s < t$ and we can write:

$$\begin{aligned}\frac{\mathrm{d}m(s)}{\mathrm{d}s} &= -AT(t-s)x(s) + T(t-s)x'(s) \\ &= -AT(t-s)x(s) + T(t-s)Ax(s) \\ &\quad + T(t-s)A_0x([s]) + T(t-s)g(s, x([s]))\end{aligned}$$

which leads to

$$\frac{\mathrm{d}m(s)}{\mathrm{d}s} = T(t-s)A_0x([s]) + T(t-s)g(s, x([s])). \tag{2}$$

The function $x([s])$ is a step function and $g(s, x([s]))$ is a continuous function in the intervals $[n, n+1)$, where $n \in \mathbb{N}$. Therefore, the functions $x([s])$ and $g(s, x([s]))$ are integrable over $[0, t]$ with $t \in \mathbb{R}^+$. Integrating both sides of (2) over $[0, t]$, yields

$$x(t) - T(t)x(0) = \int_0^t T(t-s)A_0x([s])\mathrm{d}s + \int_0^t T(t-s)g(s, x([s]))\mathrm{d}s.$$

Therefore, we give the following

Definition 3.2. *Let* $T(t)$ *be the* C_0 *semigroup generated by A. The function* $x \in \mathcal{C}(\mathbb{R}^+, \mathbb{X})$ *given by*

$$x(t) = T(t)c_0 + \int_0^t T(t-s)A_0x([s])\mathrm{d}s + \int_0^t T(t-s)g(s, x([s]))\mathrm{d}s$$

is the mild solution of the Equation (1).

Now we assume that:

(H.1) The operator A is the infinitesimal generator of an exponentially stable semigroup $(T(t))_{t\geq 0}$ such that there exist constants $M > 0$ and $\delta > 0$ with

$$\|T(t)\|_{B(\mathbb{X})} \leq M\mathrm{e}^{-\delta t}, \forall t \geq 0.$$

The proof of the main result of this paper is based on the following two lemmas.

Lemma 3.1. *Assume that* **(H.1)** *is satisfied and that* A_0 *is a linear bounded operator. Let* $\omega \in \mathbb{N}$*, we define the nonlinear operator* Γ_1 *by: for each* $\phi \in AP_{\omega ap}(\mathbb{X})$

$$(\Gamma_1 \phi)(t) = \int_0^t T(t-s) A_0 \phi([s]) \mathrm{d}s.$$

Then the operator Γ_1 maps $AP_{\omega ap}(\mathbb{X})$ into itself.

Proof. Define the function F by

$$F(t) = \int_0^t T(t-s) A_0 \phi([s]) \mathrm{d}s.$$

Since $\phi \in AP_{\omega ap}(\mathbb{X})$, it may be decomposed as $\phi = u + h$ holds, where $u \in P_{\omega ap}(\mathbb{X})$ and $h \in C_0(\mathbb{R}^+, \mathbb{X})$. We note that

$$F(t) = G(t) + H(t), \quad t \in \mathbb{R}$$

where

$$G(t) = \int_{-\infty}^t T(t-s) A_0 u([s]) \mathrm{d}s$$

and

$$H(t) = \int_0^t T(t-s) A_0 h([s]) \mathrm{d}s - \int_{-\infty}^0 T(t-s) A_0 u([s]) \mathrm{d}s.$$

