Applied Mathematics, 2016, 7, 1498-1503
Published Online August 2016 in SciRes. http://www.scirp.org/journal/am
http://dx.doi.org/10.4236/am.2016.714129

Scientific Research Publishing

Dirichlet Averages, Fractional Integral Operators and Solution of Euler-Darboux Equation on Hölder Spaces

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Received 2 June 2016; accepted 14 August 2016; published 17 August 2016

Abstract

In the present paper, we discuss the solution of Euler-Darboux equation in terms of Dirichlet averages of boundary conditions on Hölder space and weighted Hölder spaces of continuous functions using Riemann-Liouville fractional integral operators. Moreover, the results are interpreted in alternative form.

Keywords

Fractional Integral Operators, Dirichlet Averages, Hölder Space

1. Introduction

The subject of Dirichlet averages has received momentum in the last decade of 20th century with reference to the solution of certain partial differential equations. Not much work has been registered in this area of Applied Mathematics except some papers devoted to evaluation of Dirichlet averages of elementary functions as well as higher treanscendental functions interpreting the results in more general special functions. The present paper is ventured to give the interpretation of solution of a typical partial differential equation and prove its inclusion properties with respect to Hölder spaces. The Euler-Darboux equation (ED-equation) is a certain kind of degenerate hyperbolic partial differential equation of the type (see Nahušev [1]),

$$y^m \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0, \quad (m > 0). \tag{1}$$

Saigo [2]-[4] considered and studied the ED-equation given by

How to cite this paper: Vyas, D.N. (2016) Dirichlet Averages, Fractional Integral Operators and Solution of Euler-Darboux Equation on Hölder Spaces. *Applied Mathematics*, **7**, 1498-1503. http://dx.doi.org/10.4236/am.2016.714129

$$\frac{\partial^2 u}{\partial x \partial y} - \frac{\beta}{x-y}\frac{\partial u}{\partial x} + \frac{\alpha}{x-y}\frac{\partial u}{\partial y} = 0, \quad (\alpha > 0, \beta > 0, \alpha + \beta < 1), \tag{2}$$

which implies the Equation (1) for $\alpha = \beta = \dfrac{m}{(2m+4)}$ or some other degenerate hyperbolic equations described by characteristic coordinates. The boundary conditions used for the solution of Equation (2) are

$$u(x,x) = \tau(x) \text{ and } \lim (y-x)^{\alpha+\beta}\left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x}\right) = \nu(x). \tag{3}$$

The solution of ED-Equation (2), due to Saigo [22], is given by

$$\begin{aligned} u(x,y) &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 \tau\left[y+(x-y)t\right]t^{\beta-1}(1-t)^{\alpha-1}\,\mathrm{d}t \\ &+ \frac{\Gamma(1-\alpha-\beta)}{2\Gamma(1-\alpha)\Gamma(1-\beta)}(y-x)^{1-\alpha-\beta}\int_0^1 \nu\left[y+(x-y)t\right]t^{-\alpha}(1-t)^{-\beta}\,\mathrm{d}t, \end{aligned} \tag{4}$$

where x and y are restricted in the domain $\xi = \{0 < x < y < 1\}$.

Srivastava and Saigo [5] evaluated the results on multiplication of fractional integral operators and the solution of ED-equation. Deora and Banerji [6] represented the solution of Equation (2) in terms of Dirichlet averages of boundary condition functions given in (3) as follows

$$u(x,y) = \Theta(\beta,\alpha; y,x) + \frac{(y-x)^{1-\alpha-\beta}}{2\Gamma(1-\alpha-\beta)}\Xi(1-\beta, 1-\alpha, y, x), \tag{5}$$

where $\Theta(.)$ and $\Xi(.)$ denote the single Dirichlet averages of boundary functions $\tau(.)$ and $\nu(.)$, respectively.

