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Existence of Positive Solutions to Semipositone Fractional Differential Equations

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Abstract

In this paper, by means of constructing a special cone, we obtain a sufficient condition for the existence of positive solution to semipositone fractional differential equation.

Keywords

Fractional Differential Equations, Boundary Value Problems, Positive Solution, Semipositone

1. Introduction

The aim of this paper is to investigate the existence of positive solutions to the semipositone fractional differential equation

$$\begin{cases} D_{0^+}^{\alpha} u(t) + f(t, u(t)) + q(t) = 0, \ t \in (0,1), \\ u(0) = u'(0) = u(1) = 0, \end{cases} \tag{1}$$

where $2 < \alpha \le 3$, $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative of order α which is defined as follows:

$$D_{0^+}^{\alpha} h(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t (t-s)^{n-\alpha-1} h(s) \mathrm{d}s, \ n = [\alpha] + 1,$$

where Γ denotes the Euler gamma function and $[\alpha]$ denotes the integer part of number α, provided that the right side is pointwise defined on $(0, \infty)$, see [1]. Here, by a positive solution to the problem (1), we mean a function $u \in C[0,1]$, which is positive in $(0,1)$, and satisfies (1).

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Fractional differential equations have gained much importance and attention due to the fact that they have been proved to be valuable tools in the modelling of many phenomena in engineering and sciences such as physics, mechanics, economics and biology. In recent years, there exist a great deal of researches on the existence and/or uniqueness of solutions (or positive solutions) to boundary value problems for fractional-order differential equations. Sun [2] studied the existence of positive solutions for the following boundary value problems:

$$\begin{cases} D_{0^+}^{\alpha} u(t) + q(t) f(t, u(t)) = 0, \ t \in (0,1), \\ u(0) = u'(0) = 0, \quad u(1) = \int_0^1 g(s) u(s) \mathrm{d}s, \end{cases}$$

where $2 < \alpha \le 3$, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and $f(t,0) \not\equiv 0$ on $[0,1]$. But paper [2] did not give the results of the existence of positive solution when the nonlinearity can take negative value, *i.e.* semipositone problems.

The purpose of the present paper is to apply the method of varying translation together with the fixed point theorems in cone to discuss (1) without nonnegativity imposed on the nonlinearity. Meanwhile, we also allow the nonlinearity to have many finite singularities on $t \in [0,1]$.

2. Preliminaries and Lemmas

In this section, we present several lemmas that are useful to the proof of our main results. For the forthcoming analysis, we need the following assumptions:

(H_1) $f : (0,1) \times [0,+\infty) \to [0,+\infty)$ is continuous. For any $t \in (0,1)$, $f(t,1) > 0$, there exist constants $r_1 > r_2 > 1$ such that $c^{r_1} f(t,u) \le f(t,cu) \le c^{r_2} f(t,u), \forall 0 \le c \le 1, (t,u) \in (0,1) \times [0,+\infty)$.

(H_2) $q : (0,1) \to (-\infty,+\infty)$ with $q \in L[0,1]$ and $0 < \int_0^1 q_-(s) \mathrm{d}s = R_1$,

$0 < \int_0^1 G(s) [f(s,1) + q_+(s)] \mathrm{d}s < \dfrac{LR_1}{(LR_1+1)^{r_1}}$, where $q_+(s) = \max\{q(s), 0\}$, $q_-(s) = \max\{-q(s), 0\}$,

$L, G(s)$ will be defined in the following text.

In [3], the authors obtained the Green function associated with the problem (1). More precisely, the authors proved the following lemma.

