Scientific
Research
Publishing

Green's Function for the Quartic Oscillator

Robert L. Anderson

Department of Physics and Astronomy, University of Georgia, Athens, USA
Email: andersonr@hal.physast.uga.edu

Abstract

In this paper, a quantum mechanical Green's function $G_{qo}(y_b, t_b; y_a, t_a)$ for the quartic oscillator is presented. This result is built upon two previous papers: first [1], detailing the linearization of the quartic oscillator (*qo*) to the harmonic oscillator (*ho*); second [2], the integration of the classical action function for the quartic oscillator. Here an equivalent form for the quartic oscillator action function $S_{qo}(y_b, t_b; y_a, t_a)$ in terms of harmonic oscillator variables is derived in order to facilitate the derivation of the quartic oscillator Green's Function, namely in fixing its amplitude.

Keywords

Nonrelativistic, Quartic, Quantum, Mechanics, Green's, Oscillator

1. Introduction

Following Schiff [3], the quantum mechanical Green's function $G(z_2, t_2; z_1, t_1)$

$$\Psi(z_2, t_2) = \int_{-\infty}^{\infty} G(z_2, t_2; z_1, t_1) \Psi(z_1, t_1) dz_1, \tag{1.1}$$

implements the superposition principle satisfied by the wave function because it satisfies a linear partial differential equation, the Schrödinger equation. (Note $z_a = 0$ is included because of the integration.) Equation (1.1) implies that *G* also satisfies it, namely in quartic *qo* variables:

$$\frac{\hbar}{-i} \frac{\partial}{\partial t_b} G(y_b, t_b; y_a, t_a) = \left\{ \frac{1}{2m} \left(\frac{\hbar}{i} \frac{\partial}{\partial y_b} \right)^2 + V(y_b) \right\} G(y_b, t_b; y_a, t_a). \tag{1.2}$$

Here we show that the Dirac-Feynman [4] [5] form

$$G(y_b, t_b; y_a, t_a) = \frac{1}{A[t_b, t_a]} \exp \frac{i}{\hbar} S(y_b, t_b; y_a, t_a), \tag{1.3}$$

where $S(y_b, t_b; y_a, t_a)$ is the classical action for the quartic oscillator (*qo*). Thus, we will show that only the classical paths are needed in (1.1) for the *qo* as is true for the free particle and harmonic oscillator. We do not address the question of why the other non-classical paths in Feynman's path integral formulation [6] do not register.

Part II summarizes the results needed from [1].

Part III begins from first principles and expresses the Action function in terms of harmonic oscillator (*ho*) variables $S_{qo}(x_z, \hat{t}_b; x_a, \hat{t}_a)$ and then integrates it. We establish that it is equal to that given in *qo* variables in [2]. This is fundamental to obtaining the correct value of the amplitude A_{qo} appearing in the Green's function.

In Part IV, we then address the missing piece for the Green's function, namely, the Amplitude $A_{qo}(\hat{t}_b - \hat{t}_a)$. To obtain the target goals, these results are recast in terms of the *qo* variables. The Green's Function is then fixed in the final paragraph of this section.

Part V outlines the extension of these results to the hierarchy of all even power potentials.

2. Review of Linearization Map

The linearization map [1] implements the correspondence between the solutions to Newton's equations of motion for the *ho* and *qo*,

$$m \frac{d^2}{d^2\hat{t}} x(\hat{t}) = -k_2 x(\hat{t}) \Leftrightarrow m \frac{d^2}{d^2 t} y(t) = -k_4 y^3(t). \tag{2.1}$$

Note both systems are assumed to have the same mass *m*.

Specifically, the invertible linearization map to the quartic oscillator with mass *m* and space coordinate *y* is stated in two parts. First,

$$y = (2k_2/k_4)^{1/4} x \big/ (x^2)^{1/4}$$

or

$$x = (k_4/2k_2)^{1/2} (y^2)^{1/2} y, \tag{2.2}$$

where *y* is the space coordinate of the quartic oscillator and we have used the representation $\text{sgn}(x) = x/(x^2)^{1/2}$ and similarly for sgn(*y*). This implements the physical requirement that $\frac{1}{2}k_2 x^2(\hat{t}) = \frac{1}{4}k_4 y^4(t)$ *i.e.* matching the potential energies at the two different times, coupled with matching of the signs of the space coordinates. One cycle of the *qo* corresponds to one cycle of the *ho*, of course the periods are different.

