Applied Mathematics, 2016, 7, 1599-1611
Published Online August 2016 in SciRes. http://www.scirp.org/journal/am
http://dx.doi.org/10.4236/am.2016.714138

Exact Traveling Wave Solutions for Generalized Camassa-Holm Equation by Polynomial Expansion Methods

Junliang Lu[1,2], Xiaochun Hong[1]

[1]School of Statistics and Mathematics, Yunnan University of Finance and Economics, Kunming, China
[2]Collaborative Innovation Center for Development and Opening Up of Southwestern Frontier and Mountainous Areas, Kunming, China
Email: wmb0@163.com, xchhong@sina.com

Received 5 June 2016; accepted 23 August 2016; published 26 August 2016

Abstract

We formulate efficient polynomial expansion methods and obtain the exact traveling wave solutions for the generalized Camassa-Holm Equation. By the methods, we obtain three types traveling wave solutions for the generalized Camassa-Holm Equation: hyperbolic function traveling wave solutions, trigonometric function traveling wave solutions, and rational function traveling wave solutions. At the same time, we have shown graphical behavior of the traveling wave solutions.

Keywords

Camassa-Holm Equation, Partial Differential Equation, Polynomial Expansion Methods, Traveling Wave Solutions

1. Introduction

The study of dispersive waves originated from the study of water waves. To find the exact solutions of nonlinear evolution equation arising in mathematical physics plays an important role in the study of nonlinear physical phenomena. There exists an important class of solutions of nonlinear evolution equations is called traveling wave solutions which attract the interest of many mathematicians and physicists. The traveling wave solutions reduce the two variables, namely, the space variable x and the time variable t, of a partial differential equation (PDE) to an ordinary differential equation (ODE) with one independent variable $\xi = x - ct$ where $c \in \left(\mathbb{R} - \{0\}\right)$ is the wave speed with which the wave travels either to the right or to the left. There are many classical methods proposed to find exact traveling wave solutions of PDE. For example, the homogeneous balance method [1], the

How to cite this paper: Lu, J.L. and Hong, X.C. (2016) Exact Traveling Wave Solutions for Generalized Camassa-Holm Equation by Polynomial Expansion Methods. *Applied Mathematics*, **7**, 1599-1611. http://dx.doi.org/10.4236/am.2016.714138

tanh method [2] [3], the Jacobi elliptic function expansion [4]-[14], differential quadrature method [15], the truncated Painleve expansion [16], Lie classical method [17], Hirota bilinear method [18], Darboux transformation [19], the trial Equation method [20]. Recently, more and more methods to find traveling wave solutions are made. In [21]-[26] introduced a method called the $\frac{G'}{G}$-expansion method and obtained traveling solution for the four well established nonlinear evolution equation; Seadawy *et al*. [27] proposed sech-tanh method to solve the Olver equation and the fifth-order KdV equation and obtained traveling wave solutions. Those methods are very efficient, reliable, simple in solving many PDEs.

In 1993, Camassa and Holm used Hamiltonian method to derive a new completely integrable shallow water wave equation

$$u_t + 2\kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1}$$

where u is the fluid velocity in the x direction (or equivalently the height of the water's free surface above a flat bottom), κ is a constant related to the critical shallow water wave speed, and subscripts denote partial derivatives. This equation retains higher order terms (the right hand of) (1) in a small amplitude expansion of incompressible Euler's equations for unidirectional motion of wave at the free surface under the influence of gravity. Now, Equation (1) is called Camassa-Holm (CH) equation. In [28], the authors showed the smoothness of periodic traveling wave solution of the CH equation with the wave length λ, where the periodic traveling wave solution is a special solution we obtained. In recently years, CH Equation has been generalized to the following generalized Camassa-Holm (GCH) equation

$$u_t + 2\kappa u_x - u_{xxt} + \frac{1}{2}\left[f(u)\right]_x = 2u_x u_{xx} + uu_{xxx}, \tag{2}$$

where $f(u)$ is a function of u. In 2001, Dulin *et al*. considered a generalized CH equation

$$u_t + c_0 u_x + 3uu_x - \alpha^2\left(u_{xxt} + 2u_x u_{xx} + uu_{xxx}\right) + \gamma u_{xxx} = 0, \tag{3}$$

which is called CH-γ equation. Here α, c_0 and γ are constants, and $\alpha \neq 0$. The CH-γ equation becomes the CH equation when $\alpha^2 = 1, c_0 = 2\kappa$ and $\gamma = 0$. In [11] [12], the authors discussed the bifurcations of traveling wave solutions for the generalized Camassa-Holm Equation (2) and corresponding traveling wave system with $f(u) = \alpha u^2 + \beta u^3$, *i.e.*,

$$u_t + 2\kappa u_x - u_{xxt} + \frac{1}{2}\left[\alpha u^2 + \beta u^3\right]_x = 2u_x u_{xx} + uu_{xxx}. \tag{4}$$

In [13], the authors discussed the bifurcations of smooth and non-smooth traveling wave solutions for the generalized Camassa-Holm Equation (2). In [14], the author obtained the numerical solution of fuzzy Camassa-Holm equation by using homtopy analysis methods. We look for the traveling wave solutions of (4) in the form of $u(x,t) = \phi(x-ct) = \phi(\xi)$, where c is the wave speed and $\xi = x - ct$. In this paper, we pay attention to solve the (4) and get the traveling wave solutions for the Equation (4).

This paper is organized as follows. In Section 1, an introduction is presented. In Section 2, a description of the polynomial expansion method is formulated. In Section 3, the traveling wave solutions of the GCH are obtained. Finally, the paper ends with a conclusion in the Section 4.

2. Analysis of the Polynomial Expansion Methods

In this section we describe the polynomial expansion methods for finding the traveling wave solutions of nonlinear evolution equation. Suppose a nonlinear equation which has independent space variable x and time variable t is given by

$$P\left(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \cdots\right) = 0, \tag{5}$$

where $u = u(x,t)$ is an unknown function, P is a polynomial of u and its partial derivatives and the polynomial P includes the highest order derivatives and the nonlinear terms. In following, we will describe the polynomial expansion methods.

Suppose that $u(x,t) = \phi(x-ct) = \phi(\xi)$, where c is the wave speed and $\xi = x - ct$. The Equation (5) can be reduced to an ODE with variable $\phi(\xi)$

$$P(\phi, \phi', \phi'', \cdots) = 0, \tag{6}$$

where "′" is the derivative with respect to ξ.