Lemma 2.1 [3]. For any $h \in C[0,1]$, the unique solution of the boundary value problem

$$\begin{cases} D_{0^+}^{\alpha} u(t) + h(t) = 0, \ 0 \le t \le 1, \\ u(0) = u'(0) = u(1) = 0 \end{cases} \tag{2}$$

is given by

$$u(t) = \int_0^1 G(t,s) h(s) \mathrm{d}s, \ t \in [0,1], \tag{3}$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-1}, & 0 \le t \le s \le 1, \end{cases} \tag{4}$$

Lemma 2.2 [4]. The Green function $G(t,s)$ defined by (4) satisfies the inequality

$$p(t) G(s) \le G(t,s) \le G(s), \ \forall t, s \in [0,1], \tag{5}$$

here

$$p(t) = \frac{(1-t)t^{\alpha-1}}{\alpha-1}, G(s) = \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha-1)}, \ \forall t, s \in [0,1].$$

Remark 2.1. A simple computation shows that there exists a constant $L > 0$ such that $G(t,s) \le Lp(t), \forall t, s \in [0,1]$.

Remark 2.2 [5]. If $f(t,u)$ satisfies (H_1), then for any $t\in(0,1)$, $f(t,u)$ is increasing on $u\in[0,+\infty)$ and for any $[\eta_1,\eta_2]\subseteq(0,+\infty)$, $\lim_{u\to+\infty}\min_{t\in[\eta_1,\eta_2]}\frac{f(t,u)}{u}=+\infty.$

Lemma 2.3 [6]. Let X be a real Banach space, Ω be a bounded open subset of X with $\theta\in\Omega$ and $A:\bar{\Omega}\cap P\to P$ is a completely continuous operator, where P is a cone in X.

(i) Suppose that $Au\neq\lambda u, \forall u\in\partial\Omega\cap P, \lambda\geq 1,$ then $i(A,\Omega\cap P,P)=1$.

(ii) Suppose that $Au\not\leq u, \forall u\in\partial\Omega\cap P,$ then $i(A,\Omega\cap P,P)=0$.

Consider the Banach space $X=C[0,1]$ with the usual supremum norm $\|u\|=\sup_{0\leq t\leq 1}|u(t)|$ and define the cone $P=\{x\in X: x(t)\geq p(t)\|u\|, t\in[0,1]\}$. Let $w(t)=\int_0^1 G(t,s)q_-(s)\mathrm{d}s$, then $w(t)$ is the unique solution to (2) for $h(t)=q_-(t)$. Now we first consider the singular nonlinear boundary value problem

$$\begin{cases} D_{0^+}^{\alpha}u(t)+f\left(t,[u(t)-w(t)]^*\right)+q_+(t)=0, & 0\leq t\leq 1,\\ u(0)=u'(0)=u(1)=0, \end{cases} \tag{6}$$

where $[u(t)-w(t)]^*=\max\{u(t)-w(t),0\}$. We have the following Lemma.

Lemma 2.4. If the singular nonlinear boundary value problem (2) has a positive solution $u(t)$ such that $u(t)\geq w(t)$ for any $t\in[0,1]$. Then boundary value problem (1) has a positive solution $v(t)=u(t)-w(t)$.

Proof. In fact, if u is a positive solution to (6) such that $u(t)\geq w(t)$ for any $t\in[0,1]$. Let $v(t)=u(t)-w(t)$, then $v(t)\geq 0, t\in[0,1]$. Since $w(t)$ is the unique solution to (2) for $h(t)=q_-(t)$, for any $t\in[0,1]$, we have $D_{0^+}^{\alpha}(v(t)+w(t))+f(t,v(t))+q_+(t)=0$, which implies that $D_{0^+}^{\alpha}v(t)-q_-(t)+f(t,v(t))+q_+(t)=0$. So $D_{0^+}^{\alpha}v(t)+f(t,v(t))+q(t)=0$. Consequently $v(t)=u(t)-w(t)$ is positive solution to (1). This complete the proof of Lemma 2.4.