Second,

$$\frac{dt}{d\hat{t}} = 1/2 (2k_2/k_4)^{1/4} (x^2(\hat{t}))^{-1/4}$$

and

$$\frac{d\hat{t}}{dt} = (2k_2/k_4)^{1/2} (y^2(t))^{1/2}, \tag{2.3}$$

which results by requiring

$$dx(\hat{t})/d\hat{t} = dy(t)/dt. \tag{2.4}$$

Given the matching of the potential energies, the matching of the velocities and the masses of the oscillators for all values of k_2 and k_4 implies physically matching the momentum at the two different times and the kinetic energies, *i.e.* $p_{ho}(\hat{t}) = p_{qo}(t)$, $E_{ho} = E_{qo} = E$.

Further we need

$$x(\hat{t}) = \left[x_b \sin \omega(\hat{t} - \hat{t}_a) + x_a \sin \omega(\hat{t}_b - \hat{t}) \right] \big/ \sin \omega(\hat{t}_b - \hat{t}_a), \tag{2.5}$$

$$\frac{(y^2(t))^{1/2} y(t)}{(4E/k_4)^{1/2}} = \sin(\theta - \theta_o) = \frac{x_{ho}(\hat{t})}{(2E/k_2)^{1/2}}. \tag{2.6}$$

Note: Our convention $t = 0 \leftrightarrow \hat{t} = 0$. The physical significance of t_0 will be found in Part IV.

Finally, and key to the interchangeability of the *qo* and *ho* variables needed here is the standard change of variables in differentiation given by the following: First, it follows from (2.2) that

$$\frac{\partial}{\partial y} = \frac{\partial x}{\partial y}\frac{\partial}{\partial x} = 2\left(\frac{k_4}{2k_2}\right)^{1/4}\left(x^2\right)^{1/4}\frac{\partial}{\partial x}. \tag{2.7}$$

Second, it follows from (2.3) that

$$\frac{\partial}{\partial t} = \frac{\partial \hat{t}}{\partial t}\frac{\partial}{\partial \hat{t}} = 2\left(\frac{k_4}{2k_2}\right)^{1/4}\left(x^2\right)^{1/4}\frac{\partial}{\partial \hat{t}}. \tag{2.8}$$

Note from (2.7) and (2.8) that

$$\frac{\partial x}{\partial y} = \frac{\partial \hat{t}}{\partial t}. \tag{2.9}$$

3. The *qo* Action in Terms of the *ho* Variables

As stated in the Introduction, the object of this paragraph is to express the defining expression for the qo action in terms of the *ho* variables and integrate it.

We start with an expression of the *qo* action in *qo* variables and transform it to *ho* variables

$$S_{qo} = \int_{t_a}^{t_b}\left(\frac{1}{2}m\left(\frac{dy(t)}{dt}\right)^2 - \frac{1}{4}k_4 y^4(t)\right)dt$$

$$= \int_{\hat{t}_a}^{\hat{t}_b}\left(\frac{1}{2}m\left(\frac{dx(\hat{t})}{d\hat{t}}\right)^2 - \frac{1}{2}k_2 x^2(\hat{t})\right)\underbrace{\frac{1}{2}\left(\frac{2k_2}{k_4}\right)^{1/4}\left(x^2(\hat{t})\right)^{-1/4}d\hat{t}}_{dt}. \tag{3.1}$$

