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On the Asymptotic Behavior of Second Order Quasilinear Difference Equations

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Abstract

In this paper, we investigate the asymptotic behavior of the following quasilinear difference equations

$$\Delta\left(\left|\Delta y(n)\right|^{\alpha-1}\Delta y(n)\right)=p(n)\left|y(n)\right|^{\beta-1}y(n), \tag{E}$$

where $n\in N_0=\{n_0,n_0+1,n_0+2,\cdots\}$, $n_0\in N$. We classified the solutions into six types by means of their asymptotic behavior. We establish the necessary and/or sufficient conditions for such equations to possess a solution of each of these six types.

Keywords

Asymptotic Behavior, Positive Solutions, Homogeneous, Quasilinear Difference Equations

1. Introduction

Recently, the asymptotic properties of the solutions of second order differential equations [1] [2] difference equations of the type (E) and/or related equations have been investigated by many authors, for example see, [3]-[19] and the references cited there in. Following this trend, we investigate the existence of these six types of solutions of the Equation (E) showing the necessary and/or sufficient conditions can be obtained for the existence of those solutions. For the general backward on difference equations, the reader is referred to the monographs [20]-[24].

In 1996, PJY Wang and R.P. Agarwal [25] considered the quasilinear equation

$$\Delta\left(a_{n-1}\left(\Delta y_{n-1}\right)^{\sigma}\right)+q_n f\left(y_n\right)=0 \tag{1}$$

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and obtained oscillation criteria for the Equation (1).

In 1996, E. Thandapani, M.M.S. Manuel and R.P. Agarwal [26] have studied the quasi-linear difference equation

$$\Delta\left(a_{n-1}\left|\Delta x_{n-1}\right|^{\alpha-1}\Delta x_{n-1}\right)+q_n f\left(x_n\right)=0. \tag{2}$$

In 2000, Pon Sundaram and E. Thandapani [27] considered the following quasi-linear functional difference equation

$$\Delta\left(\left|\Delta y(n)\right|^{\alpha-1}\Delta y(n)\right)+f\left(n,y\left(\sigma(n)\right)\right)=0 \tag{3}$$

and they have established necessary and sufficient conditions for the solutions of Equation (3) to have various types of nonoscillatory solutions. Further they have established some new oscillation conditions for the oscillation of solutions of Equation (3).

In 1997, E. Thandapani and R. Arul [28] studied, the following quasi-linear equation

$$\Delta\left(p_n\phi\left(y_n\right)\right)+f\left(n,y_{n+1}\right)=0. \tag{4}$$

They established necessary and sufficient conditions for the solutions of (4) to have various type of nonoscillatory solutions.

In 2004, E. Thandapani *et al*. [29] studied the equation

$$\Delta\left(a_n\Delta\left(y_n-py_{n-k}^{\alpha}\right)\right)+q_n f\left(y_{n-\ell+1}\right)=0,\quad n\ge n_0\ge 0, \tag{5}$$

and established conditions for the existence of non-oscillatory solutions.

S.S. Cheng and W.T. Patula [30] studied the difference equation

$$\Delta\left(\Delta y_{k-1}\right)^{p-1}+s_k y_k^{p-1}=0 \tag{6}$$

where $p>1$ and proved an existence theorem for Equation (6).

In 2002, M. Mizukanmi *et al.* [1] discussed the asymptotic behavior of the following equation

$$\left(\left|y'\right|^{\alpha-1}y'\right)=p(t)\left|y\right|^{\beta-1}y. \tag{7}$$

Discrete models are more suitable for understanding the problems in Economics, genetics, population dynamics etc. In the qualitative theory of difference equations asymptotic behavior of solutions plays a vital role. Motivated by this, we consider the discrete analogue of (7) of the form

$$\Delta\left(\left|\Delta y(n)\right|^{\alpha-1}\Delta y(n)\right)=p(n)\left|y(n)\right|^{\beta-1}y(n) \tag{8}$$

where $n\in N_0=\{n_0,n_0+1,n_0+2,\cdots\}$, $n_0\in N$ and Δ is the forward difference operator defined by

$$\Delta y(n)=y(n+1)-y(n).$$

We assume the following conditions on Equation (8)

1) α and β are positive constants

2) $\{p(n)\}$ is a real sequence such that $p(n)>0$ for all $n\ge n_0>0$.

For simplicity, we often employ the notation

$$x^{\gamma^*}=\left|x\right|^{\gamma-1}x=\left|x\right|^{\gamma}\operatorname{sgn} x,\quad x\in R,\quad \gamma>0,$$

interms of which Equation (8) can be expressed in

$$\Delta\left(\Delta y(n)\right)^{\alpha^*}=p(n)\left(y(n)\right)^{\beta^*}.$$

By a solution of Equation (8), we mean a real sequence $y(n):N_0\to R$, together with $\left|\Delta y(n)\right|^{\alpha-1}y(n)$ exists and satisfies Equation (8) for all $n\ge n_0\in N_0$.

We here call Equation (8) super-homogeneous or sub-homogeneous according as $\alpha<\beta$ or $\alpha>\beta$ If $\alpha=\beta$ Equation (8) is often called half-linear. Our attention is mainly paid to the super-homogeneous and sub-homogeneous cases, and the half-linear is almost excluded from our consideration.

2. The Classification of All Solutions of Equation (8)

To classify all solutions of Equation (8), we need the following lemma.

Lemma 1. *Let* $y(n)$ *be a local solutions of Equation* (8) *near* $n = N \geq n_0$ *and* $[N, w)$, $w \leq \infty$, *be its right maximal interval of existence. Then we have either* $y(n) \geq 0$ *near w or* $y(n) \leq 0$ *near w. That is* $y(n)$ *does not charge strictly its sign infinitely many times as* $n \uparrow w$.

The classification of all (local) solutions of Equation (8) are given on the basis of Lemma 1. Since the proof is easy, we leave it to the reader.

Proposition 1. *Each local solution* $y(n) \not\equiv 0$ *of Equation* (8) *falls into exactly one of the following six types.*

1) Singular solution of the first kind: type (S_1) there exist a $n_1 \geq n_0$ such that

$$y(n) \not\equiv 0 \quad \text{for } n \leq n_1, \quad \text{and} \quad y(n) \equiv 0 \quad \text{for } n \geq n_1.$$

2) Decaying solution: type (D), $y(n)$ can be continued to ∞, and satisfies $y(n)\Delta y(n) < 0$ for all large *n*, and

$$\lim_{n\to\infty} y(n) = 0.$$

3) Asymptotically constant solution: type (AC) $y(n)$ can be continued to ∞, and satisfies $y(n)\Delta y(n) < 0$ for all large *n* and

$$\lim_{n\to\infty} y(n) \in R - \{0\}.$$

4) Asymptotically linear solution: type (AL) $y(n)$ can be continued to ∞ and satisfies $y(n)\Delta y(n) > 0$ for all large *n* and

$$\lim_{n\to\infty} \frac{y(n)}{n} \in R - \{0\}.$$

5) Asymptotically super-linear solution: type (AS) $y(n)$ can be continued to ∞ and satisfies $y(n)\Delta y(n) > 0$ for all large *n* and

$$\lim_{n\to\infty} \frac{y(n)}{n} = \pm\infty.$$

6) Singular solution of second kind: type (S_2) $y(n)$ has the finite escape time; that is, there exists a $n_1 > n_0$ such that

$$\lim_{n\to\infty} y(n) = \pm\infty.$$

3. Main Results for the Super-Homogeneous Equations

Before we list our main results for the case $\alpha < \beta$. Throughout this section we assume that $\alpha < \beta$.

Theorem 2. *Equation* (8) *has no solution of type* (S_1).

Theorem 3. *Equation* (8) *has a solution of type* (*D*) *if and only if*

$$\sum^{\infty}\left(\sum_{n}^{\infty} p(s)\right)^{1/\alpha} = \infty. \tag{9}$$

Theorem 4. *Equation* (8) *has a solution of type* (*AC*) *if and only if*

$$\sum^{\infty}\left(\sum_{n}^{\infty} p(s)\right)^{1/\alpha} < \infty. \tag{10}$$

Theorem 5. *Equation* (8) *has a solution of type* (*AL*) *if and only if*

$$\sum^{\infty} n^{\beta} p(n) < \infty. \tag{11}$$

Theorem 6. *Equation* (8) *has a solution of type* (*AS*) *if* (11) *holds.*

Theorem 7. *Equation* (8) *does not have solutions of type* (*AS*) *if there are constants* $\rho > 0$ *and* $\sigma \in (0,1)$ *satisfying*

$$\liminf_{n\to\infty} n^{\rho} \sum_{n}^{\infty} s^{\beta\sigma+\sigma-\rho-1} \left(p(s)\right)^{\alpha} > 0 \tag{12}$$

and

$$\left.\begin{array}{l} \beta\sigma + \sigma - \rho - 1 \ge 0 \\ 1 - \sigma - \alpha\sigma - \alpha\rho \ge 0 \end{array}\right\} \tag{13}$$

Remark 1. *The set of all pairs* $(\rho,\tau) \in (0,\infty)\times(0,1)$ *satisfying inequalities* (13) *is not empty. In fact, the pair* $(\rho,\sigma) = \left(\frac{\beta-\alpha}{\alpha\beta+2\alpha+1}, \frac{\alpha+1}{\alpha\beta+2\alpha+1}\right)$ *belongs to it.*

Theorem 8. *Equation* (8) *has a solutions of type* (S_2).

