

# A New Second Order Numerical Scheme for Solving Forward Backward Stochastic Differential Equations with Jumps

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Received 1 April 2016; accepted 26 July 2016; published 29 July 2016

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## Abstract

In this paper, we propose a new second order numerical scheme for solving backward stochastic differential equations with jumps with the generator  $f = r(t, x_t, y_t) + h(t)z_t + g(t)\Gamma_t$  linearly depending on  $z_t$ . And we theoretically prove that the convergence rates of them are of second order for solving  $y_t$  and of first order for solving  $z_t$  and  $\Gamma_t$  in  $L^p$  norm.

## Keywords

Numerical Scheme, Error Estimates, Backward Stochastic Differential Equations

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## 1. Introduction

Bismut (1973) studied the existence of the linear backward stochastic differential equation, the results could be regarded as a promotion of a famous Girsanov theorem. The existence and uniqueness of solutions for nonlinear backward stochastic differential equations (BSDEs) were first proved by Pardoux and Peng (1990). Since then, BSDEs have been extensively studied by many researchers. In [1], Peng obtained the relation between the backward stochastic differential equation and the parabolic partial differential equation (PDE), and in Peng (1990), the stochastic maximum principle for optimal control problems were based on BSDEs. The applications of BSDEs now cover many scientific fields, such as stochastic control, stock markets, risk measure, turbulence fluid flow, biology, chemical reactions, partial differential equations, and so on. Thus it is very important and useful to obtain solutions of BSDEs for real applications. However, it is often quite difficult to obtain analytic solutions of BSDEs, so computing approximate solutions of BSDEs become highly desired, by using the relation between the BSDE and PDE. As far as we know, there have been very few schemes obtained with second-order

convergence rate, such as [2] [3].

In this paper, we propose a new second order numerical scheme for the solution of forward-backward stochastic differential equations (FBSDE in short) with jumps with the following form

$$\begin{cases} x_t = x_0 + \int_0^t b(s, x_s) ds + \int_0^t \sigma(s, x_s) dW_s + \int_0^t \int_{\mathcal{E}} \beta(s, x_{s-}, e) \tilde{N}_k(ds, de) \\ y_t = g(x_T) + \int_t^T f_s ds - \int_t^T z_s dW_s - \int_t^T \int_{\mathcal{E}} U_s \tilde{N}_k(ds, dz) \end{cases} \quad (1)$$

From [4], we know that the solution  $(x_t, y_t, z_t, \Gamma_t)$  can be represented as

$$\begin{aligned} y_t &= u(t, x_t), \quad z_t = \nabla_x u(t, x_t) \sigma(t, x_t), \\ \Gamma_t &= \int_{\mathcal{E}} \{u(t, x_t + \beta(t, x_t, e)) - u(t, x_t)\} \rho(e) \lambda(de), \quad t \in [0, T], \end{aligned} \quad (2)$$

where the vector function  $u(t, x)$  is the classical solution of the following parabolic differential equation (PDE) of the form

$$\begin{cases} \mathcal{L}u(t, x) + f(t, x, u(t, x), \nabla_x u(t, x) \sigma(t, x), \mathcal{I}[u](t, x)) = 0 \\ u(T, x) = g(x) \end{cases} \quad (3)$$

where  $\nabla_x u$  denotes the gradient of  $u$  with respect to the space variable  $x$ ,

$$\begin{aligned} \mathcal{L}u(t, x) &:= \frac{\partial u}{\partial t}(t, x) \nabla_x u(t, x) b(x) + \frac{1}{2} \sum_{i,j=1}^d (\sigma \sigma^*(x)) \frac{\partial^2 u}{\partial x^i \partial x^j}(t, x) \\ &\quad + \int_{\mathcal{E}} \{u(t, x + \beta(x, e)) - u(t, x) - \nabla_x u(t, x) \beta(x, e)\} \rho(e) \lambda(de), \\ \mathcal{I}[u](t, x) &:= \int_{\mathcal{E}} \{u(t, x + \beta(x, e)) - u(t, x)\} \rho(e) \lambda(de). \end{aligned} \quad (4)$$

## 2. Preliminaries and Notation

Let  $T$  be a fixed positive number and  $\{\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{0 \leq t \leq T}\}$  be a complete, filtered probability space on which is defined a standard Brownian motion  $W_t$ , such that  $\{\mathcal{F}_t\}_{0 \leq t \leq T}$  is the natural filtration of the Brownian motion  $W_t$  and all the  $P$ -null sets are augmented to each  $\sigma$ -field  $\mathcal{F}_t$ . Denote by  $L^2$  the set of all  $\mathcal{F}_t$ -adapted and mean-square-integrable processes.