Employing $\frac{1}{2}m\left(\frac{dx(\hat{t})}{d\hat{t}}\right)^2 + \frac{1}{2}k_2\left(x(\hat{t})\right)^2 = E$, we obtain from (3.1)

$$S_{qo} = \int_{\hat{t}_a}^{\hat{t}_b}\left(m\left(\frac{dx(\hat{t})}{d\hat{t}}\right)^2 - E\right)\frac{1}{2}\left(\frac{2k_2}{k_4}\right)^{1/4}\left(x^2(\hat{t})\right)^{-1/4}d\hat{t}$$

$$= \int_{\hat{t}_a}^{\hat{t}_b}m\frac{dx(\hat{t})}{d\hat{t}}dx\frac{1}{2}\left(\frac{2k_2}{k_4}\right)^{1/4}\left(x^2(\hat{t})\right)^{-1/4}d\hat{t} - E(t_b - t_a), \tag{3.2a}$$

where

$$t_b - t_a = \int_{\hat{t}_a}^{\hat{t}_b}\frac{1}{2}\left(\frac{2k_2}{k_4}\right)^{1/4}\left(x^2(\hat{t})\right)^{-1/4}d\hat{t}. \tag{3.2b}$$

Now

$$x(\hat{t}) = \left(\frac{2E}{k_2}\right)\sin\left(\theta(\hat{t}) - \theta_o\left(\hat{t}_o\right)\right) \tag{3.3}$$

where $\theta(\hat{t}) = \omega\hat{t}$ and $k_2 = m\omega^2$.

Therefore

$$dx(\hat{t}) = \left(\frac{2E}{k_2}\right)^{1/2}\cos\left(\theta(\hat{t}) - \theta_o\left(\hat{t}_o\right)\right)\underbrace{d\theta(\hat{t})}_{\omega d\hat{t}}, \tag{3.4}$$

and

$$\frac{dx(\hat{t})}{d\hat{t}} = \left(\frac{2E}{k_2}\right)^{1/2} \cos\left(\theta(\hat{t}) - \theta(\hat{t}_o)\right)\omega. \tag{3.5}$$

Continuing, we have for the first term in the final expression (3.2a)

$$= \frac{2E}{m} \int_{\hat{t}_a}^{\hat{t}_b} \frac{1}{2}\left(\frac{2k_2}{k_4}\right)^{1/4} \left(\frac{2E}{k_2}\right)^{-1/4} \cos^2\left(\theta(\hat{t}) - \theta(\hat{t}_o)\right)\left(\sin^2\left(\theta(\hat{t}) - \theta(\hat{t}_o)\right)\right)^{-1/4} d\hat{t}. \tag{3.6}$$

Therefore

$$S_{qo} = \left[2E\left(\frac{2E}{k_2}\right)^{-1/4}\right]\left[\frac{1}{2}\left(\frac{2k_2}{k_4}\right)^{1/4}\right] \int_{\hat{t}_a}^{\hat{t}_b} \cos^2\left(\theta(\hat{t}) - \theta(\hat{t}_o)\right)\left(\sin^2\left(\theta(\hat{t}) - \theta(\hat{t}_o)\right)\right)^{-1/4} d\hat{t} - E(t_b - t_a). \tag{3.7}$$

Now paralleling the development in [2] we effect the integration by parts, where:

$$\frac{d}{d\hat{t}}(fg) = \frac{df}{d\hat{t}}g + f\frac{dg}{d\hat{t}}$$

with $f = \left(\sin^2\left(\theta(\hat{t}) - \theta(\hat{t}_o)\right)\right)^{3/4}$ and $g = \frac{2}{3}\frac{\cos\left(\theta(\hat{t}) - \theta(\hat{t}_o)\right)}{\sin\left(\theta(\hat{t}) - \theta(\hat{t}_o)\right)}$.