Remark 2. *Theorem* 7 *has the same conclusion that these are not solutions of type* (*AS*). *However, Theorem* 7 *is still valid for the case that p is nonnegative. For example, it is formed by this extended version of Theorem* 7 *that the equation*

$$\Delta\left(\left|\Delta y(n)\right|^{\alpha-1} \Delta y(n)\right) = (1+t)\left|y(n)\right|^{\beta-1} y(n), \quad n \ge 1$$

does not have solutions of type (*AS*).

Example 1 *Let* $\alpha < \beta$, *consider the Equation* (8) *with* $p(n) = n^{\sigma}$

$$\Delta\left(\left|\Delta y(n)\right|^{\alpha-1} \Delta y(n)\right) = n^{\alpha}\left|y(n)\right|^{\beta-1} y(n), \ n \ge 1 \text{ and } \sigma \in R. \tag{14}$$

For this equation, we have the following results:

1) Equation (14) has a solution of type (*D*) if and only if $\sigma \ge -\alpha - 1$ (Theorem 3).
2) Equation (14) has a solution of type (*AC*) if and only if $\sigma < -\alpha - 1$ (Theorem 4).
3) Equation (14) has a solution of type (*AL*) if and only if $\sigma < -\beta - 1$ (Theorem 5).
4) Equation (14) has a solution of type (*AS*) if and only if $\sigma < -\beta - 1$ (Theorem 6).

4. Main Results for the Sub-Homogeneous Equation

Below we list our main results for the case $\alpha > \beta$. Throughout this section we assume that $\alpha > \beta$.

Theorem 9. *Equation* (8) *has a solutions of type* (S_1).

Theorem 10. *Equation* (8) *has a solution of type* (*D*) *if*

$$\sum^{\infty}\left(\sum_{n}^{\infty} p(s)\right)^{1/\alpha} < \infty. \tag{15}$$

Theorem 11. *Equation* (8) *does not have solutions of type* (*D*) *if*

$$\liminf_{n\to\infty} n^{1+\alpha} p(n) > 0. \tag{16}$$

Theorem 12. *Equation* (8) *does not have solutions of type* (*D*) *if there are constants* $\rho > 0$ *and* $\sigma \in (0,1)$ *satisfying*

$$\liminf_{n\to\infty} n^{\rho\alpha} \sum_{n}^{\infty} (s)^{\sigma+\alpha\sigma-\alpha\rho-1} \left|p(s)\right| > 0 \tag{17}$$

and

$$\begin{cases} \beta\sigma + \sigma + \rho - 1 \le 0 \\ 1 - \sigma - \sigma\alpha + \alpha\rho \le 0. \end{cases} \tag{18}$$

Remark 3. *The set of all pairs* $(\rho,\sigma) \in (0,\infty)\times(0,1)$ *satisfying inequalities* (18) *is not empty. In fact, the* *pair*

$$(\rho,\sigma)=\left(\frac{\alpha-\beta}{\alpha\beta+2\alpha+1},\frac{\alpha+1}{\alpha\beta+2\alpha+1}\right)$$

belongs to it.

Theorem 13. *Equation* (8) *has a solution of type* (*AC*) *if and only if* (15) *holds.*

Theorem 14. *Equation* (8) *has a solution of type* (*AL*) *if and only if*

$$\sum^{\infty} n^{\beta} p(n)<\infty.$$

Theorem 15. *Equation* (8) *has a solution of type* (*AS*) *if and only if*

$$\sum^{\infty} n^{\beta} p(n)=\infty. \tag{19}$$

Theorem 16. *Equation* (8) *has no solutions of type* (S_2).

Example 2. *Let* $\alpha>\beta$ *and consider the Equation* (14) *again.*

We have the following results:

1) Equation (14) has a solution of type (*D*) if and only if $\sigma<-\alpha-1$ (Theorem 10 and 11).
2) Equation (14) has a solution of type (*AC*) if and only if $\sigma<-\alpha-1$ (Theorem 14).
3) Equation (14) has a solution of type (*AL*) if and only if $\sigma<-\beta-1$ (Theorem 15).
4) Equation (14) has a solution of type (*AS*) if and only if $\sigma\ge-\beta-1$ (Theorem 16).

5. Auxillary Lemma

In this section, we collect axillary lemmas, which are mainly concerned with local solution of Equation (8). A comparison lemma of the following type is useful, and will be used in many places.

Lemma 2. *Suppose that* $\{p_p(n)\},\{p_q(n)\}$ *are such that* $0<p_1(n)<p_2(n)$ *for* $a\le n\le b$. *Let* $y_i(n)$, $i=1,2$ *and* $a\le n\le b$ *be solutions of the equations*

$$\Delta\left(\left|\Delta y_i(n)\right|^{\alpha-1}\Delta y_i(n)\right)=p_i(n)\left|y_i(n)\right|^{\beta-1}y_i(n),\ i=1,2$$

respectively. If $y_1(a)\le y_2(a)$ and $\Delta y_1(a)\le\Delta y_2(a)$, then $y_1(n)<y_2(n)$ and $\Delta y_1(n)<\Delta y_2(n)$ for $a<n\le b$.

Proof. We have

$$\left(\Delta y_i(n)\right)^{\alpha*}=\left(\Delta y_i(a)\right)^{\alpha*}+\sum_{a}^{n-1}p_i(s)\left(y_i(s)\right)^{\beta*},\ a\le n\le b, i=1,2 \tag{20}$$

$$y_i(n)=y_i(a)+\sum_{a}^{n-1}\left(\left(\Delta y_i(a)\right)^{\alpha*}+\sum_{a}^{s-1}p_i(r)\left(y_i(r)\right)^{\beta*}\right)^{1/\alpha^*},\ a\le n\le b, i=1,2. \tag{21}$$

By the hypotheses we have $y_1(n)<y_2(n)$ in some right neighborhood of *a*. If $y_1(n)\ge y_2(n)$ for some point in $a<n\le b$, we can find a *c* such that $a<c\le b$ satisfying $y_1(n)<y_2(n)$ for $a<n<c$ and $y_1(c)=y_2(c)$. But, this yields a contradiction, because

$$\begin{aligned}0&=y_1(c)-y_2(c)\\&=y_1(a)-y_2(a)+\sum_{a}^{c-1}\left[\left(\left(\Delta y_1(a)\right)^{\alpha*}+\sum_{a}^{n-1}p_1(s)\left(y_1(s)\right)^{\beta*}\right)^{1/\alpha^*}\right.\\&\quad\left.-\left(\left(\Delta y_2(a)\right)^{\alpha*}+\sum_{a}^{n-1}p_2(s)\left(y_2(s)\right)^{\beta*}\right)^{1/\alpha^*}\right]<0.\end{aligned}$$

Hence we see that $y_1(n)<y_2(n)$ for $a<n\le b$. Returning to (20), we find that $\Delta y_1(n)<\Delta y_2(n)$ for $a\le n\le b$. The proof is complete. □

The uniqueness of local solutions with non-zero initial data can be easily proved. That is, for given $N(\geq n_0)$, y_0 and y_1, Equation (8) has a unique local solution $y(n)$ satisfying $y(N)=y_0$, $\Delta y(N)=y_1$ provided that $|y_0|+|y_1|\neq 0$. The uniqueness of the trivial solution can be concluded for the case $\alpha\leq\beta$.

Lemma 3. *Let* $\alpha\leq\beta$ *and* $N\geq n_0$. *If* $y(n)$ *is a local solution of Equation* (1) *satisfying* $y(N)=\Delta y(N)=0$ *then* $y(n)\equiv 0$ *for* $n_0\leq n<\infty$.