Kilbas *et al*. [7] studied the solution of ED-equation on Hölder Space $H^\lambda[0,1]$ or simply H^λ as well as on weighted Hölder Space of continuous functions. In the present paper we discuss the Dirichlet averages on Hölder Space via right-sided Riemann-Liouville fractional integral operators and prove the solution of Equation (2) to be justified on such spaces. In what follows are the preliminaries and definitions related to fractional integral operators, Dirichlet averages, and Hölder spaces of continuous functions.

2. Hölder Spaces

For $\lambda > 0$ and a finite interval $[a,b]$ we denote by $H^\lambda = H^\lambda[a,b]$ the space of Hölder function on $[a,b]$,

$$f(x) \in H^\lambda (\lambda = m + \sigma; m = 0,1,2,\cdots; 0 < \sigma \le 1), \tag{6}$$

if $f(x) \in C^m[a,b]$ and $f^{(m)}(x) \in H^\sigma$, *i.e.*, $f(x)$ is m-times differentiable function and its m-th derivative is continuous and satisfies the inequality

$$\left| f^{(m)}(x_1) - f^{(m)}(x_2) \right| \le A_\lambda |x_1 - x_2|^\sigma, \tag{7}$$

where $A_\lambda > 0$ for any $x_1, x_2 \in [a,b]$.

Let $a = 0$ and $b = 1$, *i.e.*, $H^\lambda[0,1](0 < \lambda \le 1)$ be the space of Hölder continuous functions and

$$H_o^\lambda[0,1] = \left\{\varphi(x) \in H^\lambda[0,1] : \phi(0) = \varphi(1) = 0\right\}. \tag{8}$$

Then we denote by $H_o^\lambda(\rho;[0,1])$ the space of functions such that $\rho(x)\varphi(x) \in H_o^\lambda[0,1]$, where $\rho(x) \ge 0$.

3. Dirichlet Averages

Carlson [8] introduced the concept of connecting elementary functions with higher transcendental functions using averaging technique. The Dirichlet average is a certain kind of integral average with respect to Dirichlet measure, which in Statistics called as beta distribution of several variables. One may refer to Banerji and Deora [9], Deora and Banerji [10] [11], Deora, Banerji and Saigo [12], Gupta and Agrawal [13] [14], Kattuveettil [15], Prabhakar [16], Chena Ram *et al*. [17], Vyas [18], Vyas and Banerji [19] [20], Vyas, Banerji and Saigo [21].

Standard Simplex: Denote the standard simplex in $R^n, n \geq 1$ by

$$E = E_n = \left(u_1, \cdots, u_n : u_1 \geq 0, \cdots, u_n \geq 0; u_1 + \cdots + u_n = 1 \text{ or } \sum_{i=1}^{n} u_i = 1 \right). \quad (9)$$

Beta Function of *k*-variables: Let $C_>^k$ denotes the k^{th} cartesian product of open right half plane and $E = E_n$ is the standard simplex in R^n. The beta function of *k*-variables can be expressed as

$$B(b_1, \cdots, b_k) = \int_E u_1^{b_1-1} \cdots u_{k-1}^{b_{k-1}-1} (1 - u_1 - \cdots - u_{k-1})^{b_k-1} \mathrm{d}u_1 \cdots \mathrm{d}u_{k-1}. \quad (10)$$

Dirichlet Measure: The complex measure μ_b, defind on *E* by

$$\mathrm{d}\mu_b(u) = [B(b)]^{-1} u_1^{b_1-1} \cdots u_{k-1}^{b_{k-1}-1} (1 - u_1 - \cdots - u_{k-1})^{b_k-1} \mathrm{d}u_1 \cdots \mathrm{d}u_{k-1} \quad (11)$$

for $b = (b_1, \cdots, b_k) \in C_>^k, k \geq 2$, is called the Dirichlet measure. Particularly, for $k = 2, b = (\alpha, \beta)$, we write by using (3), the following:

$$\mathrm{d}\mu_{\alpha,\beta}(u) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha-1} (1-u)^{\beta-1} \mathrm{d}u, \quad \Re(\alpha), \Re(\beta) \geq 0 \quad (12)$$