Therefore,

$$\int_{\hat{t}_a}^{\hat{t}_b} \cos^2\left(\theta(\hat{t}) - \theta(\hat{t}_o)\right)\left(\sin^2\left(\theta(\hat{t}) - \theta(\hat{t}_o)\right)\right)^{-1/4} d\hat{t}$$
$$= \frac{2}{3}\int_{\hat{t}_a}^{\hat{t}_b}\left(\sin^2\left(\theta(\hat{t}) - \theta(\hat{t}_o)\right)\right)^{-1/4} d\hat{t} + \frac{2}{3}\frac{1}{\omega}\left(\sin^2\left(\theta(\hat{t}) - \theta(\hat{t}_o)\right)\right)^{3/4}\frac{\cos\left(\theta(\hat{t}) - \theta(\hat{t}_o)\right)}{\sin\left(\theta(\hat{t}) - \theta(\hat{t}_o)\right)}\Bigg|_{\hat{t}_a}^{\hat{t}_b}. \tag{3.8}$$

Hence,

$$S_{qo} = \left[2E\left(\frac{2E}{k_2}\right)^{-1/4}\right]\left[\frac{1}{2}\left(\frac{2k_2}{k_4}\right)^{1/4}\right]\left\{\frac{2}{3}\int_{\hat{t}_a}^{\hat{t}_b}\left(\sin^2\left(\theta(\hat{t}) - \theta(\hat{t}_o)\right)\right)^{-1/4} d\hat{t}\right.$$
$$\left. + \frac{2}{3}\frac{1}{\omega}\left(\sin^2\left(\theta(\hat{t}) - \theta(\hat{t}_o)\right)\right)^{3/4}\frac{\cos\left(\theta(\hat{t}) - \theta(\hat{t}_o)\right)}{\sin\left(\theta(\hat{t}) - \theta(\hat{t}_o)\right)}\Bigg|_{\hat{t}_a}^{\hat{t}_b}\right\} - E(t_b - t_a)$$
$$= \underbrace{\frac{4E}{3}\int_{\hat{t}_a}^{\hat{t}_b}\left(\frac{2E}{k_2}\right)^{-1/4}\left[\frac{1}{2}\left(\frac{2k_2}{k_4}\right)^{1/4}\right]\left(\sin^2\left(\theta(\hat{t}) - \theta(\hat{t}_o)\right)\right)^{-1/4} d\hat{t}}_{(t_b - t_a)} \tag{3.9}$$
$$+ \frac{\left(\frac{2k_2}{k_4}\right)^{1/4}(k_2 m)^{1/2}}{3}\underbrace{\left(x^2(\hat{t})\right)^{3/4}}_{\left(\left(x^2(\hat{t})\right)^{1/2}\right)^{3/2}}\frac{\cos\left(\theta(\hat{t}) - \theta(\hat{t}_o)\right)}{\sin\left(\theta(\hat{t}) - \theta(\hat{t}_o)\right)}\Bigg|_{\hat{t}_a}^{\hat{t}_b} - E(t_b - t_a).$$

Finally,

$$S_{qo} = \frac{1}{3}\left(\frac{2k_2}{k_4}\right)^{1/4}(k_2 m)^{1/2}\left(\left(x^2(\hat{t})\right)^{1/2}\right)^{3/2}\frac{\cos\left(\theta(\hat{t}) - \theta(\hat{t}_o)\right)}{\sin\left(\theta(\hat{t}) - \theta(\hat{t}_o)\right)}\Bigg|_{\hat{t}_a}^{\hat{t}_b} + \frac{1}{3}E(t_b - t_a), \tag{3.10}$$

where

$$(t - t_0) = \int_{\hat{t}_0}^{\hat{t}} \frac{1}{2} \left(\frac{2k_2}{k_4} \right)^{1/4} \left(x^2 \left(\hat{t} \right) \right)^{-1/4} d\hat{t}, \tag{3.11}$$

and

$$E = \frac{1}{2} k_2 \left[\left\{ x_b^2 + x_a^2 - 2x_b x_a \cos \omega \left(\hat{t}_b - \hat{t}_a \right) \right\} / \sin^2 \omega \left(\hat{t}_b - \hat{t}_a \right) \right]. \tag{3.12}$$