Proof. Assume the contrary. We may suppose that $y(n)\not\equiv 0$ for $N\leq n<\infty$. Then, we can find n_1,n_2 such that $N\leq n_1<n_2$ satisfying $|y(n_1)|+|\Delta y(n_1)|=0$ and $|y(n)|+|\Delta y(n)|>0$ for $n_1<n\leq n_2$. Summing (8), we obtain

$$\Delta y(n)=\left(\sum_{n_1}^{n-1}p(s)\left(y(s)\right)^{\beta_*}\right)^{1/\alpha^*},$$

$$y(n)=\sum_{n_1}^{n-1}\left(\sum_{n_1}^{s-1}p(r)\left(y(r)\right)^{\beta_*}\right)^{1/\alpha^*},\quad n_1\leq n\leq n_2.$$

We therefore have

$$|\Delta y(n)|\leq\left(\sum_{n_1}^{n-1}p(s)\left\{|y(s)|+|\Delta y(s)|\right\}^{\beta}\right)^{1/\alpha}\tag{22}$$

$$|y(n)|\leq\sum_{n_1}^{n-1}\left(\sum_{n_1}^{s-1}p(r)\left\{|y(r)|+|\Delta y(r)|\right\}^{\beta}\right)^{1/\alpha},\quad n_1\leq n\leq n_2.\tag{23}$$

Put $w(n)=\max_{n_1\leq\xi\leq n}\left(|y(\xi)|+|\Delta y(\xi)|\right)$. We see that $w(n_1)=0, w(n)>0$ for $n_1<n\leq n_2$ and w is nondecreasing. From (22) and (23), we can get

$$|\Delta y(n)|\leq\left(w(n)\right)^{\beta/\alpha}\left(\sum_{n_1}^{n-1}p(s)\right)^{1/\alpha},$$

$$|y(n)|\leq\left(w(n)\right)^{\beta/\alpha}\sum_{n_1}^{n-1}\left(\sum_{n_1}^{s-1}p(r)\right)^{1/\alpha},\quad n_1\leq n\leq n_2.$$

Let $n_1\leq\tau\leq n\leq n_2$. Then from this observation we see that

$$|\Delta y(\tau)|+|y(\tau)|\leq\left(w(\tau)\right)^{\beta/\alpha}G(\tau)\leq\left(w(n)\right)^{\beta/\alpha}G(n),$$

where

$$G(v)=\left(\sum_{n_1}^{v-1}p(s)\right)^{1/\alpha}+\sum_{n_1}^{v-1}\left(\sum_{n_1}^{s-1}p(r)\right)^{1/\alpha}.$$

Consequently, we have

$$w(n)\leq\left(w(n)\right)^{\beta/\alpha}G(n),\quad n_1\leq n\leq n_2.\tag{24}$$

If $\alpha=\beta$, from (24), we have $1\leq G(n)$, $n_1<n\leq n_2$. This is a contradiction because $G(n_1)=0$. If $\alpha<\beta$, from (24) we have $\left(w(n)\right)^{-\frac{\beta-\alpha}{\alpha}}\leq G(n)$, $n_1<n\leq n_2$. This is also a contraction because $G(n_1+0)=w(n_1+0)=0$. The proof is complete.

Lemma 4. *Let* $\alpha\geq\beta$. *Then all local solutions of Equation* (8) *can be continued to* ∞ *and* n_0, *that is, all solutions of Equation* (8) *exist on the whole interval* $[n_0,\infty)$.

Proof. Let $y(n)$ be a local solution of Equation (8) is a neighborhood of $N\geq n_0$. Suppose the contrary that the right maximal interval of existence of $y(n)$ is of the form $[N,w)$, $w<\infty$. Then, it is easily seen that $y(w-0)=\pm\infty$. Summing (8) twice, we have

$$y(n) = c_0 + \sum_N^{n-1}\left(c_1^{\alpha_*} + \sum_N^{s-1} p(r)\left(y(r)\right)^{\beta_*}\right)^{1/\alpha^*}$$

where $c_0 = y(N)$ and $c_1 = \Delta y(N)$. Accordingly,

$$|y(n)| \le |c_0| + \sum_N^{n-1}\left(|c_1|^\alpha + \sum_N^{s-1} p(r)|y(r)|^\beta\right)^{1/\alpha}, \quad N \le n < w.$$

Put $z(n) = \max_{N\le\xi\le n}|y(\xi)|$. Then,

$$|y(n)| \le |c_0| + \sum_N^{n-1}\left(|c_1|^\alpha + |z(s)|^\beta \sum_N^{s-1} p(r)\right)^{1/\alpha}, \quad N \le n < w.$$

Put moreover $u(n) = \max\left\{|c_1|^{\alpha/\beta}, z(n)\right\}$. Then, as in the proof of Lemma 3, we have

$$|z(n)| \le |c_0| + \sum_N^{n-1} H(s)\left(u(s)\right)^{\beta/\alpha}, \quad N \le n < w \tag{25}$$

where $H(n) = \left(1 + \sum_N^{n-1} p(s)\right)^{1/\alpha}$. Since $y(w-0) = \pm\infty$, there is a $\bar{N}$ such that $N < \bar{N} < w$ such that $z(n) \ge |c_1|^{\alpha/\beta}$ for $\bar{N} \le n < w$. Therefore it follows from (25) that

$$u(n) \le |c_0| + \sum_N^{n-1} H(s)\left(u(s)\right)^{\beta/\alpha}, \quad \bar{N} \le n < w. \tag{26}$$

Let $\alpha = \beta$. Then, using discrete Gronwall's inequality, we see that $u(w-0) < \infty$, which is a contradiction. Next let $\alpha > \beta$. Then (26) implies that

$$u(n) \le |c_0| + \left(u(n)\right)^{\beta/\alpha} \sum_N^{n-1} H(s), \quad \bar{N} \le n < w.$$

Since $\beta/\alpha < 1$, we have $u(w-0) < \infty$. This is a contradiction too. Hence $y(n)$ can be continued to ∞. The continuability to the left end point n_0 is verified in a similar way. The proof is complete. □

The following lemma establishes more than is stated in Theorem 8. Accordingly the proof of Theorem 8 will be omitted.

Lemma 5. *Let* $\alpha < \beta$ *and* $N \ge n_0$ *and* $c > 0$ *be given. Then there exists an* $M = M(N, c) > 0$ *such that the right maximal interval of existence of each solution* $y(n)$ *of Equation* (1) *satisfying* $y(N) \ge c$ *and* $\Delta y(N) \ge M$ *is a finite interval* $[N, \bar{N})$, $\bar{N} = \bar{N}_{y(n)} < \infty$, *and* $\lim_{n\to\bar{N}-0} y(n) = \infty$.

Proof. Let $n_1 > N$ be fixed, and put $\min_{N\le n\le n_1} p(n) = m > 0$. There is an $M > 0$ satisfying

$$\sum_c^\infty \left(M^\alpha + m\left(\left(v(n)\right)^{\beta+1} - c^{\beta+1}\right)\right)^{-\frac{1}{\alpha}} < n_1 - N.$$

We first claim that the solution of Equation (8) with the initial condition $z(n) = c$, $\Delta z(n) = M$ does not exist on $[N, n_1)$; that is $z(n)$ blow up at some $\bar{N} \in (N, n_1]$. To see this suppose the contrary that $z(n)$ exists at least $[N, n_1)$. By the definition of *m*, we have

$$\Delta\left(\Delta z(n)\right)^\alpha = p(n)\left(z(n)\right)^\beta \ge m\left(z(n)\right)^\beta, \quad N \le n \le n_1.$$

Summing the inequality form *N* to $n-1$ yields

$$\left(\Delta z(n)\right)^\alpha - M^\alpha \ge m\left(\left(z(n+1)\right)^\beta - C^\beta\right), \quad N \le n \le n_1$$

and hence

$$\left(\Delta z(n)\right)^{\alpha} \geq M^{\alpha} + m\left(z(n)^{\beta} - C^{\beta}\right)$$

$$\Delta z(n) \geq \left(M^{\alpha} + m\left(\left(z(n)\right)^{\beta} - C^{\beta}\right)\right)^{\frac{1}{\alpha}}$$

$$\Delta z(n)\left[M^{\alpha} + m\left(\left(z(n)\right)^{\beta} - C^{\beta}\right)\right]^{-\frac{1}{\alpha}} \geq 1, \quad N \leq n \leq n_1.$$

Finally, summing the above inequality both sides from N to $n_1 - 1$, we obtain

$$\sum_{n}^{z(n_1)}\left[M^{\alpha} + m\left(\left(w(n)\right)^{\beta} - C^{\beta}\right)\right]^{-\frac{1}{\alpha}} \geq n_1 - N,$$

which is a contradiction to the choice of M. Hence $z(n)$ must blow up at some $\bar{N} \in (N, n_1]$, $\lim_{n\to\bar{N}-0} z(n) = \infty$.

If $y(N) \geq c$ and $\Delta y(N) \geq M$, then Lemma 2 implies that $y(n) \geq z(n)$ on the common interval of existence of y and z and therefore $y(n)$ blows up at some point before n_1. The proof is complete. □

6. Nonnegative Nonincreasing Solutions

The main objective of this section is to prove the following theorem.

Theorem 17. *For each* $y_0 > 0$, *the problem*

$$\begin{cases} \Delta\left(\left|\Delta y(n)\right|^{\alpha-1}\Delta y(n)\right) = p(n)\left|y(n)\right|^{\beta-1} y(n) \\ y(n_0) = y_0 \end{cases}$$

has exactly one solution $\bar{y}$ such that $\bar{y}$ is defined for $n \geq n_0$ and satisfies

$$\bar{y}(n) \geq 0, \quad \Delta y(n) \leq 0 \quad \text{for } n \geq n_0. \tag{27}$$

Furthermore, if $y(n)$ is a solution for $n \geq n_0$ of Equation (1) satisfying $y(n_0) = y_0$ and

$$\Delta y(n_0) > \Delta\bar{y}(n_0) \ \left[resp \ \ \Delta y(n_0) < \Delta\bar{y}(n_0)\right],$$

then

$$\lim_{n\to\infty} y(n) = \infty \ \left[resp \ \ \lim_{n\to\infty} y(n) = -\infty\right].$$

Remark 4.