Dirichlet Average: Let Ω be a convex set in $C_>$. Let $z = (z_1, \cdots, z_k) \in \Omega^k, k \geq 2$ and $u.z$ denotes a convex linear combination of $z_1, \cdots, z_k$. Then the Dirichlet average of a holomorphic function *f* is defined by (See Carlson [22])

$$F(b, z) = \int_E f(u.z) \mathrm{d}\mu_b(u), \quad (13)$$

where $b = (b_1, \cdots, b_k)$ denotes the parameters. The convex combination is given by

$$u.z = \sum_{i=1}^{k} u_i z_i, \quad u_k = 1 - \sum_{i=1}^{k-1} u_i.$$

Particularly,when $k = 2$, the Dirichlet average, so extracted out of (5), is called the single Dirichlet average of *f* over the line segment from 0 to 1. It is expressed as

$$F(\alpha, \beta; x, y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 f[ux + (1-u)y] u^{\alpha-1} (1-u)^{\beta-1} \mathrm{d}u, \quad (14)$$

where $b = (\alpha, \beta), z = (x, y)$ and $E = [0,1]$.

If we consider the continuous function $\varphi(z)$ in Hölder Space $H^\lambda[0,1]$ and $x, y \in \xi$, then without the loss of characteristics of such spaces, the Dirichlet average of $\varphi(z)$ is defined by

$$\Phi(b, z) = \int_E \varphi(u.z) \mathrm{d}\mu_b(u), \quad (15)$$

and for $k = 2$, $b = (\beta, \beta')$ and $z = (x, y)$. The equation analogous to (11), is expressed as

$$\Phi(\beta, \beta'; x, y) = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')} \int_0^1 \varphi[ux + (1-u)y] u^{\beta-1} (1-u)^{\beta'-1} \mathrm{d}u, \quad (16)$$

where $\Phi(\beta, \beta'; x, y)$ denotes the single Dirichlet average of $\varphi(z)$ in two variables *x* and *y* in $[0,1]$.

4. Fractional Integral Operators

Fractional calculus is the generalization of ordinary *n*-times iterated integrals and n^{th} derivatives of continuous functions to that of any arbitrary order real or complex. The most commonly used definition of fractional integral operators of order α is due to Riemann-Liouville. A detailed account of fractional calculus is given in Samko *et al*. [23] and the applications of it are elaborated in Hilfer [24] and Podlubney [25]. Vyas [26] interpreted the angle of collision occurring in the study of transport properties of Noble gases at low density configuration in terms of Fractional Integral Operators.

Let $H^\lambda[0,1](0 < \lambda \leq 1)$ be the Hölderian class of continuous functions and the weighted Hölder space be defined in (8). Then the right-sided Riemann-Liouville fractional integral operators are defined by

$$\left(I_{0+}^{\alpha}\varphi\right)(x)=\frac{1}{\Gamma(\alpha)}\int_0^x\frac{\varphi(t)}{(x-t)^{1-\alpha}}\mathrm{d}t,\ \ (0<x<1,\alpha>0),\tag{17}$$

$$\left(D_{0+}^{\alpha}\varphi\right)(x)=\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\left(I_{0+}^{n-\alpha}\varphi\right)(x),\ \ (0<x<1,\alpha>0,n=[\alpha]+1),\tag{18}$$

$$\left(I_{1-}^{\alpha}\varphi\right)(x)=\frac{1}{\Gamma(\alpha)}\int_x^1\frac{\varphi(t)}{(t-x)^{1-\alpha}}\mathrm{d}t,\ \ (0<x<1,\alpha>0),\tag{19}$$

$$\left(D_{1-}^{\alpha}\varphi\right)(x)=\left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\left(I_{1-}^{n-\alpha}\varphi\right)(x),\ \ (0<x<1,\alpha>0,n=[\alpha]+1).\tag{20}$$