We claim that $H \in C_0(\mathbb{R}^+, \mathbb{X})$. Since $h \in C_0(\mathbb{R}^+, \mathbb{X})$, then $\lim_{t \to +\infty} h([t]) = 0$. Therefore: $\forall \epsilon > 0$, there exists a constant $T > 0$ such that $\|h([s])\| \le \epsilon$ for all $s \ge T$. For all $t \ge 2T$, we have that

$$H(t) = \int_0^{\frac{t}{2}} T(t-s) A_0 h([s]) \mathrm{d}s + \int_{\frac{t}{2}}^t T(t-s) A_0 h([s]) \mathrm{d}s$$
$$- \int_{-\infty}^0 T(t-s) A_0 u([s]) \mathrm{d}s,$$

from which it follows that

$$\|H(t)\| \le \int_0^{\frac{t}{2}} M \mathrm{e}^{-\delta(t-s)} \|A_0\| \|h\| \mathrm{d}s + \int_{\frac{t}{2}}^t M \mathrm{e}^{-\delta(t-s)} \|A_0\| \epsilon \mathrm{d}s$$
$$+ \int_{-\infty}^0 M \mathrm{e}^{-\delta(t-s)} \|A_0\| \|u\| \mathrm{d}s$$
$$\le \frac{M}{\delta} \mathrm{e}^{-\delta \frac{t}{2}} \|A_0\| \|h\| - \frac{M}{\delta} \mathrm{e}^{-\delta t} \|A_0\| \|h\| + \frac{M}{\delta} \|A_0\| \epsilon$$
$$- \frac{M}{\delta} \mathrm{e}^{-\delta \frac{t}{2}} \|A_0\| \epsilon + \frac{M}{\delta} \mathrm{e}^{-\delta t} \|A_0\| \|u\|.$$

Hence, $\lim_{t \to +\infty} H(t) = 0$. Since H is clearly continuous, the claim is then proved. Now, we show that $G \in P_{\omega ap}(\mathbb{X})$:

$$G(t+\omega) = \int_{-\infty}^{t+\omega} T(t+\omega-s) A_0 u([s]) \mathrm{d}s = \int_{-\infty}^t T(t-s) A_0 u([s+\omega]) \mathrm{d}s$$
$$= \int_{-\infty}^t T(t-s) A_0 u([s]+\omega) \mathrm{d}s = -\int_{-\infty}^t T(t-s) A_0 u([s]) \mathrm{d}s.$$

Therefore $G(t+\omega) = -G(t)$. It follows that $G \in P_{\omega ap}(\mathbb{X})$ and $H \in C_0(\mathbb{R}^+, \mathbb{X})$ which proves that $F \in AP_{\omega ap}(\mathbb{X})$. □

Lemma 3.2. *Assume that* **(H.1)** *is satisfied and also that* $\omega \in \mathbb{N}$*. Let* $g \in BC(\mathbb{R}, \mathbb{X})$

be such that:

i) $\forall (t,x) \in \mathbb{R}\times\mathbb{X}, g(t+\omega,-x) = -g(t,x)$;

ii) $\exists K > 0, \forall (t,x,y) \in \mathbb{R}\times\mathbb{X}\times\mathbb{X}, \|g(t,x)-g(t,y)\| \le K\|x-y\|$.

Define the nonlinear operator Γ_2 by: for each $\phi \in AP_{\omega ap}(\mathbb{X})$

$$(\Gamma_2\phi)(t) = \int_0^t T(t-s)\,g\big(s,\phi([s])\big)\mathrm{d}s.$$

Then the operator Γ_2 maps $AP_{\omega ap}(\mathbb{X})$ into itself.

Proof. Let $\phi \in AP_{\omega ap}(\mathbb{X})$. Then $\phi = \phi_1 + \phi_2$ with $\phi_1 \in P_{\omega ap}(\mathbb{X})$ and $\phi_2 \in C_0(\mathbb{R}^+,\mathbb{X})$. We have

$$g\big(t,\phi([t])\big) = g\big(t,\phi_1([t])\big) + l(t)$$

with $l(t) = g\big(t,\phi([t])\big) - g\big(t,\phi_1([t])\big)$. We have

$$\|l(t)\| \le K\|\phi_2([t])\|.$$

Since $\lim_{t\to\infty}\phi_2([t]) = 0$, we deduce that $\lim_{t\to\infty} l(t) = 0$.