1) In the case $\alpha \leq \beta$, employing Lemma 3, we can strengthen (27) to the property that

$$\Delta y(n) > 0, \quad \Delta\bar{y}(n) < 0 \quad \text{for } n \geq n_0. \tag{28}$$

2) In the case $\alpha \geq \beta$, all local solutions of Equation (8) can be continued to the whole interval $[n_0, \infty)$ Hence in this case property (6.2) always holds for all solutions $y(n)$ with $y(n_0) = y_0$ and $\Delta y(n_0) > \Delta\bar{y}(n_0)$ [resp $\Delta y(n_0) < \Delta\bar{y}(n_0)$].

The property of nonnegative nonincreasing solutions $\bar{y}$ described in Theorem 17 will play important roles through the paper. This section is entirely derided to proving Theorem 17. To this end we prepare several lemmas.

Lemma 6. *Let* $A, B \in R$ *and t be a bounded function on* $[a,b] \times R$. *Then, the two point boundary value problem*

$$\begin{cases} \Delta\left(\left|\Delta y(n)\right|^{\alpha-1}\Delta y(n)\right) = f\left(n, y(n)\right), \quad a \leq n \leq b, \\ y(a) = A, \ \ y(b) = B, \end{cases} \tag{29}$$

has a solution.

Proof. Let $K > 0$ be a constant such that

$$\left|f\left(n,y(n)\right)\right|\le K,\quad \text{for } \left(n,y(n)\right)\in[a,b]\times R.$$

We first claim that with each $y(n)$, we can associate a unique constant $c(y)$ satisfying

$$\sum_a^b\left(c(y)+\sum_a^s f\left(r,y(r)\right)\right)^{1/\alpha^*}=B-A. \tag{30}$$

Further this $c(y)$ satisfies

$$-K(b-a)+\left(\frac{B-A}{b-a}\right)^{\alpha^*}\le c(y)\le K(b-a)+\left(\frac{B-A}{b-a}\right)^{\alpha^*}. \tag{31}$$

To see this let $y(n)$ be fixed, and consider the function

$$I(\lambda)=\sum_a^b\left(\lambda+\sum_a^s f\left(r,y(r)\right)\right)^{1/\alpha^*},\quad \lambda\in R.$$

If $\lambda<-K(b-a)+\left(\frac{B-A}{b-a}\right)^{\alpha_*}$, then $I(\lambda)<B-A$. If $\lambda>K(b-a)+\left(\frac{B-A}{b-a}\right)^{\alpha_*}$, then $I(\lambda)>B-A$. Since I is a strictly increasing continuous function, there is a unique constant $c(y)$ satisfying $I\left(c(y)\right)=B-A$, namely (30). Then (31) is clearly satisfied.

By (31), we see that there is a constant $M=M(a,b,A,B,K)>0$ satisfying $\left|c(y)\right|\le M$ for all $y(n)$. Choose $L>0$ so large that

$$|A|\le L \quad\text{and}\quad M+K(b-a)^{1/\alpha}(b-a)\le L.$$

Consider the Banach space B_N of all real sequences $y=\{y(n)\}_{n\ge N}$ with the supernum norm $\|y\|=\sup_{n\ge N}|y(n)|$.

Now we define the set $y\subset B_N$ and the mapping $F:Y\to B_N$ by

$$Y=\left[y(n)\in B_{N_0}:|y(n)|\le 2L \text{ for } a\le n\le b\right]$$

and

$$F\left(y(n)\right)=A+\sum_a^n\left(c(y)+\sum_a^s f\left(r,y(r)\right)\right)^{1/\alpha^*},\quad a\le n\le b$$

respectively. Then the boundary value problem (29) is equivalent to finding a fixed element of $\mathcal{F}$. We show that F has a fixed element in Y (via) the Schavder fixed point theorem

$$\begin{aligned}\left|F\left(y(n)\right)\right|&\le|A|+\sum_a^n\left(|c(y)|+\sum_a^s\left|f\left(r,y(r)\right)\right|\right)^{1/\alpha}\\&\le|A|+\sum_a^n M+K(s-a)^{1/\alpha}\\&\le|A|+M(b-a)^{1/\alpha}(b-a)\\&\le L+L=2L,\quad a\le n\le b.\end{aligned}$$

Hence F maps Y into itself.

Next, to see the continuity of F, assume that $y_k(n)$ be a sequence converging to $y\in Y$ uniformly in $[a,b]$. We must prove that $F\left(y_k(n)\right)$ converges to $Fy_k(n)$ uniformly in $[a,b]$. As a first step, we show that $\lim_{n\to\infty}c\left(y_k(n)\right)=c\left(y(n)\right)$. Assume that this is not the case. Then because of the boundedness of $\left\{c\left(y_k(n)\right)\right\}$, there is a subsequence $\left\{c\left(y_{k_i}(n)\right)\right\}$ satisfying $cy_{k_i}(n)\to\bar{c}\ne c(y)$ for some finite value $\bar{c}$. Noting the relation

$$\sum_a^b\left(c\left(y_{k_i}(n)\right)+\sum_a^s f\left(r,y_{k_i}(r)\right)\right)^{1/\alpha_*}=B-A.$$

We have

$$B - A = \lim_{K_i \to \infty} \sum_a^b \left(c\left(y_{k_i}(n)\right) + \sum_a^s f\left(r, y_{k_i}(r)\right) \right)$$
$$= \sum_a^b \left(\overline{c} + \sum_a^b f\left(r, y(r)\right) \right)^{1/\alpha^*}.$$

This contradicts the uniqueness of the number $c(y)$. Hence $\lim_{n\to\infty} c(y(n)) = c(y)$. Then we find similarly that $\lim_{n\to\infty} F(y_k(n)) = F(y(n))$ uniformly on $[a,b]$.

It will be easily seen that the sets

$$FY = \{Fy : y \in Y\} \text{ and } \{\Lambda(F(y)) : y \in Y\}$$

are uniformly bounded on $[a,b]$. Then $F\overline{Y}$ is compact.

From the above observations we see that *F* has a fixed element in *Y*. Then this fixed element is a solution of boundary value problem (29) is easily proved. The proof is now complete.

Lemma 7. *Let* $n_1 > n_0$ *and* $n_0 > 0$. *Then the two point boundary value problem*

$$\begin{cases} \Delta\left(\left|\Delta y(n)\right|^{\alpha-1} \Delta y(n)\right) = p(n)\left|y(n)\right|^{\beta-1} y(n) & \text{for } n_0 \le n \le n_1 \\ y(n_0) = y_0, \quad y(n_1) = 0 \end{cases} \tag{32}$$

has a solution $y(n)$ such that $y(n) \ge 0$ and $\Delta y(n) \le 0$ for $n_0 \le n \le n_1$.

Proof. Define the bounded function *f* on $[n_0, n_1] \times R$ by

$$f(n, y(n)) = \begin{cases} p(n) y_0^\beta & \text{for } n_0 \le n \le n_1, \ y \ge y_0 \\ p(n) y^\beta & \text{for } n_0 \le n \le n_1, \ 0 \le y \le y_0 \\ 0 & \text{for } n_0 \le n \le n_1, \ y \le 0. \end{cases}$$

By Lemma 6, the boundary value problem

$$\begin{cases} \Delta\left(\left|\Delta y(n)\right|^{\alpha-1} \Delta y(n)\right) = f(n, y(n)), & n_0 \le n \le n_1 \\ y(n_0) = y_0, \quad y(n_1) = 0 \end{cases}$$

has a solution *y*.

We show that *y* satisfies $y(n) \ge 0$ for $n_0 \le n \le n_1$. If this is not the case, we can find an interval $[\tau_0, \tau_1] \subset [n_0, n_1]$ such that $y(n) < 0$ on $[\tau_0, \tau_1]$ and $y(\tau_0) = y(\tau_1) = 0$. The definition of *f* implies that *y* satisfies the equation $\Delta\left(\left|\Delta y(n)\right|^{\alpha-1} \Delta y(n)\right) = 0$ on $\tau_0 \le n \le \tau_1$. Hence $y(n)$ is a linear function on $[\tau_0, \tau_1]$. Obviously that this is a contradiction. We see therefore that $y(n) \ge 0$ on $[n_0, n_1]$.

Since $\Delta y(n_1) \le 0$ and $\Delta\left(\left|\Delta y(n)\right|^{\alpha-1} \Delta y(n)\right) \ge 0$ on $[n_0, n_1]$, by the definition of *t*, we find that $\Delta y(n) \le 0$ on $[n_0, n_1]$. Hence $y(n) \le y_0$, which implies that *y* is a desired solution of problem (32). The proof is complete. □

Proof of Theorem 17. The uniqueness of $\overline{y}$ satisfying the properties mentored here is easily established as in the proof Lemma 2. Therefore we prove only the existence of such a $\overline{y}$.