2.1. Analysis of $\frac{G'}{G}$-Polynomial Expansion Methods

Step 1. Suppose the solution of Equation (6) can be expressed by a polynomial in $\frac{G'}{G}$ as follows,

$$\phi(\xi) = \sum_{i=0}^{N} a_i \left(\frac{G'}{G}\right)^i, \tag{7}$$

where a_i are real constants with $a_i \neq 0$ to be determined, *N* is a positive integer to be determined. The function $G(\xi)$ is the solutions of the auxiliary linear ODE

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{8}$$

where λ and μ are real constants to be determined.

Step 2. Substituting (7) into (6). At first, balancing two highest-order, get the value of *N*. Then separate all terms with same order of $\frac{G'}{G}$ together, the left hand of (6) is converted into anther polynomial of $\frac{G'}{G}$, where $G(\xi)$ is the solution of (8). Equating each coefficient of polynomial to zero. Then we obtain algebraic equations of $a_0, a_1, \cdots, a_N$, $a_i's$, c, λ and μ are solved by using Maple.

Step 3. Since we can get the general solutions of Equation (8), then substituting $a_0, a_1, \cdots, a_N, c$ and the general solutions of (8) into (7). Thus, we obtain more traveling wave solutions of nonlinear partial differential Equation (5).

2.2. Analysis of Sech-Tanh Polynomial Expansion Methods

Step 1. Suppose the solution of Equation (6) can be expressed by a polynomial in $\operatorname{sech}^i \xi \tanh^j \xi$ as follows,

$$\phi(\xi) = a_0 + \sum_{i=1}^{N} \operatorname{sech}^{i-1}(a_i \operatorname{sech} \xi + b_i \tanh \xi), \tag{9}$$

where $a_0, a_1, \cdots, a_N$ and $b_1, b_2, \cdots, b_N$ are constants to be determined.

Step 2. Equating two highest-order terms in the ODE (6) and getting the value of *N*.

Step 3. Let the coefficients of $\operatorname{sech}^i \xi \tanh^j \xi$ where $i = 1, 2, \cdots$ and $j = 0, 1$ equate to zero. We have algebraic equations about the unknowns $a_0, a_1, \cdots, a_N$ and $b_1, b_2, \cdots, b_N$.

Step 4. By using Maple, we can solve the algebraic equations in step 2 and we obtain the traveling wave solutions of (5).

3. The Traveling Wave Solutions of GCH

In this section, we will employ the proposed polynomial expansion methods to solve the generalized Camassa-Holm Equation (4). Substituting $u(x,t) = \phi(x - ct) = \phi(\xi)$ into (4), we have

$$(-c + 2\kappa)\phi' + c\phi''' + \alpha\phi\phi' + \frac{3}{2}\phi^2\phi' = 2\phi'\phi'' + \phi\phi''', \tag{10}$$

where "′" is the derivative with respect to ξ.

3.1. Application of $\frac{G'}{G}$-Polynomial Expansion Method

In this section, we apply the $\frac{G'}{G}$-polynomial expansion method to solve the Equation (10).

Balancing the terms $\phi^2\phi'$ with $\phi\phi'''$, we obtain $N = 2$. Therefore, we can write the solution of Equation

(10) in the form

$$\phi(\xi)=a_0+a_1\left(\frac{G'}{G}\right)+a_2\left(\frac{G'}{G}\right)^2, \tag{11}$$

where $a_2\neq 0$ and $G=G(\xi)$. From Equation (8) and (11), we obtain

$$\phi'(\xi)=-\sum_{i=0}^{2} i a_i\left[\left(\frac{G'}{G}\right)^{i+1}+\lambda\left(\frac{G'}{G}\right)^{i}+\mu\left(\frac{G'}{G}\right)^{i-1}\right], \tag{12}$$

$$\begin{aligned}\phi''(\xi)=\sum_{i=0}^{2} i a_i\Bigg[&(i+1)\left(\frac{G'}{G}\right)^{i+2}+(2i+1)\lambda\left(\frac{G'}{G}\right)^{i+1}+i\left(\lambda^2+2\mu\right)\mu\left(\frac{G'}{G}\right)^{i}\\&+(2i-1)\lambda\mu\left(\frac{G'}{G}\right)^{i-1}+(i-1)\mu^2\left(\frac{G'}{G}\right)^{i-2}\Bigg],\end{aligned} \tag{13}$$

$$\begin{aligned}\phi'''(\xi)=-\sum_{i=0}^{2} i a_i\Bigg[&(i+1)(i+2)\left(\frac{G'}{G}\right)^{i+3}+3(i+1)^2\lambda\left(\frac{G'}{G}\right)^{i+2}+\left(\left(3i^2+3i+1\right)(\lambda+\mu)+\mu\right)\left(\frac{G'}{G}\right)^{i+1}\\&+\left(i^2\left(\lambda^3+6\lambda\mu\right)+2\lambda\mu\right)\left(\frac{G'}{G}\right)^{i}+\left(\left(3i^2-3i+1\right)\left(\lambda^2\mu+\mu^2\right)+\mu^2\right)\left(\frac{G'}{G}\right)^{i-1}\\&+3(i-1)^2\lambda\mu^2\left(\frac{G'}{G}\right)^{i-2}+(i-1)(i-2)\mu^3\left(\frac{G'}{G}\right)^{i-3}\Bigg].\end{aligned} \tag{14}$$

Substituting (11), (12), (13), and (14) into Equation (10), let the coefficients of $\left(\frac{G'}{G}\right)^i (i=0,1,2,3,4,5,6,7)$ be zero, we obtain the algebraic equation system for $a_0,a_1,a_2,c,\alpha,\beta,\lambda$ and μ as follows:

$$\left(\frac{G'}{G}\right)^7: -3a_2^3\beta+48a_2^2;$$

$$\left(\frac{G'}{G}\right)^6: -\frac{15}{2}a_1a_2^2\beta+118a_2^2\lambda+50a_1a_2-3a_2^3\beta\lambda;$$

$$\begin{aligned}\left(\frac{G'}{G}\right)^5: &-6a_0a_2^2\beta-3a_2^3\mu\beta+118a_1a_2\lambda+94a_2^2\lambda^2-\frac{15}{2}a_1a_2^2\beta\lambda\\&+96a_2^{\mu}+24a_0a_2+10a_1^2-6a_1^2a_2\beta-24ca_2-2a_2^2\alpha;\end{aligned}$$

$$\begin{aligned}\left(\frac{G'}{G}\right)^4: &-54ca_2\lambda-6a_1^2a_2\beta\lambda-3a_1a_2\alpha+68a_1a_2\mu+148a_2^2\mu\lambda\\&-9a_0a_1a_2\beta+89a_1a_2\lambda^2+54a_0a_2\lambda-6a_1c-6a_0a_2^2\beta\lambda\\&+6a_0a_1-\frac{9}{2}a_1a_2^2\mu\beta-\frac{3}{2}a_1^3\beta+24a_2^2\lambda^3+22a_1^2\lambda-2a_2^2\alpha\lambda;\end{aligned}$$