Now we verify that this is indeed S_{qo}; but expressed in *ho* variables. To do this we use the linearization tools described in Part II. In particular, from (2.6)

$$\begin{aligned}
\frac{\partial S_{qo}}{\partial y_b} &= 2 \left(\frac{k_4}{2k_2} \right)^{1/4} \left(x_b^2 \right)^{1/2} \frac{\partial}{\partial x_b} S_{qo} \\
&= 2 \left(\frac{k_4}{2k_2} \right)^{1/4} \left(x_b^2 \right)^{1/2} \frac{1}{3} \left(\frac{2k_2}{k_4} \right)^{1/4} \left(k^2 m \right)^{1/2} \frac{3}{4} \left(x_b^2 \right)^{1/4} 2x_b \frac{\cos \omega \left(\hat{t}_b - \hat{t}_o \right)}{\sin \omega \left(\hat{t}_b - \hat{t}_o \right)} \\
&= \left(k_2 m \right)^{1/2} x_b \frac{\cos \omega \left(\hat{t}_b - \hat{t}_o \right)}{\sin \omega \left(\hat{t}_b - \hat{t}_o \right)} \\
&= \left(\frac{mk_4}{2} \right)^{1/2} \left(y_b^2 \right)^{1/2} y_b \frac{\cos \int_{t_o}^{t_b} \left(\frac{2k_4}{m} \right)^{1/2} \left(y^2 \left(t \right) \right)^{1/2} dt}{\sin \int_{t_o}^{t_b} \left(\frac{2k_4}{m} \right)^{1/2} \left(y^2 \left(t \right) \right)^{1/2} dt} = p_b.
\end{aligned} \tag{3.13}$$

This checks with (4.3) in [2].
Next,

$$\begin{aligned}
\frac{\partial S_{qo}}{\partial t_b} &= 2 \left(\frac{k_4}{2k_2} \right)^{1/4} \left(x_b^2 \right)^{1/4} \frac{1}{3} \left(\frac{2k_2}{k_4} \right)^{1/4} \left(k^2 m \right)^{1/2} \left(\left(x_b^2 \right)^{1/2} \right)^{3/2} \left(\frac{-1}{\sin^2 \omega \left(\hat{t}_b - \hat{t}_o \right)} \right) \\
&= -\frac{4}{3} E + \frac{1}{3} E = -E.
\end{aligned} \tag{3.14}$$

This checks with (4.3) in [2].

$$\left(\text{Thus, } S \text{ obeys the H-J equation } 0 = \frac{\partial S}{\partial t_b} + H \left(y_b, \frac{\partial S}{\partial y_b} \right). \right)$$

Similarly

$$\frac{\partial S_{qo}}{\partial x_a} = -p_a \quad \text{and} \quad \frac{\partial S_{qo}}{\partial t_a} = E. \tag{3.15}$$

Using the results in Part II, (3.10) can be directly shown to be equal to the result (4.2) in [2], namely, it is equal to

$$\begin{aligned}
&S_{qo} \left(y_b, t_b; y_a, t_a \right) \\
&= \frac{1}{3} \left(\frac{mk_4}{2} \right)^{1/2} \left[\left(y_a^2 \right)^{3/2} \frac{\cos \left(\theta_b - \theta_o \right)}{\sin \left(\theta_b - \theta_o \right)} - \left(y_a^2 \right)^{3/2} \frac{\cos \left(\theta_b - \theta_o \right)}{\sin \left(\theta_b - \theta_o \right)} \right] + \frac{E}{3} \left(t_b - t_a \right),
\end{aligned} \tag{3.16}$$

$$\left[\left(y_a^2 \right)^{3/2} \frac{\cos \int_{t_o}^{t_b} \left(\frac{2k_4}{m} \right)^{1/2} \left(y^2 \left(t' \right) \right)^{1/2} dt'}{\sin \int_{t_o}^{t_b} \left(\frac{2k_4}{m} \right)^{1/2} \left(y^2 \left(t' \right) \right)^{1/2} dt'} - \left(y_a^2 \right)^{3/2} \frac{\cos \int_{t_a}^{t_o} \left(\frac{2k_4}{m} \right)^{1/2} \left(y^2 \left(t' \right) \right)^{1/2} dt'}{\sin \int_{t_a}^{t_o} \left(\frac{2k_4}{m} \right)^{1/2} \left(y^2 \left(t' \right) \right)^{1/2} dt'} \right] + \frac{E}{3} \left(t_b - t_a \right),$$

where

$$E = \frac{1}{4} k_4 \left[y_b^4 + y_a^4 - 2 \left(y_b^2 \right)^{1/2} y_b \left(y_a^2 \right)^2 y_a \cos \left(\theta(t_b) - \theta(t_a) \right) \right],$$ (3.17)

and

$$\theta(t) - \theta(t_o) = \int_{t_o}^{t} \left(\frac{2k_4}{m} \right)^{1/2} \left(y^2(t') \right)^{1/2} dt'.$$ (3.18)

The significance of t_0 will be discussed in the latter paragraph of Part IV.