By Lemma 7, for each $k \in N$, we have a solution $y = y_0$ of the boundary value problem

$$\begin{cases} \Delta\left(\left|\Delta y(n)\right|^{\alpha-1} \Delta y(n)\right) = p(n)\left|y(n)\right|^{\beta-1} y(n) \\ y(n_0) = y_0, \quad y(n_0 + k) = 0, \ \text{for } n_0 \le n \le n_0 + k \end{cases}$$

satisfying $y_k(n) \ge 0$ and $\Delta y_k(n) \le 0$ for $n_0 \le n \le n_0 + k$ let us extend each y_k over the interval $[n_0, \infty)$

by defining $y_k \equiv 0$ for $n \ge n_0 + k$. Below we show that $\{y_k(n)\}$ contains a subsequence converging to a desired solution of (8).

As a first step, we prove that

$$\Delta y_1(n_0) \le \Delta y_2(n_0) \le \cdots \le y_k(n_0) \le 0. \tag{33}$$

In fact, if this is not case, then $\Delta y_i(n_0) > \Delta y_{i+1}(n_0)$ for some *i*. Since $y_i(n_0) = y_{i+1}(n_0)$. Lemma 2 implies that $y_i(n) > y_{i+1}(n)$ for $n_0 \le n \le n_0 + i$. Putting $n = n_0 + i$, we have $0 = y_i(n_0 + i) > y_{i+1}(n_0 + i) \ge 0$ a contradiction. Accordingly (33) holds, and so $\lim_{n\to\infty} \Delta y_k(n_0) = l \in (-\infty, 0]$ exists, since $0 \le y_k(n) \le y_0$ on $[n_0, n_0 + k]$ for any $k \in N$, $\{y_k(n)\}$ is uniformly bounded on each compact subinterval of $[n_0, \infty)$. Noting that $\Delta y_k(n)$ is nondecreasing and nonpositive on $[n_0, n_0 + k]$, we have

$$\Delta y_1(n_0) \le \Delta y_k(n_0) \le \Delta y_k(n) \le 0 \quad \text{on } [n_0, n_0 + k], k \in N.$$

Hence $\{y_k(n)\}$ is equicontinuous on each compact subinterval of $[n_0, \infty)$. From these consideration we find that there is a subsequence $\{y_{k_i}(n)\} \subset \{y_k(n)\}$ and a function $\bar{y}$ satisfying $\lim_{k_i\to\infty}(y_{k_i}(n)) = \bar{y}(n)$ uniformly on each compact subinterval of $[n_0, \infty)$. Finally we shall show that $\bar{y}$ is a desired solution of Equation (8). Let $n \in [n_0, \infty)$ be fixed arbitrarily. For all sufficiently large k_i's we have

$$y_{k_i}(n) = y_0 + \sum_{n_0}^{n-1}\left(\left|\Delta y_{k_i}(n_0)\right|^{\alpha *} + \sum_{n_0}^{s} p(r)\left|y_{k_i}(r)\right|^{\beta}\right)^{1/\alpha^*}$$

letting $k_i \to \infty$, we obtain

$$\bar{y}(n) = y_0 + \sum_{n_0}^{n-1}\left(l^{\alpha *} + \sum_{n_0}^{s} p(r)\left[\bar{y}(r)\right]^{\beta}\right)^{1/\alpha^*}.$$

Taking difference in this above equality, we are that $\bar{y}$ solves Equation (8) on $[n_0, \infty)$. That $\bar{y}$ satisfies (27) is evident. The proof of Theorem 17 is complete.

7. Proofs of Main Results for the Super-Homogeneous Equations

Throughout this section, we assume that $\alpha < \beta$.

Proof of Theorem 2. The theorem is an immediate consequence of the uniqueness of the trivial solution (Lemma 3).

Proof of Theorem 4. Necessity Part: Let $y(n)$ be a positive solution of Equation (8) for $n \ge n_1$ of type (*AC*). It is easy to see that $\Delta y(n) \uparrow 0$ and $y(n) \downarrow y(\infty) > 0$ as $n \uparrow \infty$. Hence summing (8) twice, we have

$$-y(\infty) + y(n_1) = \sum_{n_1}^{\infty}\left(\sum_{s}^{\infty} p(r)\left|y(r)\right|^{\beta}\right)^{1/\alpha}$$

from which we find that

$$\left(y(\infty)\right)^{\beta/\alpha} \sum_{n_1}^{\infty}\left(\sum_{s}^{\infty} p(r)\right)^{1/\alpha} < \infty.$$

This is equivalent to (10).

Sufficiency Part: Let (10) hold. Fix an $l > 0$ and choose $n_1 \ge n_0$ so that

$$\sum_{n_1}^{\infty}\left(\sum_{s}^{\infty} p(r)\right)^{1/\alpha} \le \frac{(2l)^{\frac{\alpha-\beta}{\alpha}}}{2}.$$

We introduce the Banach space l^{∞} of all bounded, real sequences $\{y(n)\}$ with norm $\|y\| = \sup_n |y(n)|$. Define the set $Y = \{y(n) \in l^{\infty} : l \le y(n) \le 2l, n \ge n_1\}$ and the mapping $F : Y - l^{\infty}$ by

$$Fy(n) = l + \sum_{n}^{\infty}\left(\sum_{s}^{\infty} p(r)|y(r)|^{\beta}\right)^{1/\alpha}, \ n \ge n_1.$$

We below show via the Schauder-Tychonoff fixed point theorem that *F* has at least one fixed element in *Y*. Firstly, let $y(n) \in Y$. Then

$$l \le Fy(n) \le l + (2l)^{\beta/\alpha} \sum_{n_1}^{\infty}\left(\sum_{s}^{\infty} p(r)\right)^{1/\alpha} \le l + l = 2l, \ n \ge n_1.$$

Thus $Fy \in Y$, and hence $FY \subset Y$. Secondly, to see the continuity of *F*, let $\{y_k(n)\}$ be a sequence in *Y* covering to $y(n) \in Y$ uniformly on each compact subinterval of $[n_1, \infty)$ since $p(n)$ is bounded for $n_1 \le n \le \infty$ and

$$0 \le \sum_{s}^{\infty} p(r)[y_k(r)]^{\beta} \le (2l)^{\beta} \sum_{s}^{\infty} p(r) < \infty.$$

The Lebesgue dominated convergence theorem implies that $Fy_k(n) \to Fy(n)$ uniformly on each compact subinterval of $[n_1, \infty)$ since for $y(n) \in Y$,

$$\left|\Delta(Fy(n))\right| \le \left(\sum_{n}^{\infty} p(s)[y(s)]^{\beta}\right)^{1/\alpha} \le (2l)^{\beta/\alpha}\left(\sum_{n_1}^{\infty} p(s)\right)^{1/\alpha}, \ n \ge n_1.$$

The set $\{\Delta(Fy(n)) : y \in Y\}$ is uniformly bounded on $[n_1, \infty)$. This implies that $\overline{F}_Y$ is compact.

From there observations we find that *F* has a proved element *y* in *Y* such that $Fy = y$. That this *y* is a solution of Equation (1) of type (*AC*) is easily proved. The proof is complete.

Proof of Theorem 3. Sufficiency Part: Let $\overline{y}(n)$ be a solution of Equation (8) satisfying $\overline{y}(n) > 0$, $\Delta\overline{y}(n) < 0$ for $n \ge n_0$. The existence of such a solution is ensured by Theorem 17. Obviously, $\overline{y}(n)$ is either of type (*D*) or type (*AC*). Theorem 4 shows that under assumption (9), Equation (8) does not posses solutions of type (*AC*). Hence $\overline{y}(n)$ must be of type (*D*).

Necessity Part: Let $y(n)$ be a positive solution of Equation (8) for $n \ge n_1$ of type (*D*). Clearly $y(n)$ satisfies

$$y(n) = \sum_{n}^{\infty}\left(\sum_{s}^{\infty} p(r)|y(r)|^{\beta}\right)^{1/\alpha}, \ n \ge n_1.$$

To verify (9), suppose the contrary that (9) fails to hold. Then, nothing that $y(n)$ is decreasing for $n \ge n_1$, we have

$$y(n) \le |y(n)|^{\beta/\alpha} \sum_{n}^{\infty}\left(\sum_{s}^{\infty} p(r)\right)^{1/\alpha}, \ n \ge n_1.$$

Accordingly,

$$|y(n)|^{1-\beta/\alpha} \le \sum_{n}^{\infty}\left(\sum_{s}^{\infty} p(r)\right)^{1/\alpha}, \ n \ge n_1.$$

The left hand side tends to ∞ as $n \to \infty$ because of $\alpha < \beta$, where as the right hand side tends to 0 as $n \to \infty$. This contradiction verifies (9). The proof is complete.

Proof of Theorem 5. Necessity Part: Let $y(n)$ be a positive solution of Equation (8) near ∞ of type (*AL*). There is a constant $c > 0$ and $n_1 \ge n_0$ satisfying

$$y(n) \ge cn, \quad n \ge n_1. \tag{34}$$

Summation of Equation (8) from n_1 to $n-1$ yields

$$\left[\Delta y(n)\right]^{\alpha *} - \left[\Delta y(n_1)\right]^{\alpha *} = \sum_{n_1}^{n-1} p(s)(y(s))^{\beta}, \quad n \ge n_1.$$

Since $\lim_{n\to\infty}\Delta y(n)=\Delta y(\infty)\in(0,\infty)$, this in equality implies that

$$\sum_{n_1}^{\infty}p(n)\left[y(n)\right]^{\beta}<\infty. \tag{35}$$

Combining (35) with (34), we find that (11) holds.