$$\begin{aligned}\left(\frac{G'}{G}\right)^3: &56\mu^2a_2^2+40a_0\mu a_2+38a_0a_2\lambda^2-12a_1c\lambda-6\beta a_0\mu a_2^2\\&+54a_2^2\mu\lambda^2-3a_1a_2\alpha\lambda-3\beta a_0^2a_2+92a_1a_2\mu\lambda-40c\mu a_2-\frac{3}{2}a_1^3\beta\lambda\\&-9a_0a_1a_2\beta\lambda+12a_0a_1\lambda-3a_0a_1^2\beta+8a_1^2\mu-38ca_2\lambda^2-a_1^2\alpha\\&-2a_0a_2\alpha+21a_1a_2\lambda^3+2ca_2-4a_2\kappa-2a_2^2\mu\alpha+15a_1^2\lambda^2;\end{aligned}$$

$$\left(\frac{G'}{G}\right)^2 : -3a_0^2a_2\beta\lambda - a_1a_2\mu\alpha - 52ca_2\mu\lambda - 2a_0a_2\alpha\lambda - 3a_0a_1^2\beta\lambda$$
$$+27a_1a_2\mu\lambda^2 + 52a_0a_2\mu\lambda - 3a_0a_1a_2\mu\beta + 3a_1^2\lambda^3 - 2a_1\kappa - 8a_1c\mu$$
$$-\frac{3}{2}a_0^2a_1\beta - 8ca_2\lambda^3 + 8a_0a_2\lambda^3 + 8a_1^2\mu\lambda + 14a_1a_2\mu^2 + \frac{3}{2}a_1^3\mu\beta$$
$$+7a_0a_1\lambda^2 - a_0a_1\alpha + 8a_0a_1\mu - a_1^2\alpha\lambda - 4a_2\kappa\lambda + 2ca_2\lambda + 38a_2^2\mu^2\lambda$$
$$-7a_1c\lambda^2 + a_1c;$$

$$\left(\frac{G'}{G}\right)^1 : a_1c\lambda + 2c\mu a_2 + 16a_0a_2\mu^2 - a_1c\lambda^3 - 8a_1c\mu\lambda - 2a_0a_2\mu\alpha$$
$$-16ca_2\mu^2 - 2a_1^2\mu^2 - a_0a_1\alpha\lambda + 2a_1a_2\mu^2\lambda - 4a_2\mu\kappa + a_1^2\mu\alpha$$
$$-14ca_2\mu\lambda^2 + 8a_2^2\mu^3 + a_1^2\mu\lambda^2 - 3a_0^2a_2\mu\beta + 3a_0a_1^2\mu\beta - 2a_1\kappa\lambda$$
$$-\frac{3}{2}a_0^2a_1\beta\lambda + a_0a_1\lambda^3 + 14a_0a_2\mu\lambda^2 + 8a_0a_1\mu\lambda;$$

$$\left(\frac{G'}{G}\right)^0 : -4a_1a_2\mu^3 - 6ca_2\mu^2\lambda - a_1c\mu\lambda^2 + 2a_1\mu\kappa + a_0a_1\mu\alpha - 2a_1^2\mu^2\lambda$$
$$-a_1c\mu + a_1a_1\mu\lambda^2 + 2a_1a_1\mu^2 + 6a_0a_2\mu^2\lambda + \frac{3}{2}a_0^2a_1\mu\beta - 2a_1c\mu^2.$$

Solving the algebraic equation system by Maple we obtained six types of solutions:

$$\textbf{I}: a_0 = a_0, a_1 = a_1, a_2 = a_2, c = -\frac{2\left(6a_0^2a_2 - a_0a_1^2 - a_2^2\kappa\right)}{12a_0a_2 - a_1^2 + a_2^2}, \lambda = \frac{a_1}{a_2}, \mu = 0,$$
$$\alpha = -\frac{3\left(96a_0^2a_2^2 - 16a_0a_1^2a_2 + 12a_0a_2^3 + a_1^4 - a_1^2a_2^2 + 8a_2^3\kappa\right)}{a_2^2\left(12a_0a_2 - a_1^2 + a_2^2\right)}, \beta = \frac{16}{a_2}, \tag{15}$$

where a_0, a_1, a_2 and κ are arbitrary constants.

$$\textbf{II}: a_0 = a_0, a_1 = 0, c = c, a_2 = -\frac{12a_0\left(a_0 + c\right)}{c - 2\kappa}, \lambda = 0,$$
$$\mu = 0, \alpha = \frac{\left(3a_0 + c\right)\left(c - 2\kappa\right)}{a_0\left(a_0 + c\right)}, \beta = -\frac{4\left(c - 2\kappa\right)}{3a_0\left(a_0 + c\right)}, \tag{16}$$

where a_0, c and κ are arbitrary constants.

$$\textbf{III}: a_0 = a_0, a_1 = 0, a_2 = a_2, c = -\frac{2\left(2\mu^2a_2^2 - 8a_0\mu a_2 + a_0^2 - a_2\kappa\right)}{12a_0 - 8\mu a_2 + a_2}, \lambda = 0,$$
$$\mu = \mu, \alpha = -\frac{12\left(12a_2^2\mu - 32a_0a_2\mu - 2a_2^2\mu + 24a_0^2 + 3a_0a_2 + 2a_2\kappa\right)}{a_2\left(12a_0 - 8\mu a_2 + a_2\right)}, \beta = \frac{16}{a_2}, \tag{17}$$

where a_0, a_2, μ and κ are arbitrary constants.

$$\textbf{IV}: a_0 = -\frac{2\kappa\left(8\mu - 1\right)}{\left(4\mu - 1\right)\left(4\mu + 1\right)}, a_1 = 0, c = c, a_2 = -\frac{24\kappa}{\left(4\mu - 1\right)\left(4\mu + 1\right)},$$
$$\lambda = 0, \mu = \mu, \alpha = \frac{16c\mu^2 - c + 6\kappa}{2\kappa}, \beta = -\frac{2\left(4\mu - 1\right)\left(4\mu + 1\right)}{3\kappa}, \tag{18}$$

where c, μ and κ are arbitrary constants.

$$\mathbf{V}: a_0 = 0, a_1 = 0, c = 2\kappa, a_2 = a_2, \lambda = 0, \mu = 0, \alpha = -\frac{24\mu}{a_2}, \beta = \frac{16}{a_2}, \tag{19}$$

where a_2 and κ are arbitrary constants.

$$\begin{gathered}\mathbf{VI}: a_0 = a_0, a_1 = 0, a_2 = a_2, c = -\frac{2\left(6a_0^2 - a_2\kappa\right)}{12a_0 + a_2}, \lambda = 0, \\ \mu = 0, \alpha = -\frac{12\left(24a_0^2 + 3a_0a_2 + 2a_2\kappa\right)}{\left(12a_0 + a_2\right)a_2}, \beta = \frac{16}{a_2},\end{gathered} \tag{20}$$

where a_0, a_2 and κ are arbitrary constants.