(It is important to correct some exponent typos in [2] (arXiv:1207.4376v2 [math-ph]). These corrections do not affect any of the results reported there or here. The correct exponents were used in arriving at the results reported. The minus sign on lhs of (2.8) should be a plus. The exponent in (2.9a) and (3.1) in [2] on the $\sin^2$ terms should read $-1/4$. The terms involving $\left(\frac{2k}{m} \right)$ in the integrands on pp 120-121 should all carry an exponent of 1/2. The corresponding equations and pages should be corrected in the arXiv article. Sorry for any inconvenience, but again no errors in the final results!)

4. Green's Function for the Quartic Oscillator

Here, (assuming the Dirac-Feynman form of the Green's function) an amplitude A_{qo} is identified such that the expression $A_{qo}^{-1} \exp \left(\frac{i}{\hbar} S_{qo} \right)$ satisfies the Schrödinger equation:

$$\frac{\hbar}{-i} \frac{\partial}{\partial t_b} A_{qo}^{-1} \exp \left(\frac{i}{\hbar} S_{qo} \right) = \left\{ \frac{1}{2m} \left(\frac{\hbar}{i} \frac{\partial}{\partial y_b} \right)^2 + \frac{1}{4} k_4 y_b^4 \right\} A_{qo}^{-1} \exp \left(\frac{i}{\hbar} S_{qo} \right),$$ (4.1)

where S_{qo} is given by (3.16)-(3.18) and A is to be determined.

Thus we have on the *lhs* of (4.1)

$$= \left\{ \frac{\hbar}{i} \left(\frac{\partial A_{qo}}{\partial t_b} \bigg/ A_{qo}^2 \right) + \frac{\hbar}{-i} \frac{i}{\hbar} (-E) A_{qo}^{-1} \right\} \exp \frac{i}{\hbar} (S),$$ (4.2)

where $\dfrac{\partial S_{qo}}{\partial t_b} = -E$ from (3.14).

And we have the *rhs* of (3.1)

$$= \left\{ \hbar / 2mi \frac{\partial p_b}{\partial y_b} + \frac{1}{2} m p_b^2 + V(y) \right\} A_{qo}^{-1} \exp \frac{i}{\hbar} (S),$$ (4.3)

where $\dfrac{\partial S}{\partial y_b} = p_b$ from (3.13).

The 2nd term in the *lhs* of (4.1) is equal to the sum of the 2nd and 3rd terms in the *rhs* of (4.1) for our conservative system.

This leaves only the 1st term of the *lhs* of (4.1) and the 1st term in the *rhs* of (4.1). Equating their coefficients and cancelling common factors we obtain

$$\frac{\partial A_{qo}}{\partial t_b} \bigg/ A_{qo} = 2m \frac{\partial p_b}{\partial y_b}.$$ (4.4)

Proceeding with the evaluation of (4.4), we have for the *lhs*

$$\frac{1}{A_{qo}} 2 \left(\frac{k_4}{2k_2} \right)^{1/4} \left(x_b^2 \right)^{1/4} \frac{\partial}{\partial \hat{t}_b} A_{qo},$$ (4.5)

and for the *rhs*

$$\frac{1}{2m} 2\left(\frac{k_4}{2k_2}\right)^{1/4} \left(x_b^2\right)^{1/4} \frac{\partial}{\partial \hat{x}_b} \left(k^2 m\right)^{1/2} \frac{\cos \omega\left(\hat{t}_b - \hat{t}_o\right)}{\sin \omega\left(\hat{t}_b - \hat{t}_o\right)}. \tag{4.6}$$