A process  $(x_t, y_t, z_t, \Gamma_t) : [0, T] \times \Omega \rightarrow R^m \times R^{m \times d}$  is called an  $L^2$ -adapted solution of the FBSDE(1) if it's  $\{\mathcal{F}_t\}$ -adapted and  $L^2$ -integrable, and satisfies (1). Under some standard conditions on the functions  $f$  and  $h$ , there is a unique adapted random process.

Now we introduce a new probability space: for  $\{\Lambda_t : t \geq 0\}$  is an exponential martingale and satisfies  $\mathbb{E}_{t_n}^x(\Lambda_t) = 1$ , we define  $\tilde{\mathbb{E}}[X] = \tilde{\mathbb{E}}[X \Lambda_{t_n}^{t_n}]$ . The random processes  $\Lambda_{t_n}^t$ , it is easy to verify that  $\Lambda_{t_n}^t$  is an exponential martingale.

$$\Lambda_0^t = \exp \left\{ \int_0^t h(s) dW_s - \frac{1}{2} \int_0^t h^2(s) ds \right\} \prod_{0 \leq s \leq t} \left[ 1 + \int_{\mathcal{E}} g(s) \rho(e) N_k(\{s\}, de) \right] e^{-\int_0^t \int_{\mathcal{E}} g(s) \rho(e) \pi(de) ds}. \quad (5)$$

Let us first introduce the following lemma.

**Lemma 1.** Given the time partition  $0 \leq t_n \leq t_{n+1} \leq T$ ,  $X$  is a  $\mathcal{F}_t$ -measurable random variable, and satisfies  $\tilde{\mathbb{E}}[X] < \infty$ .

$$\tilde{\mathbb{E}}_{t_n}^{x^n}[X] = \frac{\mathbb{E}_{t_n}^{x^n}[X \Lambda_0^{t_{n+1}}]}{\Lambda_0^{t_n}} = \mathbb{E}_{t_n}^{x^n}[X \Lambda_{t_n}^{t_{n+1}}], \text{ a.s.} \quad (6)$$

We use the following Itô-Taylor approximation to solve the forward SDEs with jumps

$$X^{n+1} = X^n + \sum_{\alpha \in \mathcal{I}_2} I_{\alpha} \left[ f_{\alpha}(t_n, X^n) \right]_{t_n, t_{n+1}}, \quad (7)$$

where

$$I_\alpha [g(\cdot)]_{t_n, \tau} = \begin{cases} g(\tau), l = 0, \alpha = \nu \\ \int_{t_n}^\tau I_{\alpha-} [g(\cdot)]_{t_n, r} dr, l \geq 1, j_l = 0 \\ \int_{t_n}^\tau I_{\alpha-} [g(\cdot)]_{t_n, r} dW_r^{j_l}, l \geq 1, j_l \in 1, 2, \dots, m \\ \int_{t_n}^\tau \int_{\mathcal{E}} I_{\alpha-} [g(\cdot)]_{t_n, r-} \tilde{N}(de, dr), l \geq 1, j_l = -1 \end{cases}, \tag{8}$$

and the coefficient function

$$f_\alpha(t, x, e) = \begin{cases} f(t, x), l(\alpha) = 0 \\ L^{j_l} f_{-\alpha}(t, x, e), l(\alpha) \geq 1, j_l \in 0, 1, \dots, m, \\ L_e^{-1} f_{-\alpha}(t, x, e), l(\alpha) \geq 1, j_l = -1 \end{cases} \tag{9}$$

with

$$\begin{aligned} L^0 f(t, x, e) &= \frac{\partial}{\partial t} f(t, x, e) + \sum_{i=1}^d a^i(t, x) \frac{\partial}{\partial x^i} f(t, x, e) + \frac{1}{2} \sum_{i,l=1}^d \sum_{j=1}^m b^{i,j}(t, x) b^{l,j}(t, x) \frac{\partial^2}{\partial x^i \partial x^l} f(t, x, e), \\ L^k f(t, x, e) &= \sum_{i=1}^d b^{i,k}(t, x) \frac{\partial}{\partial x^i} f(t, x, e), \\ L_e^{-1} f(t, x, e) &= f(t, x + c(t, x, \nu), e) - f(t, x, e), t \in [0, T], x \in R^d, e \in \mathcal{E}^s. \end{aligned} \tag{10}$$

Now we introduce some basic notations.