We note also that $g\big(t+\omega,\phi_1([t+\omega])\big) = -g\big(t,\phi_1([t])\big)$. In fact

$$\begin{aligned} g\big(t+\omega,\phi_1([t+\omega])\big) &= g\big(t+\omega,\phi_1([t]+\omega)\big) \\ &= g\big(t+\omega,-\phi_1([t])\big) \\ &= -g\big(t,\phi_1([t])\big). \end{aligned}$$

We put

$$F(t) = \int_0^t T(t-s)\,g\big(s,\phi([s])\big)\mathrm{d}s.$$

Since the function g is lipschitzian, then the function $t \to g\big(t,\phi([t])\big)$ is piecewise continuous. Therefore the function F is well defined. Since $\phi = \phi_1 + \phi_2$ with $\phi_1 \in P_{\omega ap}(\mathbb{X})$ and $\phi_2 \in C_0(\mathbb{R}^+,\mathbb{X})$, we observe that

$$F(t) = G(t) + H(t)$$

where

$$G(t) = \int_{-\infty}^t T(t-s)\,g\big(s,\phi_1([s])\big)\mathrm{d}s, \quad t \in \mathbb{R}$$

and

$$H(t) = \int_0^t T(t-s)\,l(s)\,\mathrm{d}s - \int_{-\infty}^0 T(t-s)\,g\big(s,\phi_1([s])\big)\mathrm{d}s, \quad t \in \mathbb{R}^+.$$

The functions $G(t)$ and $l(t)$ are well defined because the function $l(t)$ and $g\big(t,\phi_1([t])\big)$ are continuous on $[n,n+1)$ where n is an integer. Since $\lim_{t\to\infty} l(t) = 0$ and $g\big(t+\omega,\phi_1([t+\omega])\big) = -g\big(t,\phi_1([t])\big)$, it follows that $G \in P_{\omega ap}(\mathbb{X})$ and $H \in C_0(\mathbb{R}^+,\mathbb{X})$. □

Now we can state and prove the main result of this work.

Theorem 3.3. *We assume that the hypothesis* **(H.1)** *is satisfied. We assume also that* $\omega \in \mathbb{N}$. *Let* $g \in BC(\mathbb{R},\mathbb{X})$ *such that*:

i) $\forall (t,x) \in \mathbb{R}\times\mathbb{X}, g(t+\omega,-x) = -g(t,x)$.

ii) $\exists K > 0, \forall (t,x,y) \in \mathbb{R}\times\mathbb{X}\times\mathbb{X}, \|g(t,x)-g(t,y)\| \le K\|x-y\|$.

Then the Equation (1) has a unique asymptotically ω antiperiodic solution if

$$\rho := \frac{M}{\delta}\left(\|A_0\| + K\right) < 1.$$

Proof. Define the nonlinear operator $\Gamma : AP_{\omega ap}(\mathbb{X}) \mapsto AP_{\omega ap}(\mathbb{X})$,

$$(\Gamma u)(t) := L(t) + (\Gamma_1 u)(t) + (\Gamma_2 \phi)(t)$$

for every $u \in AP_{\omega ap}(\mathbb{X})$, where

$$L(t) = T(t)c_0$$

$$(\Gamma_1 \phi)(t) = \int_0^t T(t-s) A_0 \phi([s]) \mathrm{d}s$$

and

$$(\Gamma_2 \phi)(t) = \int_0^t T(t-s) g(s, \phi([s])) \mathrm{d}s.$$

Since $L(t) \in C_0(\mathbb{R}^+, \mathbb{X})$ we have $\|L(t)\| \le M\mathrm{e}^{-\delta t} c_0, \forall t \ge 0$. Then, using Lemma 3.1 and Lemma 3.2, it follows that the operator Γ maps $AP_{\omega ap}(\mathbb{X})$ into itself.