Sufficiency Part: We fix $l>0$ arbitrarily, and choose $n_1\geq n_0$ large enough so that

$$\sum_{n_1}^{\infty}n^{\beta}p(n)\leq(2l)^{\alpha-\beta}\left(1-\frac{1}{2^{\alpha}}\right).$$

Let l^{α} be the Banach space as in the proof Theorem 4. Define the set $Y\in l^{\infty}$ as follows

$$Y=\left\{y(n)\in l^{\infty}:l(n-n_1)\leq y(n)\leq 2l(n-n_1),n\geq n_1\right\}.$$

The mapping $F:Y\to l^{\infty}$ defined by

$$Fy(n)=\sum_{n_1}^{n-1}\left((2l)^{\alpha}-\sum_{s}^{\infty}p(r)\left[y(r)\right]^{\beta}\right)^{1/\alpha},\quad n\geq n_1.$$

As in the proof of the sufficiency part of Theorem 4, we can show that F has a fixed element $y(n)\in Y$ by the Schavder-Tyehonoff fixed point Theorem

$$y(n)=\sum_{n_1}^{n-1}\left((2l)^{\alpha}-\sum_{s}^{\infty}p(r)\left[y(r)\right]^{\beta}\right)^{1/\alpha},\quad n\geq n_1.$$

Taking Δ twice for this formula we see that $y(n)$ is a positive solution of Equation (8) for $n\geq n_1$. L'Hospital's rule shows that $\lim_{n\to\infty}\frac{y(n)}{n}=2l$. Thus $y(n)$ is a solution of Equation (8) of type (*AL*). The proof is complete.

Lemma 8. *Let* $y(0)>0$. *If* (11) *holds, then there is a positive solution of Equation* (8) *for* $n\geq n_0$ *of type* (*AL*) *satisfying* $y(n_0)=y(0)$.

Proof. By Theorem 5, there is an (*AL*)-type positive solution $z(n)$ of Equation (8) defined in some neighborhood of $\infty:0<\lim_{n\to\infty}\frac{z(n)}{n}=\lim_{n\to\infty}\Delta z(n)<\infty$. Let $\overline{y}(n)$ be a positive solution of Equation (8) for $n\geq n_0$ satisfying $\overline{y}(n_0)=y(0)$ and $\overline{y}(n)>0$, $\Delta\overline{y}(n)<0$ for $n\geq n_0$. Take a $n_1>n_0$ such that $\overline{y}(n)<z(n)$ and $\Delta\overline{y}(n)<\Delta z(n)$ for $n\geq n_1$. By Lemma 2 if $\lambda>\Delta\overline{y}(n_0)$ is sufficiently elver to $\Delta\overline{y}(n_0)$, then the solution $y(n)$ of Equation (8) with $y(n_0)=y(0)$ and $\Delta y(n_0)=\lambda$ exists at least on $[n_0,n_1]$ and satisfies

$$\overline{y}(n_1)<y(n_1)<z(n_1),\quad \Delta\overline{y}(n_1)<\Delta y(n_1)<\Delta z(n_1).$$

Then Lemma 2 again implies that $\overline{y}(n)<y(n)<z(n)$ as long as $y(n)$ exists. Since $\overline{y}(n)$ and $z(n)$ exists for $n\geq n_1$, this means that $y(n)$ exists for $n\geq n_1$ and satisfies $\overline{y}(n)<y(n)<z(n)$, $n\geq n_1$. Then we have

$$\frac{\overline{y}(n)}{n}<\frac{y(n)}{n}<\frac{z(n)}{n},\quad n\geq n_1.$$

Noting that $\overline{y}(n)$ is the unique solution of (8) satisfying $\lim_{n\to\infty}\frac{\overline{y}(n)}{n}=0$ and passing through the point $(n_0,y(0))$ we have $\lim\lim_{n\to\infty}\frac{y(n)}{n}\in(0,\infty)$. Therefore $y(n)$ is of type (*AL*). The proof is complete.

Proof of Theorem 6. For $\lambda>0$, we denote by $y_{\lambda}(n)$, the unique solution of Equation (8) with in initial condition $y(n_0)=y(0)$ and $\Delta y(n_0)=\lambda$. The maximal interval of existence of $y_k(n)$ may be finite or infinite.

Define the set $S\subset(0,\infty)$ by

$$S = \{\lambda > 0 : y_\lambda(n) \text{ exists for } n \ge n_0 \text{ and is of type } (AL)\}$$

We know by Lemma 8 that $S \ne Q$ and by Lemma 5 that $\lambda \notin S$ for all sufficiently large $\lambda > 0$. Hence $\sup S = \bar{\lambda} \in (0,\infty)$ exists. For $\bar{\lambda}$ there are three possibilities:

1) $\bar{\lambda} \in S$
2) $\bar{\lambda} \notin S$ and $y_{\bar{\lambda}}(n)$ is of type (*AS*)
3) $\bar{\lambda} \notin S$ and $\lambda_{\bar{\lambda}}(n)$ is of type (S_2).

To prove the theorem, we below show that case (b) occurs. For simplicity, we write $\bar{y}$ for $y_{\bar{\lambda}}$ below. Suppose that the case (a) occurs. Then $\lim_{n\to\infty} \Delta\bar{y}(n) = \Delta\bar{y}(\infty) = l \in (0,\infty)$ and $\Delta\bar{y}(n) < l, n \ge n_0$. By condition (11) we can find a $n_1 > n_0$ satisfying

$$\sum_{n_0}^{\infty} p(s)(y_0 + 2ls)^\beta < (2l)^\alpha - l^\alpha$$

Choose $\lambda > \bar{\lambda}$ close enough to $\bar{\lambda}$ so that $y_\lambda(n)$ exists at least on $[n_0, n_1]$ and $\Delta y_\lambda(n_1) < l$. Then, for such a λ, $y_\lambda(n)$ can be extended to ∞, and satisfies $\Delta y_\lambda(n) < 2l, n \ge n_1$. In fact, if this is not the case, there is $\bar{n} > n_1$ satisfying $\Delta y_\lambda(n) < 2l$ for $n_0 \le n < \bar{n}$ and $\Delta y_\lambda(\bar{n}) = 2l$. It follows therefore that $y_\lambda(n) \le y(0) + 2lt$ for $n_0 \le n \le \bar{n}$. Summing the Equation (8) (with $y = y_\lambda$) for $n_1 \le n \le \bar{n}$ yields

$$(2l)^\alpha = (\Delta y_\lambda(\bar{n}))^\alpha = (\Delta y_\lambda(n_1))^\alpha + \sum_{n_1}^{\bar{n}-1} p(s)\left|y_{\lambda(s)}\right|^\beta$$
$$\le l^\alpha + \sum_{n_1}^{\bar{n}-1} p(s)(y(0) + 2ls)^\beta \le l^\alpha + \sum_{n_1}^{\infty} p(s)(y(0) + 2ls)^\beta < (2l)^\alpha.$$

This contradiction implies that $y_\lambda(n)$ exists for $n \ge n_0$ and satisfies $\Delta y_\lambda(n) < 2l, n \ge n_0$. These observations show that $S \ni \lambda > \bar{\lambda}$, which contradicts the definition of $\bar{\lambda}$. Hence case (a) does not occur.

Next, suppose that case (c) occurs. Let $N > n_0$ be the point such that $\bar{y}(N-0) = \Delta\bar{y}(N-0) = \infty$. By Lemma 5, there is an $M > 0$ such that solution $y(n)$ of Equation (8) satisfying $y(N) \ge 1$, $\Delta y(N) \ge M$ must blow up at some finite $\bar{N} = \bar{N}(y) \in (N,\infty) : y(\bar{N}-0) = \Delta\bar{y}(\bar{N}-0) = \infty$. For sufficiently small $\epsilon > 0$, we have $\bar{y}(N-\epsilon) > 1$ $\Delta\bar{y}(N-\epsilon) > M$. Then if $\lambda < \bar{\lambda}$ is sufficiently close to $\bar{\lambda}$, then $y_\lambda(n)$ can be continued at least to $N-\epsilon$, and satisfies $y_\lambda(N-\epsilon) > 1$, $\Delta y_\lambda(N-\epsilon) > M$. Then, even through y_λ can be continued to *N*, $y_\lambda(n)$ blows up at some finite point by the definition of *M*. This fact shows that such a $\lambda(<\bar{\lambda})$ does not belong to *S*, contradicting the definition of $\bar{\lambda}$, again. Consequently case (b) occurs, and hence the proof of Theorem 6 is complete.