- $\{\mathcal{F}_t\}_{t \leq T}$ : the  $\sigma$ -field generated by the Brownian motion.
- Throughout this paper, we denote by  $C$  a generic constant depending only on  $T$ , the upper bounds of the derivatives of the functions  $f$ .

### 3. Numerical Schemes for Solving BSDE

From the time interval  $[0, T]$ , we introduce the following time partition:  $0 = t_0 < \dots < t_N = T$ , let  $\Delta t_n = t_{n+1} - t_n$  and  $\max_{0 \leq n \leq N-1} \Delta t_n = h$ . According to (1), it's easy to obtain that for  $0 \leq n \leq N - 1$ ,

$$y_{t_n} = y_{t_{n+1}} + \int_{t_n}^{t_{n+1}} (r(s, x_s, y_s) + h(s)z_s + g(s)\Gamma_s) ds - \int_{t_n}^{t_{n+1}} z_s dW_s - \int_{t_n}^{t_{n+1}} \int_{\mathcal{E}} U_s \tilde{N}_k(ds, dz). \tag{11}$$

From (5) and (11), we have

$$d\Lambda_{t_n}^t = \Lambda_{t_n}^t \left[ \int_{\mathcal{E}} \rho(e) g(t) d\tilde{N}_t + h(t) dW_t \right], \tag{12}$$

$$dy_t = -[r(t, x_t, y_t) + h(t)z_t + g(t)\Gamma_t] dt + z_t dW_t + \int_{\mathcal{E}} U_t d\tilde{N}_t, \tag{13}$$

$$d\Lambda_{t_n}^t y_t = \Lambda_{t_n}^t dy_t + y_t d\Lambda_{t_n}^t + d\Lambda_{t_n}^t dy_t. \tag{14}$$

From (12), (13) and (14), by applying Itô formula to  $\Lambda_{t_n}^t y_t$ , we obtain the equation

$$d\Lambda_{t_n}^t y_t = -\Lambda_{t_n}^t r(t, x_t, y_t) dt + \Lambda_{t_n}^t [z_t + h(t)y_t] dW_t + \Lambda_{t_n}^t \int_{\mathcal{E}} [U_t + y_t \rho(e) g(t)] d\tilde{N}_t. \tag{15}$$

From (15), it is easy to obtain that for  $0 \leq n \leq N - 1$ ,

$$\begin{aligned} y_{t_n}^{t_n, x^n} &= \Lambda_{t_n}^{t_{n+1}} y_{t_{n+1}}^{t_n, x^n} + \int_{t_n}^{t_{n+1}} \Lambda_{t_n}^s r(s, x_s^{t_n, x^n}, y_s^{t_n, x^n}) ds - \int_{t_n}^{t_{n+1}} \Lambda_{t_n}^s [z_s^{t_n, x^n} + h(s)y_s^{t_n, x^n}] dW_s \\ &\quad - \int_{t_n}^{t_{n+1}} \int_{\mathcal{E}} \Lambda_{t_n}^s [U_s^{t_n, x^n} + y_s^{t_n, x^n} \rho(e) g(s)] d\tilde{N}_s, \end{aligned} \tag{16}$$

Taking the conditional mathematical expectation  $\mathbb{E}_{t_n}^x[\cdot]$  on both side of the obtained equation, and by the nature of the conditional mathematical expectation, we deduce

$$y_{t_n}^{t_n, x^n} = \mathbb{E}_{t_n}^{x^n} \left[ \Lambda_{t_n}^{t_{n+1}} y_{t_{n+1}}^{t_n, x^n} \right] + \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{x^n} \left[ \Lambda_{t_n}^s r \left( s, x_s^{t_n, x^n}, y_s^{t_n, x^n} \right) \right] ds. \tag{17}$$