Therefore equating (4.5) to (4.6) and canceling the common factors including one given by (2.9), we have

$$A_{qo} \frac{\partial}{\partial \hat{t}} A_{qo} = \frac{1}{2}\omega \frac{\cos \omega\left(\hat{t}_b - \hat{t}_o\right)}{\sin \omega\left(\hat{t}_b - \hat{t}_o\right)},$$

or

$$\frac{\partial}{\partial \hat{t}} \ell n A_{qo} = \frac{1}{2}\frac{\partial}{\partial \hat{t}} \ln \alpha \sin \omega\left(\hat{t}_b - \hat{t}_o\right), \tag{4.7}$$

or

$$A_{qo} = \left(\alpha \sin \omega\left(\hat{t}_b - \hat{t}_o\right)\right)^{1/2}, \tag{4.8}$$

where $\alpha = $ constant.

Before completing our discussion of the amplitude, we start with the observation that there exists a $z_1 = 0$ at t_1. Now this $z_1 = 0$ corresponds to a $\hat{z}_1 = 0$ at $\hat{t}_1$ via (2.2). This implies via (2.6) that $\hat{t}_o = \hat{t}_1$.

Now set $\hat{t}_1$. This implies t_1 via the quadrature (3.11).

Now set $\hat{t}_2$ and this implies t_2 via another application of the quadrature (3.11).

$x_2 = x\left(\hat{t}_2\right)$ implies $y_2 = y\left(t_2\right) = z_2\left(t_2\right)$ via (2.2). Therefore returning to the amplitude, we have

$$A_{qo} = \left(\alpha \sin \omega\left(\hat{t}_2 - \hat{t}_1\right)\right)^{1/2}. \tag{4.9}$$

Here using A_{qo} in the limit, which is ($\alpha \omega \varepsilon$), where ε is the increment in *ho* time, we link up with the development in F-H [6] for any system for which $H = T + V$ on pp 76 - 78 and find

$$\alpha = \frac{2\pi i\hbar}{m\omega}. \tag{4.10}$$

Equivalently from (3.18)

$$A_{qo} = \left(\frac{2\pi i\hbar}{m\omega}\right) \sin\left[\int_{t_1}^{t_2} \left(\frac{2k_4}{m}\right)^{1/2} \left(y^2(t)\right)^{1/2} \, dt\right]. \tag{4.11}$$

We turn to how do we use this structure of the Green's function to bring it to the form (1.1).

There are two quadratures necessary to fix the connection between the coordinates.

To obtain (1.1) we set $z_2 = y_b$, $t_2 = t_b$, $z_1 = y_a$, $t_1 = t_a$ and note that the integrations overall z_1 technically eliminates knowing the exact connection; except for $z_1 = 0_1$, between the integration variable z_1 and t_1 because t_1 is fixed and the integration is over all values of z_1, where z_1 appears in E given by (3.12). Therefore with these substitutions understand the $G\left(y_b, t_b, y_a, t_0\right)$ in our Green's function (1.3) is given by (3.16)-(3.18) and the amplitude $A\left[t_b, t_0\right]$ is given by (4.8) and (4.10)-(4.11) where $\omega\left(\hat{t}_b - \hat{t}_a\right)$ equals the rhs of (3.18). Again, there are only two quadratures with this application fixed by $\left(y_b = z_2, t_b = t_2\right)$ and $\left(y_a = z_1 = 0, t_a = t_1\right)$.

5. Extremal Mapping for $\left.\frac{1}{2n}k_{2n}y_{2n}^{2n}(t)\right|_{n>1}$ Hierarchy

In this section we present a brief outline of the extension of these results to the hierarchy of attractive potentials given by even powers of the space coordinate [1].