For any $u\in X$, define an operator

$$Tu(t)=\int_0^1 G(t,s)\left[f\left(s,[u(s)-w(s)]^*\right)+q_+(s)\right]\mathrm{d}s,\ t\in[0,1]. \tag{7}$$

Since for any fixed $u\in X$, we can choose $0<c<1$ such that $c\|u\|<1$. Note that $c[u(t)-w(t)]^*\leq cu(t)\leq c\|u\|<1,$ so by (H_1), we have

$$f\left(t,[u(t)-w(t)]^*\right)\leq\left(\frac{1}{c}\right)^{r_1}f\left(t,c[u(t)-w(t)]^*\right)\leq c^{r_2-r_1}\|u\|^{r_2}f(t,1).$$

Consequently, for any $t\in[0,1]$, we have

$$\begin{aligned} &\int_0^1 G(t,s)\left[f\left(s,[u(s)-w(s)]^*\right)+q_+(s)\right]\mathrm{d}s\\ &\leq\int_0^1 G(s)\left[f\left(s,[u(s)-w(s)]^*\right)+q_+(s)\right]\mathrm{d}s\\ &\leq\int_0^1 G(s)\left[c^{r_2-r_1}\|u\|^{r_2}f(s,1)+q_+(s)\right]\mathrm{d}s\\ &\leq\left(c^{r_2-r_1}\|u\|^{r_2}+1\right)\int_0^1 G(s)\left[f(s,1)+q_+(s)\right]\mathrm{d}s. \end{aligned} \tag{8}$$

Therefore, the operator T is well defined and $T:X\to X$.

Lemma 2.5. Assume that (H_1), (H_2) hold. Then $T:P\to P$ is a completely continuous operator.

Proof. For any $u\in P$, in view of (2) we conclude that

$$\begin{aligned} Tu(t)&=\int_0^1 G(t,s)\left[f\left(s,[u(s)-w(s)]^*\right)+q_+(s)\right]\mathrm{d}s\\ &\geq p(t)\int_0^1 G(s)\left[f\left(s,[u(s)-w(s)]^*\right)+q_+(s)\right]\mathrm{d}s,\ t\in[0,1]. \end{aligned}$$

Whence, it follows from (8) that $Tu(t)\geq\|Tu\|p(t)$, which implies $T(P)\subseteq P$.

Next we show that $T: P \to P$ is continuous. Suppose $\{u_m\} \subseteq P, u_0 \in P$, and $\lim_{m\to+\infty} u_m = u_0$. Then, there exists a constant $M > 0$ such that $\|u_m\| \le M, m = 0,1,2,\cdots$. Since for any $t \in [0,1]$,
$[u_m(s) - w(s)]^* \le u_m(s) \le \|u\| \le M \le M+1$, by Remark 2.2, we have

$$\begin{aligned} f\left(s,[u_m(s)-w(s)]^*\right)+q_+(s) &\le f(s,M+1)+q_+(s) \\ &\le (M+1)^{r_1} f(s,1)+q_+(s) \\ &\le \left[(M+1)^{r_1}+1\right]\left(f(s,1)+q_+(s)\right). \end{aligned} \tag{9}$$

Thus, we have

$$\begin{aligned} &\int_0^1 G(t,s)\left[f\left(s,[u_m(s)-w(s)]^*\right)+q_+(s)\right]\mathrm{d}s \\ &\le \left[(M+1)^{r_1}+1\right]\int_0^1 G(s)\left(f(s,1)+q_+(s)\right)\mathrm{d}s, \end{aligned}$$

and $\lim_{m\to+\infty} f\left(s,[u_m(s)-w(s)]^*\right) = f\left(s,[u_0(s)-w(s)]^*\right)$. It follows from the Lebesgue control convergence theorem that

$$\begin{aligned} \lim_{m\to+\infty}\|Tu_m - Tu_0\| &= \sup_{t\in[0,1]}\left|Tu_m(t)-Tu_0(t)\right| \\ &\le \lim_{m\to+\infty}\int_0^1 G(s)\left|f\left(s,[u_m(s)-w(s)]^*\right)-f\left(s,[u_0(s)-w(s)]^*\right)\right|\mathrm{d}s \\ &= 0, \end{aligned}$$

which implies $T: P \to P$ is continuous.