Next, we use the solution sets from **I** to **VI** and the solutions of (8) to obtain the solutions of (10).

For **I**, substituting the solution set (15) and the corresponding solutions of (8) into (11), we obtain the hyperbolic function traveling wave solutions of (10) as follows:

$$\begin{aligned}\phi_1(\xi) = a_0 + a_1 &\frac{-\lambda K_2\left(\cosh(-\lambda\xi) - \sinh(-\lambda\xi)\right)}{K_1 + K_2\left(\cosh(-\lambda\xi) - \sinh(-\lambda\xi)\right)} \\ &+ a_2\lambda^2 K_2^2\left(\frac{\cosh(-\lambda\xi) - \sinh(-\lambda\xi)}{K_1 + K_2\left(\cosh(-\lambda\xi) - \sinh(-\lambda\xi)\right)}\right)^2,\end{aligned} \tag{21}$$

where K_1 and K_2 are arbitrary constants. When $a_0 = 4, a_1 = 2, a_2 = 3, \kappa = 2, K_1 = 6, K_2 = 5,$ the figure of **I** is like to **Figure 1**.

For **II**, substituting the solution set (16) and the corresponding solutions of (8) into (11), we obtain the rational function traveling wave solutions of (10) as follows:

$$\phi_2(\xi) = a_0 + a_2\left(\frac{K_2}{K_1 + K_2\xi}\right)^2, \tag{22}$$

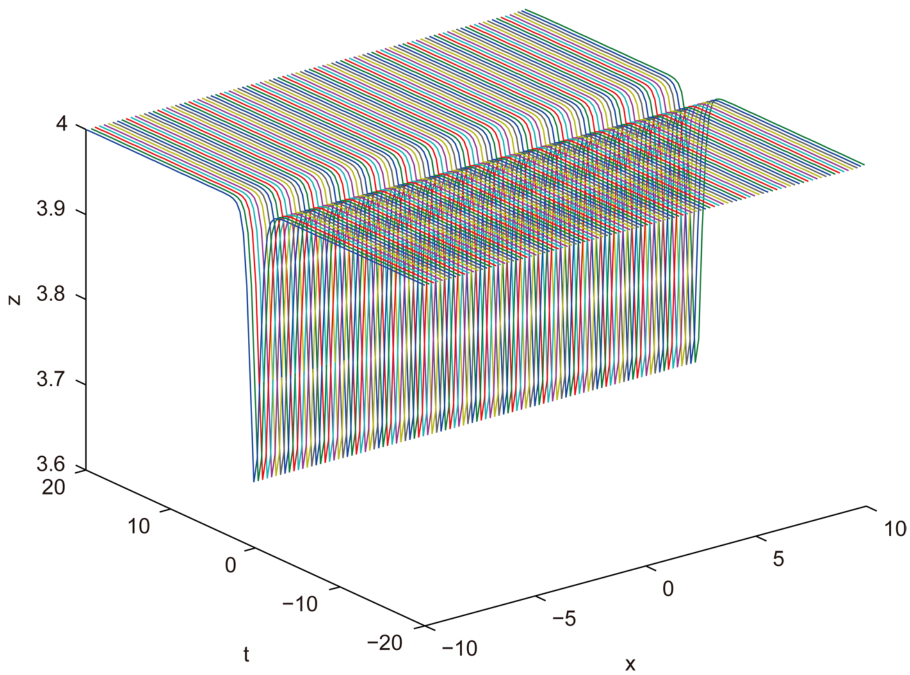

Figure 1. The figure of (10) for **I** applied $\frac{G'}{G}$-polynomial expansion method.

where K_1 and K_2 are arbitrary constants. When $a_0 = 3, a_1 = 0, c = 2, \kappa = 2, K_1 = 6, K_2 = 5$, the figure of **II** is like to **Figure 2**.

For **III**, substituting the solution set (17) and the corresponding solutions of (8) into (11), we obtain the traveling wave solutions of (10) as follows:

When $\mu < 0$, we have the hyperbolic function traveling wave solutions

$$\phi_{31}(\xi) = a_0 - a_2\mu\left(\frac{(K_1+K_2)\sinh\left(\sqrt{-\mu}\xi\right)-(K_1-K_2)\cosh\left(\sqrt{-\mu}\xi\right)}{(K_1-K_2)\sinh\left(\sqrt{-\mu}\xi\right)+(K_1+K_2)\cosh\left(\sqrt{-\mu}\xi\right)}\right)^2, \tag{23}$$

where K_1 and K_2 are arbitrary constants. When $\mu = -2, \kappa = 2, a_2 = 2, K_1 = 6, K_2 = 5$, the figure of **III** is like to **Figure 3**.

When $\mu > 0$, we have the trigonometric function traveling wave solutions

$$\phi_{32}(\xi) = a_0 + a_2\mu\left(\frac{K_1\cos\left(\sqrt{\mu}\xi\right)-K_2\sin\left(\sqrt{\mu}\xi\right)}{K_1\sin\left(\sqrt{\mu}\xi\right)+K_2\cos\left(\sqrt{\mu}\xi\right)}\right)^2, \tag{24}$$

where K_1 and K_2 are arbitrary constants. When $\mu = 5, \kappa = 4, a_2 = 2, K_1 = 6, K_2 = 5$, the figure of **III** is like to **Figure 3**.

For **IV**, when $\mu < 0$, we have the hyperbolic function traveling wave solutions of (10) like the solution (23).

When $\mu > 0$, we have the trigonometric function traveling wave solutions of (10) like the solution (24).

For **V** and **VI**, we have the rational function traveling wave solutions of (10) like (22).

In addition, the figures of **IV** are similar to the figures of **III**, and the figures of **V** and **VI** are similar to the figure of **II**.

3.2. Application of Sinh-Tanh Polynomial Expansion Method

In this section, we apply the sinh-tanh polynomial expansion method to solve the Equation (10).