Proof of Theorem 7. The proof is done by contradiction. Let $y(n)$ be a solution of Equation (1) of type (AS). We suppose that $y(n)$ exists for $n \ge n_1$ and satisfies

$$y(n) \ge C_1 n,\ \Delta y(n) \ge C_1,\ n \ge n_1 \quad \text{for some } C_1 > 0 \tag{36}$$

Put $z(n) = y(n)(\Delta y(n))^\alpha\ (>0)\quad n \ge n_1$. Then

$$\begin{aligned}
\Delta z(n) &= y_{n+1}(\Delta y(n))^\alpha + y(n)\Delta(\Delta y(n))^\alpha \\
&= y_{n+1}(\Delta y(n))^\alpha + (y(n))^{\beta+1} p(n) \\
&= (\Delta y(n))^\alpha \left[y_{n+1} + p(n)\frac{y_n^{\beta+1}}{(\Delta y(n))^\alpha} \right] \\
&= \frac{(\Delta y(n))^\alpha}{y(n)} \left[\frac{y(n+1)}{y(n)} + p(n)\frac{(y(n))^\beta}{(\Delta y(n))^\alpha} \right] \\
&\ge z(n)\left[\frac{\Delta y}{y} + p(n)\frac{y^\beta}{(\Delta y)^\alpha} \right].
\end{aligned}$$

Now, we employ the Young inequality of the form

$$X+Y\ge \sigma^{-\sigma}(1-\sigma)^{-(1-\sigma)}X^{1-\sigma}Y^{\sigma} \quad \text{for } X,Y>0 \text{ and } 0<\sigma<1 \tag{37}$$

in the last inequality. It follows therefore that

$$\Delta z(n)\ge C_2 z(n)(\Delta y(n))^{1-\sigma-\sigma\alpha}(y(n))^{\beta\sigma+\sigma-1}(p(n))^{\sigma},\ n\ge n_1$$

where $C_2=C_2(\sigma,\alpha,\beta)>0$ is a constant. We rewrite is inequality as

$$\Delta z(n)\ge C_2(y(n))^{\beta\sigma-\sigma-\rho-1}(\Delta y(n))^{1-\sigma-\sigma\alpha-\rho\alpha}(p(n))^{\sigma}(z(n))^{1+\rho},\ n\ge n_1.$$

Noting (7.3) and condition (13), we obtain

$$\Delta z(n)\ge C_3 n^{\beta\sigma+\alpha-\rho-1}(p(n))^{\sigma}(z(n))^{1+\rho},\quad n\ge n_1,$$

where $C_3=C_3(\alpha,\beta,\sigma,\rho,C_2,C_1)>0$ is a constant. Dividing both sides by $(z(n))^{1+\rho}$ and summing from n to ∞, we have

$$\frac{1}{\rho}(z(n))^{-\rho}\ge C_3\sum_{n}^{\infty}(s)^{\beta\sigma+\sigma-\rho-1}[p(s)]^{\sigma},\quad n\ge n_1$$

because $\lim_{n\to\infty} z(n)=\infty$. Consequently, we have

$$\frac{1}{\rho}\left(\frac{n}{y(n)}\right)^{\rho}(\Delta y(n))^{-\alpha\rho}\ge C_3 n^{\rho}\sum_{n}^{\infty}(s)^{\beta\alpha+\sigma-\rho-1}[p(s)]^{\sigma},\quad n\ge n_1.$$

Letting $n\to\infty$, we get a contradiction to assumption (12). This completes the proof.

As was mentioned in Section 5, the proof of Theorem 8 is omitted. In fact, a more general result is proved in Lemma 5.

8. Proofs of Main Results for the Sub-Homogeneous Equations

Throughout this section, we assume that $\alpha>\beta$.

Proof of Theorem 9. Let n_1,n_2 be fixed so that $n_0\le n_1\le n_2$ and put

$$m=\min_{n_1\le n\le n_2} p(n)>0 \quad\text{and}\quad \rho=\frac{\alpha+1}{\alpha-\beta}>0.$$

Then there are constants $L>0$ and $c>0$ satisfying

$$L^{\beta/\alpha}\sum_{n_1}^{n_2-1}\left(\sum_{s}^{n_2-1}p(r)\right)^{1/\alpha}\le L,$$

$$\frac{C^{\beta/\alpha}m^{1/\alpha}}{(\rho\beta+1)^{1/\alpha}}\cdot\frac{\alpha}{\alpha+\rho\beta+1}\ge c$$

and

$$c(n_2-n)^{\rho}\le L\quad \text{for } n_1\le n\le n_2.$$

Consider the Banach space B_N of all real sequences $y=\{y(n)\}_{n\ge N}$ with sup norm $\|y\|=\sup_{n\ge N}|y|$. Define the subset Y of B_N by

$$Y=\left\{y^{(n)}\in C[n_1,n_2]: c(n_2-n)^{e}\le y(n)\le L \text{ for } n_1\le n\le n_2\right\}$$

and

$$Fy^{(n)}\begin{cases}=\sum_{n}^{n_2-1}\left(\sum_{s}^{n_2-1}p(r)\left[y(r)\right]^{\beta}\right)^{1/\alpha}, & n_1\le n\le n_2,\\ =0, & \text{other wise.}\end{cases}$$

We show that the hypothesis of the Schavder fixed point theorem is satisfied for Y and F. Let $y\in Y$. Then, obviously $Fy(n)\le L$ for $n_1\le n\le n_2$. Moreover

$$\begin{aligned}Fy(n)&\ge m^{1/\alpha}c^{\beta/\alpha}\sum_{n}^{n_2-1}\left(\sum_{s}^{n_2-1}(n_2-r)^{\rho\beta}\right)^{1/\alpha}\\&\ge\frac{m^{1/\alpha}c^{\beta/\alpha}\alpha}{(\rho\beta+1)^{1/\alpha}(\alpha+\rho\beta+1)}(n_2-n)^{\frac{\alpha+\rho\beta+1}{\alpha}}\\&\ge c(n_2-n)^{\rho},\quad \text{for } n_1\le n\le n_2.\end{aligned}$$

Hence $FY\subset Y$. The continuity of F and the boundedness of the sets FY and $\{\Delta FY: y\in Y\}$ can be easily established. Accordingly there is a $\bar{y}\in Y$ satisfying $F\bar{Y}=\bar{y}$. By taking difference twice, we find that $\bar{y}(n)$ is a solution of Equation (1) for $n_1\le n\le n_2$ and that $\bar{y}(n)>0$ for $n_1\le n\le n_2$ and $\bar{y}(n_2)=\Delta\bar{y}(n_2)=0$. Now, we put

$$y(n)=\begin{cases}\bar{y}(n) & n_1\le n\le n_2\\ 0 & n\ge n_2.\end{cases}$$

It is easy to see that $y(n)$ is a solution of equation (8) for $n\ge n_1$ and is of type (S_1). The proof is complete.

Theorems 14 and 15 can be proved easily as in the proofs of Theorems 4 and 5 respectively. We therefore omit the proofs.

Proof of Theorem 10. By our assumption we can find a positive solution $y_k(n), k\in N$ of Equation (8) satisfying $y_k(\infty)=\frac{1}{K}$. Since $\alpha>\beta$, we see by Lemma 4 that each $y_k(n)$ exists for $n\ge n_0$. We show that the sequence $\{y_k(n)\}$ has the limit function $y(n)$, and it gives rise to a positive solution of Equation (8) of type (*D*).

We first claim that

$$y_1(n)>y_2(n)>\cdots>y_k(n)>y_{k+1}(n)>\cdots>0,\ n\ge n_0. \tag{38}$$

If this is not true, then $y_i(\bar{n})=y_{i+1}(\bar{n})$ for some $i\in N$ and $\bar{n}\ge n_0$. This means however that there are two nonnegative nonincreasing solutions of Equation (8) passing through the point $(\bar{n}, y_i(\bar{n}))$. This contradiction to Theorem 17. We therefore have (38) and so $\lim_{k\to\infty}y_k(n)=y(n)$ exists observe that $y_k(n)$ satisfies

$$y_k(n)=\frac{1}{K}+\sum_{n}^{\infty}\left(\sum_{s}^{\infty}p(r)\left[y_i(r)\right]^{\beta}\right)^{1/\alpha},\quad n\ge n_0.$$

Letting $k\to\infty$, we obtain via the dominated convergence theorem

$$y(n)=\sum_{n}^{\infty}\left(\sum_{s}^{\infty}p(r)\left[y(r)\right]^{\beta}\right)^{1/\alpha},\quad n\ge n_0.$$

We see that $y(n)$ is a nonnegative solution of Equation (8) satisfying $y(\infty)=0$. It remains to prove that $y(n)>0$ for $n\ge n_0$ Fix $N>n_0$ arbitrarily. The proof of Theorem 2 implies that there is a solution $y_N(n)>0$ for $n_0\le n<N$ and $y_N(n)=0$ for $n\ge N$. We claim that

$$y_k(n)>y_N(n)\quad \text{for } n_0\le n\le N \text{ for all } k\in N. \tag{39}$$

In fact, if this fails to hold, then

$$y_i(\bar{n}) = y_N(\bar{n}) \quad \text{for some } i \in N \quad \text{and} \quad n_0 \le \bar{n} < N.$$

By this means, as before, that there are two nonnegative nonincreasing solution fo Equation (8) passing through the point $(\bar{n}, y_N(\bar{n}))$. This contradiction shows that (39) holds. Hence by letting $i \to \infty$ in (39) we have $y(n) \ge y_N(n) > 0$ for $n_0 \le n < N$. Since $N > n_0$ is arbitrary, we see that $y(n) > 0$ for $n \ge n_0$. The proof is complete.