Proposition 1: Let $0<\alpha<1, 0<\lambda<1, \lambda+\alpha<1$ and $\rho(x)=x^{\mu}$ with $0\le\mu<\lambda+\alpha+1$. Then for $\varphi(x)\in H_o^{\lambda}(\rho;[0,1])$, we have

$$\left(I_{0+}^{\alpha}\right)(x)\in H_o^{\lambda+\alpha}\left(x^{\mu};[0,1]\right).\tag{21}$$

Proposition 2: Let $0<\alpha<1, 0<\lambda<1, \lambda-\alpha<1$ and $\rho(x)=x^{\mu}$ with $0\le\mu<\lambda-\alpha+1$. Then for $\varphi(x)\in H_o^{\lambda}(\rho;[0,1])$, we have

$$\left(I_{0+}^{\alpha}\right)(x)\in H_o^{\lambda-\alpha}\left(x^{\mu};[0,1]\right).\tag{22}$$

Generalization of fractional integral operators is due to Saigo [27]. Let α,β and η be real numbers and $F(a,b;c;z)$ be the Gauss' hypergeometric function. One may refer Erdélyi [28] [29] and Slater [30]. Then the fractional integral operator involving Gauss' hypergeometric function on Hölder space is defined by

$$\left(I_{0+}^{\alpha,\beta,\eta}\varphi\right)(x)=\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}F\left(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x}\right)\varphi(t)\mathrm{d}t,\ \ (0<x<1,\alpha>0)\tag{23}$$

$$\left(I_{0+}^{\alpha,\beta,\eta}\varphi\right)(x)=\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\left(I_{0+}^{\alpha+n,\beta-n,\eta-n}\varphi\right)(x),\ \ (0<x<1,\alpha>0,n=[-\alpha]+1),\tag{24}$$

$$\left(I_{1-}^{\alpha,\beta,\eta}\varphi\right)(x)=\frac{(1-x)^{-\alpha-\beta}}{\Gamma(\alpha)}\int_x^1(t-x)^{\alpha-1}F\left(\alpha+\beta,-\eta;\alpha;1-\frac{t-x}{1-x}\right)\varphi(t)\mathrm{d}t,\ \ (0<x<1,\alpha>0)\tag{25}$$

$$\left(I_{1-}^{\alpha,\beta,\eta}\varphi\right)(x)=\left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\left(I_{1-}^{\alpha+n,\beta-n,\eta-n}\varphi\right)(x),\ \ (0<x<1,\alpha>0,n=[-\alpha]+1).\tag{26}$$

Proposition 3: Let $0<-\alpha<\lambda\le 1$ and $\eta>\beta-1$. Then for $\varphi(x)\in H_o^{\lambda}(\rho;[0,1])$, we have

$$x^{\beta}\left(I_{0+}^{\alpha,\beta,\eta}\varphi\right)(x),\ \ (1-x)^{\beta}\left(I_{1-}^{\alpha,\beta,\eta}\varphi\right)(x)\in H_o^{\lambda+\alpha}\left(\rho;[0,1]\right).\tag{27}$$

Proposition 4: Let $0<-\alpha<\lambda\le 1$ and $\beta<\min[0,\eta+1]$. Then for $\varphi(x)\in H_o^{\lambda}(\rho;[0,1])$, we have

$$x^{\beta}\left(I_{0+}^{\alpha,\beta,\eta}\varphi\right)(x),\ \ (1-x)^{\beta}\left(I_{1-}^{\alpha,\beta,\eta}\varphi\right)(x)\in H_o^{\min[\lambda+\alpha,-\beta]}\left(\rho;[0,1]\right).\tag{28}$$

By setting $\beta=-\alpha$, the generalized fractional integral operators defined in (23) to (26) reduce to right-sided Riemann-Liouville fractional integral operators defined in (17) to (20).