Balancing the terms $\phi^2\phi'$ with $\phi\phi'''$, we obtain $N = 2$. Therefore, we can write the solution of Equation (10) in the form

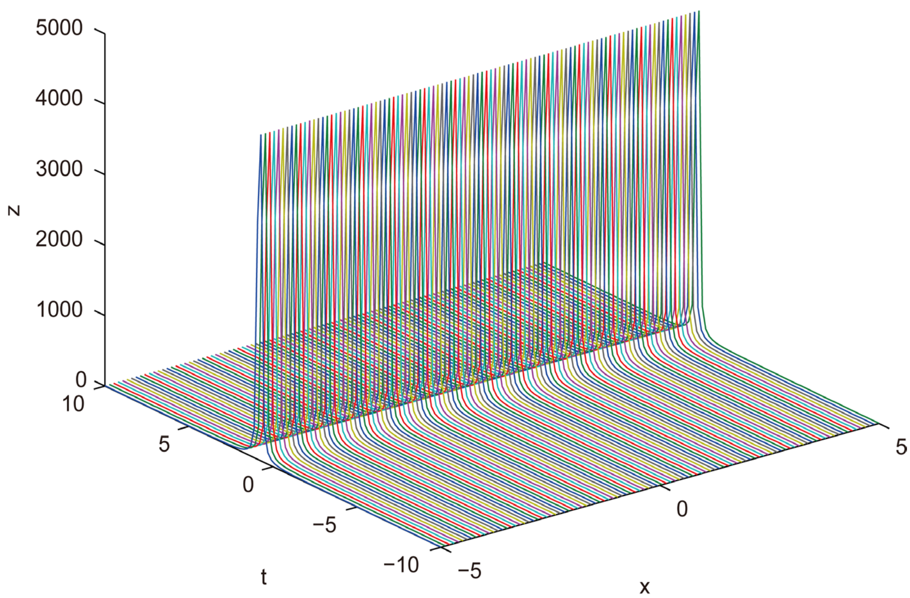

Figure 2. The figure of (10) for **II** applied $\frac{G'}{G}$-polynomial expansion method.

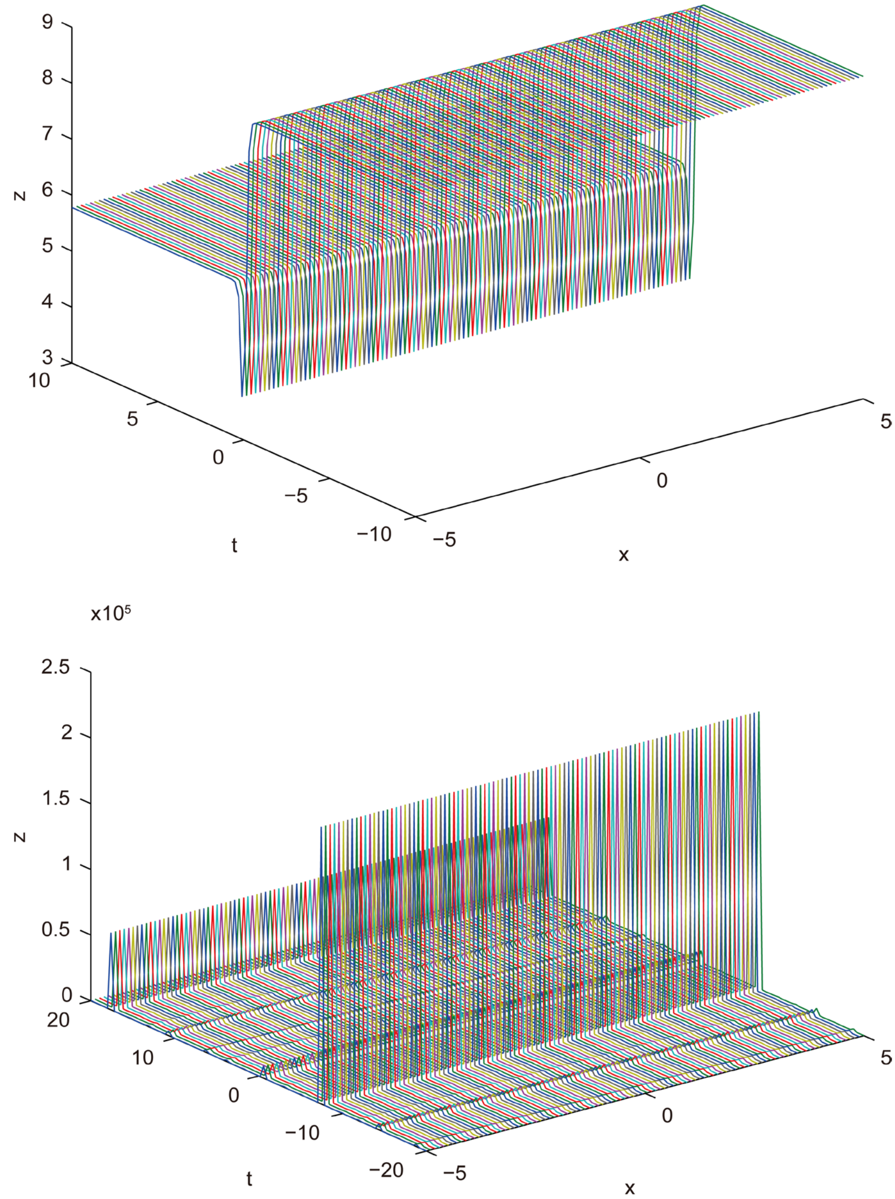

Figure 3. The figure of (10) for **III** applied $\frac{G'}{G}$-polynomial expansion method. The first figure satisfies $\mu < 0$ and the second one satisfies $\mu > 0$.

$$\phi(\xi) = a_0 + \sum_{i=1}^{2} \operatorname{sech}^{i-1}(a_i \operatorname{sech}\xi + b_i \tanh\xi), \tag{25}$$

where a_0, a_1, a_2, b_1, b_2 are constants to be determined, and a_2, b_2 at least one is not zero. From (25), we have

$$\phi'(\xi) = \sum_{i=1}^{2} \left(i a_i \operatorname{sech}^i \xi \tanh\xi + (i-1) b_i \operatorname{sech}^{i-1}\xi - (i-2)\operatorname{sech}^{i+1} \right); \tag{26}$$

$$\begin{aligned}\phi''(\xi) = \sum_{i=1}^{2} \Big(& i^2 a_i \operatorname{sech}^i \xi - (i^2 - 1)\operatorname{sech}^{i+2}\xi + (i-1)^2 b_i \operatorname{sech}^{i-1}\xi \tanh\xi \\ & - (i-2)(i+1) b_i \operatorname{sech}^{i+1} \tanh\xi \Big);\end{aligned} \tag{27}$$

$$\phi'''(\xi)=\sum_{i=1}^{2}\Big(i^3 a_i \operatorname{sech}^i\xi\tanh\xi-\left(i^2-i\right)\left(i+2\right)\operatorname{sech}^{i+2}\xi\tanh\xi$$
$$+\left(i-1\right)^3 b_i\operatorname{sech}^{i-1}\xi-2\left(i-2\right)\left(i^2+1\right)b_i\operatorname{sech}^{i+1}\xi \tag{28}$$
$$+i\left(i-2\right)\left(i+1\right)b_i\operatorname{sech}^{i+3}\xi\Big).$$