For every $\phi, \psi \in AP_{\omega ap}(\mathbb{X})$,

$$\begin{aligned}
\|\Gamma(\phi)(t) - \Gamma(\psi)(t)\| &\le \left\| \int_0^t T(t-s) A_0 \left(\phi([s]) - \psi([s])\right) \mathrm{d}s \right\| \\
&\quad + \left\| \int_0^t T(t-s) \left(g(s, \phi(s)) - g(s, \psi(s))\right) \mathrm{d}s \right\| \\
&\le \int_0^t \|T(t-s)\| \|A_0\| \|\phi([s]) - \psi([s])\| \mathrm{d}s \\
&\quad + \int_0^t \|T(t-s)\| \left\| \left(g(s, \phi(s)) - g(s, \psi(s))\right) \right\| \mathrm{d}s \\
&\le \int_0^t \|T(t-s)\| \|A_0\| \|\phi - \psi\|_\infty \mathrm{d}s + \int_0^t \|T(t-s)\| K \|\phi - \psi\|_\infty \mathrm{d}s \\
&\le \int_0^t M\mathrm{e}^{-\delta(t-s)} \mathrm{d}s \|A_0\| \|\phi - \psi\|_\infty + \int_0^t MK\mathrm{e}^{-\delta(t-s)} \mathrm{d}s \|\phi - \psi\|_\infty \\
&\le \frac{M}{\delta} \|A_0\| \|\phi - \psi\|_\infty + \frac{M}{\delta} K \|\phi - \psi\|_\infty \\
&\le \frac{M}{\delta} \left(\|A_0\| + K\right) \|\phi - \psi\|_\infty .
\end{aligned}$$

Therefore, since $\rho < 1$, using the Banach fixed point Theoren we conclude that Equation (1) has a unique asymptotically ω-antiperiodic solution. □

4. Application

As an application, consider for $t \in \mathbb{R}^+, x \in [0, \pi]$ and $\alpha \in \mathbb{R}$, the Cauchy problem:

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x) + \alpha u([t], x) + g(t, u([t], x)); \\ u(t, 0) = u(t, \pi) = 0. \end{cases} \tag{3}$$

We take $(\mathbb{X}, \|\cdot\|) = \left(L^2([0, \pi]), \|\cdot\|_2\right)$ and we define the linear operator A by

$$D(A) = \left\{ v \in L^2([0, \pi]), v'' \in L^2([0, \pi]), v(0) = v(\pi) = 0 \right\}$$

$$Av = v''.$$

where the derivatives are taken in the distributional sense. Then, A is the infinitesimal generator of a semigroup $T(t)$ on $L^2([0,\pi])$ satisfying $\|T(t)\| \le e^{-t}$ for $t \ge 0$ (see [20]). The operator $A_0 : L^2([0,\pi]) \to L^2([0,\pi])$ defined by $A_0(v) = \alpha v$ is linear and bounded with $\|A_0\| = |\alpha|$. Therefore (3) takes the abstract form (1). Assume that the function $g \in BC(\mathbb{R}, \mathbb{X})$ satisfies the following:

i) $\forall (t,x) \in \mathbb{R} \times \mathbb{X}, g(t+\omega, -x) = -g(t,x)$,

ii) $\exists K > 0, \forall (t,x,y) \in \mathbb{R} \times \mathbb{X} \times \mathbb{X}, \|g(t,x) - g(t,y)\| \le K\|x-y\|$.

Note that such a function exists. Take for instance $g(t,x) = f(t)x$ where f is a ω-periodic function from $\mathbb{R}$ into $\mathbb{R}$. Then we have

$$g(t+\omega, -x) = f(t)(-x) = -g(t,x)$$

and

$$\|g(t,x) - g(t,y)\| = \|f(t)(x-y)\| \le \|f\|_\infty \|x-y\|.$$

Theorem 4.1. *We assume that* $\omega \in \mathbb{N}$. *Then System* (3) *has a unique asymptotically ω-antiperiodic if* $|\alpha| + K < 1$.

Proof. We have $M = 1$, $\delta = 1$, $\|A_0\| = |\alpha|$ and we apply Theorem 3.3. □

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