Based on (17), we have

$$y_{t_n}^{t_n, x^n} = \mathbb{E}_{t_n}^{x^n} \left[ \Lambda_{t_n}^{t_{n+1}} y_{t_{n+1}}^{t_{n+1}, x^{n+1}} \right] + \frac{1}{2} \Delta t_n \left( r \left( t_n, x^n, y_{t_n}^{t_n, x^n} \right) + \mathbb{E}_{t_n}^{x^n} \left[ \Lambda_{t_n}^{t_{n+1}} r \left( t_{n+1}, x^{n+1}, y_{t_{n+1}}^{t_{n+1}, x^{n+1}} \right) \right] \right) + \sum_{j=1}^2 R_{yj}^n, \tag{18}$$

where

$$R_{y1}^n = \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{x^n} \left[ \Lambda_{t_n}^t r_t^{t_n, x^n} \right] dt - \frac{1}{2} \Delta t_n \mathbb{E}_{t_n}^{x^n} \left[ \Lambda_{t_n}^{t_{n+1}} r_{t_{n+1}}^{t_n, x^n} \right] - \frac{1}{2} \Delta t_n r_{t_n}^{t_n, x^n}, \tag{19}$$

and

$$R_{y2}^n = \mathbb{E}_{t_n}^{x^n} \left[ \Lambda_{t_n}^{t_{n+1}} \left( y_{t_{n+1}}^{t_n, x^n} - y_{t_{n+1}}^{t_{n+1}, x^{n+1}} \right) \right] + \frac{1}{2} \Delta t_n \mathbb{E}_{t_n}^{x^n} \left[ \Lambda_{t_n}^{t_{n+1}} \left( r_{t_{n+1}}^{t_n, x^n} - r_{t_{n+1}}^{t_{n+1}, x^{n+1}} \right) \right]. \tag{20}$$

According to Lemma 1, the equality  $\mathbb{E}_{t_n}^{x^n} \left[ \Lambda_{t_n}^{t_{n+1}} y_{t_{n+1}}^{t_n, x^n} \right] = \tilde{\mathbb{E}}_{t_n}^{x^n} \left[ y_{t_{n+1}}^{t_n, x^n} \right]$ , we have

$$y_{t_n}^{t_n, x^n} = \tilde{\mathbb{E}}_{t_n}^{x^n} \left[ y_{t_{n+1}}^{t_n, x^n} \right] + \frac{1}{2} \Delta t_n r \left( t_n, x^n, y_{t_n}^{t_n, x^n} \right) + \frac{1}{2} \Delta t_n \tilde{\mathbb{E}}_{t_n}^{x^n} \left[ r \left( t_{n+1}, x^{n+1}, y_{t_{n+1}}^{t_{n+1}, x^{n+1}} \right) \right] + \sum_{j=1}^2 R_{yj}^n, \tag{21}$$

Let  $\Delta W_t = W_t - W_{t_n}$  for  $t_n \leq t \leq t_{n+1}$ . Then  $\Delta W_t$  is a standard Brownian motion with mean zero and variance  $t - t_n$ . Now multiply (11) by  $\Delta W_t$ , taking the conditional mathematical expectation  $\mathbb{E}_{t_n}^{x^n} [\cdot]$  on both sides of the obtained equation, and using the Itô isometric formula, we deduce

$$0 = \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{x^n} \left[ f_s^{t_n, x^n} \Delta W_{t_{n+1}} \right] ds + \mathbb{E}_{t_n}^{x^n} \left[ y_{t_{n+1}}^{t_n, x^n} \Delta W_{t_{n+1}} \right] - \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{x^n} \left[ z_s^{t_n, x^n} \right] ds. \tag{22}$$