Substituting (25), (26), (27), and (28) into Equation (10), let the coefficients of $\operatorname{sech}^i\xi\tanh^j\xi$ $\left(i=0,1,2,3,4,5,6,7;\ j=0,1\right)$ be zero, we obtain the algebraic equation system with the unknowns $a_0,a_1,a_2,b_1,b_2,\alpha,\beta$ and c. Like above section, we solve the algebraic equation system by Maple, we get four types of solutions as follows:

$$\mathbf{i}: a_0=a_0, a_1=0, a_2=a_2, b_1=0, b_2=0, c=-\frac{2\left(2a_0^2+4a_0a_2+a_2\kappa\right)}{4a_0+3a_2},$$
$$\alpha=\frac{4\left(8a_0^2+13a_0a_2+9a_2^2-2a_2\kappa\right)}{a2\left(4a_0+3a_2\right)}, \beta=\frac{16}{3a_2}, \tag{29}$$

where a_0, a_2 and κ are arbitrary constants;

$$\mathbf{ii}: a_0=-\frac{2}{5}\kappa, a_1=0, a_2=\frac{8}{15}\kappa, b_1=0, b_2=0, c=c,$$
$$\alpha=\frac{3\left(2\kappa+5c\right)}{2\kappa}, \beta=-\frac{10}{\kappa}, \tag{30}$$

where c and κ are arbitrary constants;

$$\mathbf{iii}: a_0=-\kappa, a_1=a_1, a_2=0, b_1=0, b_2=b_2, c=c, \alpha=3, \beta=0, \tag{31}$$

where a_1, b_2 and κ are arbitrary constants;

$$\mathbf{iv}: a_0=0, a_1=0, a_2=a_2, b_1=0, b_2=0, c=-\frac{2}{3}\kappa,$$
$$\alpha=\frac{4\left(9a_2-2\kappa\right)}{3a_2}, \beta=-\frac{16}{3a_2}, \tag{32}$$

where a_2 and κ are arbitrary constants.

Therefore, we obtain the solutions of (10) by the solution sets from case 1 to case 4.

For **i**, substituting the solution set (29) into (11), we obtain the hyperbolic function traveling wave solutions of (10) as follows:

$$\phi_1(\xi)=a_0+a_2\operatorname{sech}^2\xi=a_0+a_2\operatorname{sech}^2\left(x-ct\right), \tag{33}$$

where a_0 and a_2 are arbitrary constants. When $a_0=2, a_2=3$, the figure of **i** is like to **Figure 4**.

For **ii**, substituting the solution set (30) into (11), we obtain the hyperbolic function traveling wave solutions of (10) as follows:

$$\phi_2(\xi)=-\frac{2}{5}\kappa+\frac{8}{15}\kappa\operatorname{sech}^2\xi=-\frac{2}{5}\kappa+\frac{8}{15}\kappa\operatorname{sech}^2\left(x-ct\right), \tag{34}$$

where κ and c are arbitrary constants. When $\kappa=2, c=4$, the figure of **ii** is like to **Figure 5**.

For **iii**, substituting the solution set (31) into (11), we obtain the hyperbolic function traveling wave solutions of (10) as follows:

$$\phi_3(\xi)=-\kappa+a_1\operatorname{sech}\xi+b_2\operatorname{sech}\xi\tanh\xi$$
$$=-\kappa+a_1\operatorname{sech}\left(x-ct\right)+b_2\operatorname{sech}\left(x-ct\right)\tanh\left(x-ct\right), \tag{35}$$

where κ, a_1, b_2 and c are arbitrary constants. When $\kappa=2, a_1=1, b_2=3, c=0.5$, the figure of **iii** is like to **Figure 6**.

For **iv**, substituting the solution set (32) into (11), we obtain the hyperbolic function traveling wave solutions

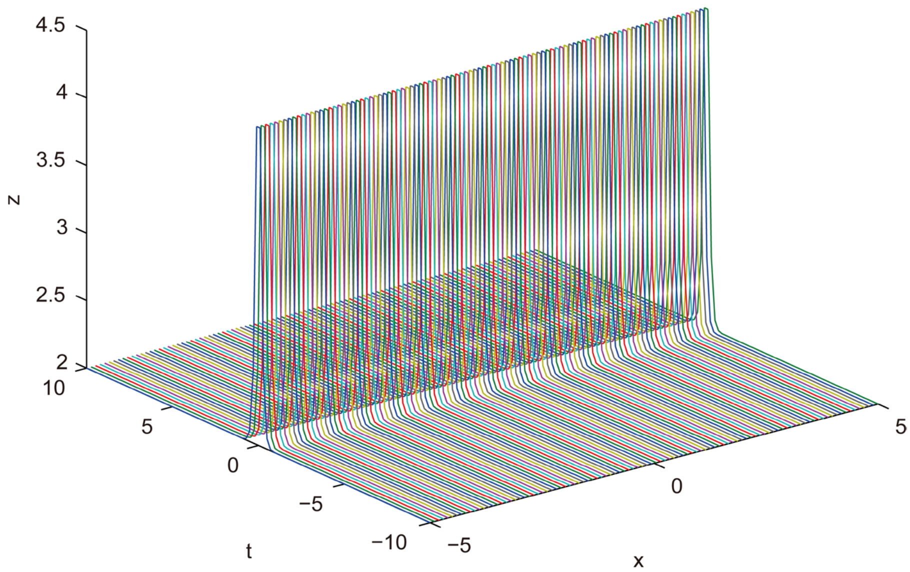

Figure 4. The figure of (10) for **i** applied sinh-tanh polynomial expansion method.