In what follows, we need to prove that $T: P \to P$ is relatively compact.

Let $D \subseteq P$ be any bounded set. Then there exists a constant $M_1 > 0$ such that $\|u\| \le M_1$ for any $u \in D$. Similarly as (9), for any $u \in D, t \in [0,1]$ we have

$$f\left(s,[u(s)-w(s)]^*\right)+q_+(s) \le \left[(M_1+1)^{r_1}+1\right]\left(f(s,1)+q_+(s)\right). \tag{10}$$

Consequently

$$\begin{aligned} |Tu(t)| &= \int_0^1 G(t,s)\left[f\left(s,[u(s)-w(s)]^*\right)+q_+(s)\right]\mathrm{d}s \\ &\le \int_0^1 G(s)\left[(M_1+1)^{r_1}+1\right]\left(f(s,1)+q_+(s)\right)\mathrm{d}s \\ &\le \left[(M_1+1)^{r_1}+1\right]\int_0^1 G(s)\left(f(s,1)+q_+(s)\right)\mathrm{d}s \\ &< +\infty. \end{aligned} \tag{11}$$

Therefore $T(D)$ is uniformly bounded.

Now we show that $T(D)$ is equicontinuous on $[0,1]$. For any $u \in D, t_1, t_2 \in [0,1]$, by (9), (11) and the Lebesgue control convergence theorem, and noticing the continuity of $G(t,s)$, we have

$$\begin{aligned} |Tu(t_1)-Tu(t_2)| &\le \int_0^1 |G(t_1,s)-G(t_2,s)|\left[f\left(s,[u(s)-w(s)]^*\right)+q_+(s)\right]\mathrm{d}s \\ &\le \left[(M_1+1)^{r_1}+1\right]\int_0^1 |G(t_1,s)-G(t_2,s)|\left[f(s,1)+q_+(s)\right]\mathrm{d}s \\ &\to 0, (t_1 \to t_2). \end{aligned} \tag{12}$$

Thus, $T(D)$ is equicontinuous on [0,1]. The Arezlà-Ascoli Theorem guarantees that $T(D)$ is relatively compact set. Therefore $T: P \to P$ is completely continuous operator.

Lemma 2.6. Let $\Omega_1 = \{u \in P : \|u\| < LR_1\}$, then $i(T, \Omega_1, P) = 1$.

Proof. Assume that there exists $\mu \geq 1, z_0 \in \partial\Omega_1$ such that $\mu z_0 = Tz_0$. Then $z_0 = \frac{1}{\mu}Tz_0$ and $0 < \frac{1}{\mu} \leq 1$. Thus we have

$$LR_1 = \|z_0\| = \sup_{t\in[0,1]}\left|\int_0^1 G(t,s)\left[f\left(s,\left[z_0(s)-w(s)\right]^*\right)+q_+(s)\mathrm{d}s\right]\right|$$
$$\leq (LR_1+1)^{\eta_1}\int_0^1 G(s)\left[f(s,1)+q_+(s)\right]\mathrm{d}s.$$

This contradiction shows that $i(T,\Omega_1,P)=1$.

Lemma 2.7. There exists a constant $R_2 > LR_1$ such that $i(T,\Omega_2,P)=0$, where $\Omega_2 = \{u\in P:\|u\|<R_2\}$.

Proof. Choose constants η_1,η_2 and N such that

$$[\eta_1,\eta_2]\subseteq(0,1), N > 2\left[\min_{t\in[\eta_1,\eta_2]} p(t)\right]^{-2}\left[\int_{\eta_1}^{\eta_2}G(s)\right]^{-1}.$$