From (22) we have,

$$0 = \mathbb{E}_{t_n}^{x^n} \left[ y_{t_{n+1}}^{t_{n+1}, x^{n+1}} \Delta W_{t_{n+1}} \right] - \Delta t_n z_{t_n}^{t_n, x^n} + \sum_{j=1}^2 R_{zj}^n, \tag{23}$$

where

$$\begin{aligned} R_{z1}^n &= \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{x^n} \left[ f_s^{t_n, x^n} \Delta W_{t_{n+1}} \right] ds - \int_{t_n}^{t_{n+1}} \left[ \mathbb{E}_{t_n}^{x^n} \left[ z_s^{t_n, x^n} \right] - z_{t_n}^{t_n, x^n} \right] ds, \\ R_{z2}^n &= \mathbb{E}_{t_n}^{x^n} \left[ \left( y_{t_{n+1}}^{t_n, x^n} - y_{t_{n+1}}^{t_{n+1}, x^{n+1}} \right) \Delta W_{t_{n+1}} \right]. \end{aligned} \tag{24}$$

Let  $\Delta \tilde{N}_{t_{n+1}} = \Delta N_{t_{n+1}} - \lambda \Delta t_n$ , similarly multiplying (11) by  $\Delta \tilde{N}_{t_{n+1}}$  yields

$$0 = \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{x^n} \left[ f_s^{t_n, x^n} \Delta \tilde{N}_{t_{n+1}} \right] ds + \mathbb{E}_{t_n}^{x^n} \left[ y_{t_{n+1}}^{t_n, x^n} \Delta \tilde{N}_{t_{n+1}} \right] - \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{x^n} \left[ \Gamma_s^{t_n, x^n} \right] ds. \tag{25}$$

From (25) we have,

$$0 = \mathbb{E}_{t_n}^{x^n} \left[ y_{t_{n+1}}^{t_{n+1}, x^{n+1}} \Delta \tilde{N}_{t_{n+1}} \right] - \Delta t_n \Gamma_{t_n}^{t_n, x^n} + \sum_{j=1}^2 R_{\Gamma j}^n, \tag{26}$$

where

$$\begin{aligned} R_{\Gamma 1}^n &= \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{x^n} \left[ f_s^{t_n, x^n} \Delta \tilde{N}_{t_{n+1}} \right] ds - \int_{t_n}^{t_{n+1}} \left[ \mathbb{E}_{t_n}^{x^n} \left[ \Gamma_s^{t_n, x^n} \right] - \Gamma_{t_n}^{t_n, x^n} \right] ds, \\ R_{\Gamma 2}^n &= \mathbb{E}_{t_n}^{x^n} \left[ \left( y_{t_{n+1}}^{t_n, x^n} - y_{t_{n+1}}^{t_{n+1}, x^{n+1}} \right) \Delta \tilde{N}_{t_{n+1}} \right]. \end{aligned} \tag{27}$$

Based on (21), (23) and (26), for solving the BSDE (1) we propose the following scheme.

**Scheme 1.** Given  $(x^0, y^N, z^N, \Gamma^N)$ , solve  $(x^{n+1}, y^n, z^n, \Gamma^n)$  backwardly by

$$\begin{cases} x^{n+1} = x^n + \sum_{\alpha \in I_2} I_\alpha \left[ f_\alpha(t_n, x^n, y^n) \right]_{t_n, t_{n+1}}, \end{cases} \quad (28)$$

$$\begin{cases} y^n = \tilde{\mathbb{E}}_{t_n}^{x^n} \left[ y^{n+1} \right] + \frac{1}{2} \Delta t_n f(t_n, x^n, y^n) + \frac{1}{2} \Delta t_n \tilde{\mathbb{E}}_{t_n}^{x^n} \left[ f(t_{n+1}, x^{n+1}, y^{n+1}) \right], \end{cases} \quad (29)$$

$$\begin{cases} \Delta t_n z^n = \mathbb{E}_{t_n}^{x^n} \left[ y^{n+1} \Delta W_{t_{n+1}} \right], \end{cases} \quad (30)$$

$$\begin{cases} \Delta t_n \Gamma^n = \mathbb{E}_{t_n}^{x^n} \left[ y^{n+1} \Delta \tilde{N}_{t_{n+1}} \right]. \end{cases} \quad (31)$$