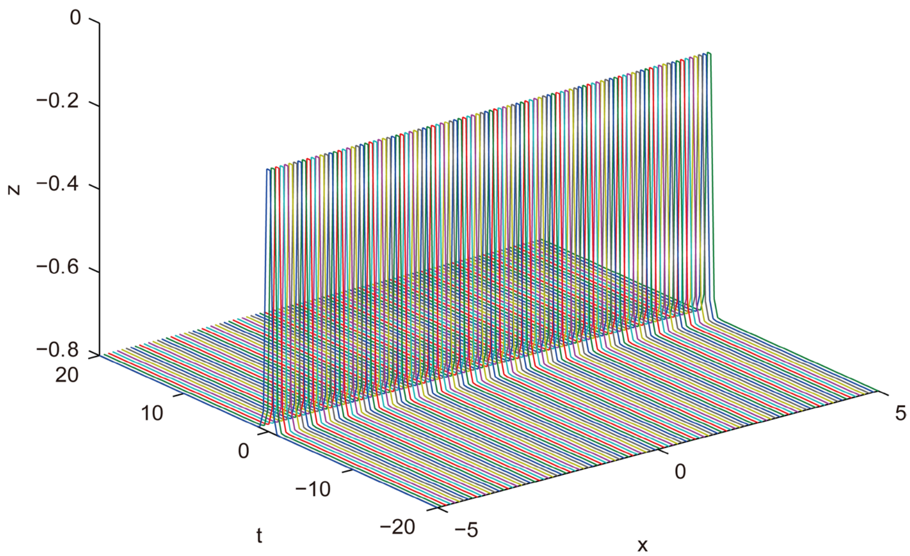

Figure 5. The figure of (10) for **ii** applied sinh-tanh polynomial expansion method.

of (10) as follows:

$$\phi_4(\xi) = a_2 \operatorname{sech}^2 \xi = a_2 \operatorname{sech}^2 \left(x + \frac{2}{3} \kappa t \right), \tag{36}$$

where a_2 and κ are arbitrary constants. When $\kappa = 2, a_2 = 1$, the figure of **iv** is like to **Figure 7**.

4. Conclusions and Remarks

We proposed efficient polynomial expansion methods and obtained the exact traveling wave solutions of generalized Camassa-Holm equation. By polynomial expansion method we obtain hyperbolic function traveling wave solutions, trigonometric function traveling wave solutions, and rational function traveling wave solutions. On comparing with the polynomial expansion methods and other methods to find out the traveling wave for PDEs, the polynomial expansion methods are more effective, powerful and convenient. Moreover, the polynomial expansion methods can be used to solve any high-order degree PDEs.

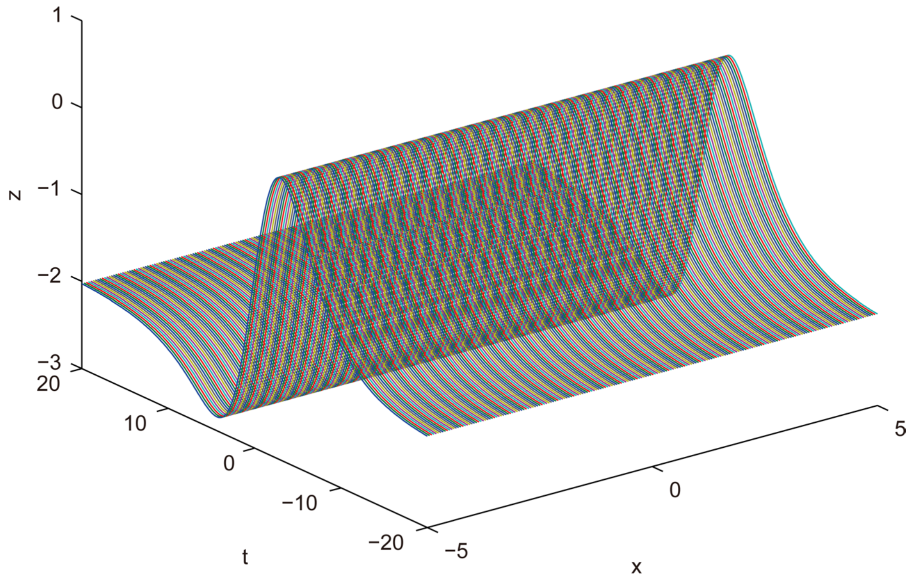

Figure 6. The figure of (10) for **iii** applied sinh-tanh polynomial expansion method.

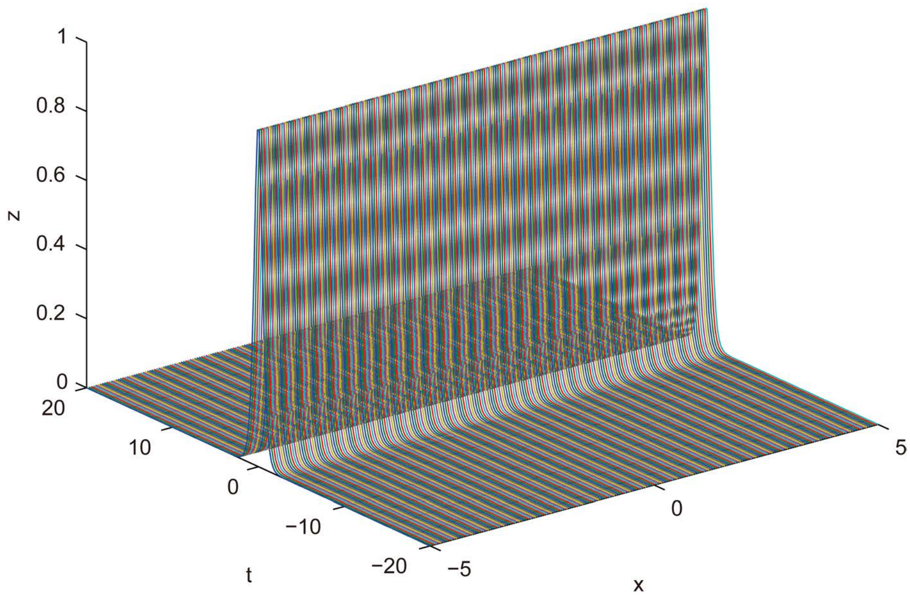

Figure 7. The figure of (10) for **iv** applied sinh-tanh polynomial expansion method.

Acknowledgements

The research is supported in part by the Science and Research Foundation of Yunnan Province Department of Education under grant No. 2015Y277, in part by the Natural Science Foundation of China under grant No. 11161038 and in part by Yunnan Province and Shanghai University of Finance and Economics Education Co-operation consulting Project under grant No. 42111217003.

References

[1] Wang, M., Li, Z. and Zhou, Y. (1999) The Homogenous Balance Principle and Its Application. *Physics Letters of Lanzhou University*, **35**, 8-16.