Proof of Theorem 11. The proof is done by contradiction. Let $y(n)$ be a positive solution of equation (8) for $n \ge n_0$ of type (D). Using (16). We obtain from Equation (8)

$$\Delta(\Delta y(n))^\alpha \ge C_1 n^{-\alpha} (y(n))^\beta, \quad n \ge n_1. \tag{40}$$

where C_1 is a positive constant. We fix a $N \ge n_1$ arbitrary and consider inequality (42) only on the interval $[N, 2N]$ for a moment. A summation of (42) from *n* to 2*N*, given

$$\left(\Delta y(2N+1)^\alpha\right) - (\Delta y(n))^\alpha \ge C_1 \sum_{n}^{2N} s^{-\alpha} (y(s))^\beta$$

$$-(\Delta y(n))^\alpha \ge C_1 \frac{1}{N^\alpha} (y(2N))^\beta - (y(n))^\beta$$

$$-\Delta y(n) \ge C_1 \frac{1}{N} \left[(y(2N))^\beta - (y(n))^\beta\right]^{\frac{1}{\alpha}}, \quad N \le n \le 2N$$

$$\frac{-\Delta y(n)}{\left[(y(2N))^\beta - (y(n))^\beta\right]^{\frac{1}{\alpha}}} \ge C_1 N^{-1}, \quad N \le n \le 2N.$$

From which, we have

$$\int_{y(N)}^{y(2N)} \frac{\mathrm{d}u}{\left[u^\beta - (y(2N))^\beta\right]^{\frac{1}{\alpha}}} \ge C_2, \quad N \ge n_1$$

$$(y(2N))^{\frac{\alpha-\beta}{\alpha}} \int_{y(N)/y(2N)}^{1} \left(v^\beta - 1\right)^{\frac{\alpha-\beta}{\alpha}} \mathrm{d}v \ge C_2, \quad N \ge n_1. \tag{41}$$

We can find a constant $C > 0$ satisfying

$$\int_x^1 \left(v^\beta - 1\right)^{-\frac{1}{\alpha}} \mathrm{d}v \le C(1-x)^{\frac{\alpha-\beta}{\alpha+1}}, \quad x \to 0.$$

Therefore (43) implies that

$$C(y(2N))^{\frac{\alpha-\beta}{\alpha}} \left[1 - \left(\frac{y(N)}{y(2N)}\right)\right]^{\frac{\alpha-\beta}{\alpha}} \ge C_2, \quad N \ge n_1$$

from which we have

$$C\left[(y(2N)) - (y(N))\right]^{\frac{\alpha-\beta}{\alpha}} \ge C_2, \quad N \ge n_1.$$

Letting $N \to \infty$, we have a contradiction. The proof is complete.

Proof of Theorem 12. The proof is done by contradiction. Let $y(n)$ be a solution of Equation (8) of type (*D*). We notice first that

$$\lim_{n\to\infty} n\Delta y(n) = 0. \tag{42}$$

In fact, since $\Delta^2 y(n) > 0$, we can compute as follows

$$\Delta y(n)=\sum_{n}^{\infty}(-\Delta y(s))\geq\sum_{n}^{2n}(-\Delta y(s))$$
$$\geq-\Delta y(2n)\sum_{n}^{2n}p(s)=-\Delta y(2n)\cdot n\geq 0 \quad \text{for large } n.$$

Therefore (42) holds.

We may suppose that for some $C_1>0$ and $n_1\geq n_0$

$$0<y(n)\leq C_1,\ 0<-n\Delta y(n)\leq C_1,\ n\geq n_1. \tag{43}$$

But $z(n)=y(-\Delta y)^{\alpha}\ (>0)\ n\geq n_1$. Then

$$-\Delta z(n)=(-\Delta y(n))(-\Delta y(n))^{\alpha}-y_{n+1}\Delta(-\Delta y_n)^{\alpha}$$
$$=(-\Delta y(n))^{\alpha+1}+y_{n+1}p(n)(y(n))^{\beta}$$
$$=y(-\Delta y(n))^{\alpha}\left[\frac{(-\Delta y(n))}{y(n)}+p(n)(y(n))^{\beta}\left(\frac{y(n)}{-\Delta y(n)}-1\right)\right]$$
$$\geq y(-\Delta y(n))^{\alpha}\left[\frac{(-\Delta y(n))}{y(n)}+p(n)\frac{(y(n))^{\beta}}{(-\Delta y(n))^{\alpha}}\right]$$

proceeding as in the proof of Theorem 8, we obtain

$$-\Delta z(n)\geq C_2(y(n))^{\sigma+\sigma\alpha-\rho\alpha-1}(-\Delta y(n))^{1-\sigma-\sigma\alpha+\rho\sigma}[p(n)]^{\alpha}(z(n))^{1-\rho},\quad n\geq n_1$$

where C_2 is a constant. We obtain from (43) and assumption (18)

$$-\Delta z(n)\geq C_3 n^{\sigma+\sigma\alpha-\rho\alpha-1}[p(n)]^{\sigma}(z(n))^{1-\rho},\quad n\geq n_1$$

where $C_3>0$ is a constant. Dividing both sides by $(z(n))^{1-\rho}$ and summing from n to ∞, we have

$$\frac{1}{\rho}(z(n))^{\rho}\geq C_3\sum_{n}^{\infty}(s)^{\sigma+\sigma\alpha-\rho\alpha-1}[p(s)]^{\sigma},\quad n\geq n_1$$

that is,

$$\frac{1}{\rho}(y(n))^{\rho}(-n\Delta y(n))^{\sigma\rho}\geq C_3 n^{\sigma\rho}\sum_{n}^{\infty}(s)^{\sigma+\sigma\alpha-\rho\alpha-1}[p(s)]^{\sigma},\quad n\geq n_1.$$

Letting $n\to\infty$, we get a contradiction to assumption (17) by (42). The proof is complete.

Proof of Theorem 16. Sufficiency Part: By Theorem 17 and (2) of Remark 6.2, there is a positive solution $y(n)$ of equation (8) satisfying $y(\infty)=\infty$. This $y(n)$ is either of type (*AL*) or of type (*AS*). But by Theorem 15, we see that $y(n)$ must be of type (*AS*).

Necessity Part: Let $y(n)$ be a positive solution of Equation (8) for $n\geq n_1$ of type (*AS*). To prove (19), we suppose the contrary that $\sum^{\infty}n^{\beta}p(n)<\infty$. As in the proof of Lemma 5.3, we have

$$|y(n)|=|\rho_0|+\sum_{n_1}^{n-1}\left(|C_1|^{\alpha}+\sum_{n_1}^{s-1}p(r)|y(r)|^{\beta}\right)^{1/\alpha},\ n\geq n_1$$

where $c_0=y(n_1)$ and $c_1=\Delta y(n_1)$ let $z(n)=\max\limits_{n_1\leq\xi\leq n-1}\frac{|y(\xi)|}{\xi}$. It follows that

$$\frac{|y(n)|}{n} \le C_2 + \frac{1}{n}\sum_{n_1}^{n-1}\left(|c_1|^\alpha + [z(s)]^\beta \sum_{n_1}^{s-1} r^\beta p(r)\right)^{1/\alpha}$$

$$\le c_2 + \left(|c_1|^\alpha + [z(s)]^\beta \sum_{n_1}^{n-1} r^\beta p(r)\right)^{1/\alpha}, \quad n \ge n_1$$

where c_2 is a constant. put $w(n) = \max\left\{|c_1|^{\alpha/\beta}, z(n)\right\}$. We then have

$$z(n) \le c_2 + [w(n)]^{\beta/\alpha}\left(1 + \sum_{n_1}^{n-1} r^\beta f(r)\right)^{1/\alpha}, \quad n \ge m.$$

Since $y(n)$ is of type (*AS*), $\frac{|y(n)|}{n}$ is unbounded for $n \ge n_1$ and so is $z(n)$. Accordingly, there is a $n_2 \ge n$, satisfying $w(n) \equiv z(n)$ for $n \ge n_2$. Thus

$$w(n) \le c_2 + (w(n))^{\beta/\alpha}\left(1 + \sum_{n_1}^{n-1} r^\beta p(r)\right)^{1/\alpha}$$

$$\le c_2 + (w(n))^{\beta/\alpha}\left(1 + \sum_{n_1}^{\infty} r^\beta p(r)\right)^{1/\alpha}, \quad n \ge n_2.$$

Since $\beta/\alpha < 1$, this implies the boundedness of *w*, which is a contraction. Hence, we must have (19). The proof is complete.

Theorem 16 is clear because of all solutions of equation (8) with $\alpha < \beta$ exist for $n \ge n_0$ [see Lemma 5].

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