From Remark (2.2), there exists $\bar{R} > 2LR_1$, such that

$$f(t,u)\geq Nu, t\in[\eta_1,\eta_2], u\geq\bar{R}. \tag{13}$$

Let $R_2 \geq \max\left\{\frac{2\bar{R}}{\min_{t\in[\eta_1,\eta_2]} p(t)},\bar{R}\right\}$. Obviously, $R_2 > \bar{R} > 2LR_1$. Now we show that $u \ngeq Tu, u\in\partial\Omega_2$. In fact, otherwise, there exists $y_1\in\partial\Omega_2$ such that $y_1 \geq Ty_1$. By (2), for any $t\in[\eta_1,\eta_2]$, we have

$$y_1(t)-w(t)\geq y_1(t)-Lp(t)\int_0^1 q_-(s)\mathrm{d}s = y_1(t)-Lp(t)R_1$$
$$\geq y_1(t)-\frac{y_1(t)}{\|y_1\|}LR_1 = y_1(t)-\frac{LR_1}{R_2}y_1(t)$$
$$\geq\frac{1}{2}y_1(t)\geq\frac{1}{2}p(t)\|y_1(t)\|\geq\frac{1}{2}R_2\min_{t\in[\eta_1,\eta_2]}p(t)\geq\bar{R}$$

So

$$R_2\geq y_1(t)\geq Ty_1(t)=\int_0^1 G(t,s)\left[f\left(s,\left[y_1(s)-w(s)\right]^*\right)+q_+(s)\right]\mathrm{d}s$$
$$=\int_0^1 G(t,s)\left[f(s,y_1(s)-w(s))+q_+(s)\right]\mathrm{d}s$$
$$\geq\int_{\eta_1}^{\eta_2}G(t,s)\left[f(s,y_1(s)-w(s))+q_+(s)\right]\mathrm{d}s$$
$$\geq\int_{\eta_1}^{\eta_2}G(t,s)f(s,y_1(s)-w(s))\mathrm{d}s$$
$$\geq\int_{\eta_1}^{\eta_2}G(t,s)N\left[y_1(s)-w(s)\right]\mathrm{d}s$$
$$\geq\frac{1}{2}N\min_{t\in[\eta_1,\eta_2]}p(t)R_2\int_{\eta_1}^{\eta_2}P(t)G(s)\mathrm{d}s.$$

Consequently, $R_2\geq\frac{1}{2}N\left[\min_{t\in[\eta_1,\eta_2]}p(t)\right]^2 R_2\int_{\eta_1}^{\eta_2}G(s)\mathrm{d}s$. That is $N\leq 2\left[\min_{t\in[\eta_1,\eta_2]}p(t)\right]^{-2}\left[\int_{\eta_1}^{\eta_2}G(s)\right]^{-1}$. This contradiction shows that $i(T,\Omega_2,P)=0$.

3. Main Results

Theorem 3.1. Suppose that (H_1), (H_2) hold. Then, the boundary value problems (1) has at least one positive solution $z_0(t)$, and exists a constant $k>0$ such that $z_0(t)\geq kp(t), t\in[0,1]$.

Proof of Theorem 3.1. Applying Lemma 2.6 and Lemma 2.7 and the definition of the fixed point index, we have $i\left(T,\Omega_{R_2}\setminus\bar{\Omega}_{R_1},P\right)=-1$. Thus T has a fixed point $z_0(t)$ in $\Omega_{R_2}\setminus\bar{\Omega}_{R_1}$ with $R_1<\|z_0\|<R$. Since $R_1<\|z_0\|$, we have

$$z_0(t)-w(t)\geq\|z_0\|p(t)-\int_0^1 G(t,s)q_-(s)\,\mathrm{d}s\geq\|z_0\|p(t)-Lp(t)\int_0^1 q_-(s)\,\mathrm{d}s$$
$$\geq(\|z_0\|-LR_1)p(t)=kp(t)\geq 0,\ t\in[0,1].$$

Let $y_0(t)=z_0(t)-w(t)$. It follows from Lemma (2.4) that $y_0(t)$ is a positive solution to boundary value problem (1), and there exists a constant $k>0$ such that $y_0(t)\geq kp(t), t\in[0,1]$.

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