Fundamental to this outline the mapping of the harmonic oscillator extremals onto the extremals of a each member of an hierarchy of attractive oscillators with coordinates $y_{2n}(t)$; $n = 2, 3, 4, \cdots$ characterized by even positive power law potentials. (The case, $y \equiv y_4$ which is included in the hierarchy, has been the subject of the

preceding paragraph.) In a straight forward manner the mappings in Part II and Part 1, generalize and yield the following relationships:

$$y_{2n} = \left(nk_2/k_{2n}\right)^{1/2n} x \left(x^2\right)^{(1/2)(1-n)/n}$$

$$x = \left(k_{2n}/nk_2\right)^{1/2} y_{2n} \left(y_{2n}^2\right)^{(n-1)/2}, \tag{5.1}$$

which is the generalization of (2.1). The generalization of (2.2) is given by:

$$\frac{dt}{d\hat{t}} = n^{-(2n-1)/2n} \left(k_2/k_{2n}\right)^{1/2n} \left(x^2\left(\hat{t}\right)\right)^{-(n-1)/2n} \left(\hat{t}\right),$$

and

$$\frac{d\hat{t}}{dt} = \sqrt{n} \left(k_{2n}/k_2\right)^{1/2} \left(y_{2n}^2\right)^{(n-1)/2}, \tag{5.2}$$

and

$$\hat{t}_o = t_o. \tag{5.3}$$

These mappings take the space-time extremals of the linear oscillator with coordinates $\left(x,\hat{t}\right)$ and map them onto the space-time extremals of the $(2n)$th oscillator with coordinates $\left(y_{2n},t\right)$.

With these mapping in hand, all of the analyses presented in Parts II - IV can then be extended to the members of the hierarchy including the analysis of the corresponding A_{2n}. It will be tedious.

6. Conclusions

A quantum mechanical Green's function $G\left(y_b,t_b;y_a,t_a\right)$ (1.3) for the quartic oscillator is presented where $S_{qo}\left(y_b,t_b;y_a,t_a\right)$ is given by (3.16)-(3.18) and $A_{qo}\left(t_b,t_a\right)$ is given by (4.9)-(4.10) with $y_a = 0$, $t_a = 0$. See the end of Part V. This result is built upon two previous papers: first [1], detailing the linearization of the quartic oscillator (qo) to the harmonic oscillator (ho); second [2], the integration of the classical action function for the quartic oscillator. Here an equivalent form for the quartic oscillator action function $S_{qo}\left(y_b,t_b;y_a,t_a\right)$ in terms of harmonic oscillator variables is given by (3.10)-(3.12) in order to facilitate the derivation of the quartic oscillator Green's Function amplitude. To implement (1.1) set $y_b = y_2$, $t_o = t_z$, $y_a = y_1$, $t_a = t_1$ in the above.

The linearization map originally given in [1] and described in parts in Part II and IV of this paper describes the difference between our approach and that of R.C. Santos, J. Santos and J.A.S. Lima [7] who first reported the possibility of linearization of the *qo* to the *ho*.

References

[1] Anderson, R.L. (2010) An Invertible Linearization Map for the Quartic Oscillator. *Journal of Mathematical Physics*, **51**, Article ID: 122904. http://dx.doi.org/10.1063/1.3527070

[2] Anderson, R.L. (2013) Integration of the Classical Action for the Quartic Oscillator in 1+1 Dimensions. *Applied Mathematics*, **4**, 117-122. http://dx.doi.org/10.4236/am.2013.410A3014

[3] Schiff, L.I. (1955/1968) Quantum Mechanics. 3rd Edition, McGraw-Hill, New York, p. 306.

[4] Dirac, P.A.M. (1958) The Principle of Quantum Mechanics. 4th Edition, Oxford at the Clarendon Press, p. 128.

[5] Brown, L.M. (Ed.) (2005) Feynman's Thesis—A New Approach to Quantum Theory. World Scientific, Singapore.

[6] Feynman, R.P. and Hibbs, A.R. (1965) Quantum Mechanics and Path Integrals. McGraw-Hill, New York.

[7] Santos, R.C., Santos, J. and Lima, J.A.S. (2006) Hamilton-Jacobi Approach for Power-Law Potentials. *Brazilian Journal of Physics*, **36**, 1257-1261. http://dx.doi.org/10.1590/S0103-97332006000700024