5. Main Results

Theorem 1: Let $\varphi(x)\in H^{\lambda}[0,1]$, $b=(\beta,\beta'), z=(x,y)(0<x<y<1)$. Then the single Dirichlet average of $\varphi(x)$ is expressed as

$$\Phi(\beta,\beta';x,y)=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)}(y-x)^{1-\beta-\beta'}\left(I_{0+}^{\beta'}(y-t)^{\beta-1}\varphi\right)(y-x),\tag{29}$$

where $I_{0+}^{\beta'}(.)$ is the right-sided Riemann-Liouville fractional integral operator defined in (17).

Proof: Using Equation (16), we write

$$\Phi(\beta,\beta';x,y)=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\int_0^1\varphi\left[ux+(1-u)y\right]u^{\beta-1}(1-u)^{\beta'-1}\,\mathrm{d}u \tag{30}$$

Using the transformation $u=\frac{t}{y-x}$ in (30) and adjusting the terms involved, we obtain

$$\Phi(\beta,\beta';x,y)=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}(y-x)^{1-\beta-\beta'}\int_0^{y-x}\varphi(y-t)t^{\beta-1}\left[(y-x)-t\right]^{\beta'-1}\mathrm{d}u, \tag{31}$$

which upon using (17), can be expressed as

$$\Phi(\beta,\beta';x,y)=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)}(y-x)^{1-\beta-\beta'}\left(I_{0+}^{\beta'}t^{\beta-1}\varphi(y-t)\right)(y-x), \tag{32}$$

which, for $y=1$, can also be put in the form

$$\Phi(\beta,\beta';x,1)=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)}\left(\wp_{0+}^{\beta,\beta'}t^{\beta-1}\varphi(y-t)\right)(1-x), \tag{33}$$

where $\left(\wp_{0+}^{\beta,\beta'}\varphi\right)(x)$ denotes the new fractional operator defined by

$$\left(\wp_{0+}^{\beta,\beta'}\varphi\right)(x)=(1-x)^{1-\beta-\beta'}\left[I_{0+}^{\beta'}\varphi\right](x). \tag{34}$$

Owing to the proposition 1 to proposition 4 we conclude the proof of theorem 1.

Corollary 1: If $\varphi\in H^{\lambda}[0,1]$ and restrictions on parmeters hold true, then for $y=1$

$$\Phi(\beta,\beta';x,1)\in H^{\lambda+\beta'}(\rho;[0,1]). \tag{35}$$

Proof: Invoking the proposition 1 and using the result (32), we find that the fractional integral representation of single Dirichlet average of $\varphi(z)$, for $y=1$ gives rise to the result (35). This justifies that the Dirichlet averages, so evaluated, belong to the Hölderian class.

Theorem 2: Let $\Theta(\beta,\alpha;y,x)$ and $\Xi(1-\beta,1-\alpha,y,x)$ denote the Dirichlet averages of boundary functions $\tau(.)$ and $\nu(.)$, respectively associatd with the ED-Equation (2) and if $u(x,y)$ denotes the solution of ED-equation, given by (5), in terms of $\Theta(.)$ and $\Xi(.)$. Then, for $y=1$, the solution belongs to the Hölderian class $H^{\lambda+\beta}(\rho;[0,1])$.

Proof: Using Equation (5), Theorem 1 and the Corollary 1, theorem 2 can be proved easily under the proposition 4.

Acknowledgements

The author is indebted to P. K. Banerji, Jodhpur, India for fruitful discussions during the preparation of this paper. Financial support under Technical Education Quality Improvement Programme (TEQIP)-II, a programme of Ministry of Human Resource Development, Government of India is highly acknowledged. Author is also thankful to worthy refree for his/her valuable suggestions upon improvement.

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