[2] Malfielt, W. and Hereman, W. (1996) The tanh Method: I. Exact Solutions of Nonlinear Evolution and Wave Equations. *Physica Scripta*, **54**, 563-568. http://dx.doi.org/10.1088/0031-8949/54/6/003

[3] Ablowita, M.J. and Clarkson, P.A. (1991) Solitons, Nonlinear Evolution Equations and Inverse Scattering. Cambridge University Press, Cambridge. http://dx.doi.org/10.1017/CBO9780511623998

[4] Li, J. and Liu, Z. (2000) Smooth and Non-Smooth Travelling Waves in a Nonlinearly Dispersive Equation. *Applied Mathematical Modelling*, **25**, 41-56. http://dx.doi.org/10.1016/S0307-904X(00)00031-7

[5] Li, J. and Chen, G. (2005) Bifurcations of Travelling Wave and Breather Solutions of a General Class of Nonlinear Wave Equations. *International Journal of Bifurcation and Chaos*, **15**, 2913-2926. http://dx.doi.org/10.1142/S0218127405013770

[6] Lu, J., He, T. and Feng, D. (2007) Persistence of Traveling Waves for a Coupled Nonlinear Wave System. *Applied Mathematics and Computation*, **191**, 347-352. http://dx.doi.org/10.1016/j.amc.2007.02.092

[7] Feng, D., He, T. and Lu, J. (2007) Bifurcations of Traveling Wave Solutions for (2+1)-Dimensional Boussinesq-Type Equation. *Applied Mathematics and Computation*, **185**, 402-414. http://dx.doi.org/10.1016/j.amc.2006.07.039

[8] Feng, D., Lu, J., Li, J. and He, T. (2007) Bifurcation Studies on Traveling Wave Solutions for Nonlinear Intensity Klein-Gordon Equation. *Applied Mathematics and Computation*, **1891**, 271-284. http://dx.doi.org/10.1016/j.amc.2006.11.106

[9] Feng, D., Li, J., Lu, J. and He, T. (2007) The Improved Fan Sub-Equation Method and Its Application to the Boussinseq Wave Equation. *Applied Mathematics and Computation*, **194**, 309-320. http://dx.doi.org/10.1016/j.amc.2007.04.026

[10] Feng, D., Li, J., Lu, J. and He, T. (2008) New Explicit and Exact Solutions for a System of Variant RLW Equations. *Applied Mathematics and Computation*, **198**, 715-720. http://dx.doi.org/10.1016/j.amc.2007.09.009

[11] Li, J. and Qiao, Z. (2013) Bifurcation and Exact Traveling Wave Solutions for a Generalized Camassa-Holm Equation. *International Journal of Bifurcation and Chaos*, **23**, Article ID: 1350057.

[12] Li, J. (2013) Singular Nonlinear Travelling Wave Equations: Bifurcations and Exact Solutions. Science Press, Beijing.

[13] Shen, J. and Xu, W. (2005) Bifurcations of Smooth and Non-Smooth Travelling Wave Solutions in the Generalized Camassa-Holm Equation. *Chaos*, *Solitons and Fractals*, **26**, 1149-1162. http://dx.doi.org/10.1016/j.chaos.2005.02.021

[14] Behzadi, S.S. (2011) Numerical Solution of Fuzzy Camassa-Holm Equation by Using Homtopy Analysis Methods. *Journal of Applied Analysis and Computation*, **1**, 315-323.

[15] Jiwari, R., Pandit S. and Mittal, R.C. (2012) Numerical Simulation of Two-Dimensional Sine-Gorden Solitions by Differential Quadrature method. *Computer Physics Communiciations*, **183**, 600-616. http://dx.doi.org/10.1016/j.cpc.2011.12.004

[16] Mohammad, A.A. and Can, M. (1996) Painlevé Analysis and Symmetries of the Hirota-Satsuma Equation. *Journal of Nonlinear Mathematical Physics*, **3**, 152-155. http://dx.doi.org/10.2991/jnmp.1996.3.1-2.15

[17] Goyal, N. and Gupta, R.K. (2012) On Symmetries and Exact Solutions of Einsein Vacuum Quations for Axially Symmetric Gravitional Fields. *International Journal of Mathematical*, *Computational*, *Physical*, *Electrical and Computer Engineering*, **6**, 838-841.

[18] Hirota, R. (1971) Exact Solution of the Korteweg-de Vries Equation for Multiple Collsions of Soliton. *Physical Review Letters*, **27**, 1192-1194. http://dx.doi.org/10.1103/PhysRevLett.27.1192

[19] Matveev, V.B. and Salle, M.A. (1991) Darboux Transformations and Solitons. Springer, Berlin.

[20] Gurefe, Y., Sonmezoglu, A. and Misirli, E. (2011) Application of the Trial Equation Method for Solving Some Nonlinear Evolution Equations Arising in Mathematical Physics. *Pramana—Journal of Physics*, **77**, 1023-1029. http://dx.doi.org/10.1007/s12043-011-0201-5

[21] Zhang, S., Dong, L., Ba, J. and Sun, Y. (2009) The (G'/G)-Expansion Method for Nonlinear Differential-Difference Equation. *Physics Letters A*, **373**, 905-910. http://dx.doi.org/10.1016/j.physleta.2009.01.018

[22] Wang, M., Li, X. and Zhang, J. (2008) The (G'/G)-Expansion Method and Travelling Wave Solutions of Nonlinear Evolution Equations in Mathematical Physics. *Physics Letters A*, **372**, 417-423. http://dx.doi.org/10.1016/j.physleta.2007.07.051

[23] Naher, H., Abdullah, F.A. and Bekir, A. (2015) Some New Traveling Wave Solutions of the Modified Benjamin-Bona-Mahony Equation via the Improved (G'/G)-Expansion Method. *New Trends in Mathematical Sciences*, **3**, 78-89.

[24] Alam, M.N., Akbar, M.A. (2014) Traveling Wave Solutions for the mKdV Equation and the Gardner Equations by New Approach of the Generalized (G'/G)-Expansion Method. *Journal of the Egyptian Mathematical Society*, **22**, 402-406. http://dx.doi.org/10.1016/j.joems.2014.01.001

[25] Alam, M.N., Akbar, M.A. and Mohyud-Din, S.T. (2015) General Traveling Wave Solutions of the Strain Wave Equation in Microstructured Solids via the New Approach of Generalized (G'/G)-Expansion Method. *Alexandria Engineering Journal*, **53**, 233-241. http://dx.doi.org/10.1016/j.aej.2014.01.002

[26] Kaplan, M., Bekir, A. and Akbulut, A. (2016) A Generalized Kudryashov Method to Some Nonlinear Evolution Equations in Mathematical Physics. *Nonlinear Dynamics*, **85**, 2843-2850. http://dx.doi.org/10.1007/s11071-016-2867-1

[27] Seadawy, A.R., Amer, W. and Sayed, A. (2014) Stability Analysis for Travelling Wave Solutions of the Olver and Fifth-Order KdV Equations. *Jorunal of Applied Mathematics*, **2014**, Article ID: 839485. http://dx.doi.org/10.1155/2014/839485

[28] Geyer, A. and Villadelprat, J. (2015) On the Wave Length of Smooth Periodic Traveling Waves of the Camassa-Holm Equation. *Journal of Differential Equations*, **259**, 2317-2332. http://dx.doi.org/10.1016/j.jde.2015.03.027