### 4. Error Estimates

In this section, we will give the error estimates of Scheme 1 proposed in Section 3. Now we introduce the error  $e_y^n = y_{t_n}^{t_n, x^n} - y^n$ ,  $e_z^n = z_{t_n}^{t_n, x^n} - z^n$  and  $e_\Gamma^n = \Gamma_{t_n}^{t_n, x^n} - \Gamma^n$  in  $L^p$  norm, where  $(x_t^{t_n, x^n}, y_t^{t_n, x^n}, z_t^{t_n, x^n}, \Gamma_t^{t_n, x^n})$  is the solution of the FBSDEs (1), and  $(x^{n+1}, y^n, z^n, \Gamma^n)$  is the solution of Scheme 1. For the sake of simplicity, we only consider one-dimensional BSDEs (i.e.,  $m = d = 1$ ). However, all error estimate that we obtain in the sequel also hold for general multidimensional BSDEs. In our error analysis, we will use a constraint on the time partition step  $\Delta t_n$ :

$$\frac{\max_{0 \leq n \leq N-1} \Delta t_n}{\min_{0 \leq n \leq N-1} \Delta t_n} \leq C_0. \quad (32)$$

Let us introduce the following Lemma, its proof can be found in the reference [2].

**Lemma 2.** Let  $R_{y_j}^n$ ,  $R_{z_j}^n$  and  $R_{\Gamma_j}^n$  ( $j = 1, 2$ ) be the truncation errors defined in (21), (23) and (26), respectively. It holds that

$$\max_{0 \leq n \leq N-1} \left\{ |R_{y1}^n|, |R_{z1}^n|, |R_{\Gamma1}^n| \right\} \leq C (\Delta t)^3, \quad (33)$$

$$\max_{0 \leq n \leq N-1} \left\{ |R_{y2}^n|, |R_{z2}^n|, |R_{\Gamma2}^n| \right\} \leq C (\Delta t)^3. \quad (34)$$

Here  $C$  is a positive constant depending on  $T$ . We first give the error estimate for  $y_{t_n}^{t_n, x^n} - y^n$  in the following theorem.

**Theorem 1.** Let  $(x_t^{t_n, x^n}, y_t^{t_n, x^n}, z_t^{t_n, x^n}, \Gamma_t^{t_n, x^n})_{0 \leq t \leq T}$  ( $0 \leq n \leq N-1$ ) be the solution of the FBSDE (1) and  $(x^{n+1}, y^n, z^n, \Gamma^n)$  be the solution of Scheme 1. Assume  $y^N = \varphi(X^N)$ . Then for sufficiently small time step  $\Delta t_n$ , we have

$$\max_{0 \leq n \leq N} \mathbb{E} \left[ \left| y_{t_n}^{t_n, x^n} - y^n \right|^p \right] \leq C h^{2p} \quad (35)$$

for  $1 \leq p \leq \infty$ , where  $C$  is a constant depending on  $T$ .

*Proof.* Let  $e_y^n = y_{t_n}^{t_n, x^n} - y^n$ . Subtracting (29) from (21) to get

$$e_y^n = \tilde{\mathbb{E}}_{t_n}^{x^n} \left[ y_{t_{n+1}}^{t_n, x^n} - y^{n+1} \right] + \frac{1}{2} \Delta t_n \left( e_f^n + \tilde{\mathbb{E}}_{t_n}^{x^n} \left[ e_f^{n+1} \right] \right) + \sum_{j=1}^2 R_{y_j}^n. \quad (36)$$

Under the conditions of the theorem and by Lemma 2, we deduce that,

$$\begin{aligned} e_y^n &\leq \tilde{\mathbb{E}}_{t_n}^{x^n} \left[ e_y^{n+1} \right] + \frac{1}{2} \Delta t_n L |e_y^n| + \frac{1}{2} \Delta t_n L \tilde{\mathbb{E}}_{t_n}^{x^n} \left[ |e_y^{n+1}| \right] + C (\Delta t_n)^3 \\ &\leq (1 + C \Delta t_n) \tilde{\mathbb{E}}_{t_n}^{x^n} \left[ |e_y^{n+1}| \right] + C \Delta t_n |e_y^n| + C (\Delta t_n)^3, \end{aligned} \quad (37)$$

where  $L$  is the Lipschitz constant of  $f(t, x, y)$  with respect to  $y$ . Applying the inequality

$$(a + b)^p \leq a^p \left( 1 + \epsilon (2^{p-1}) \right) + b^p \left( 1 + \frac{2^{p-1} - 1}{\epsilon^{p-1}} \right) \text{ for } 0 \leq \epsilon \leq 1 \text{ with } a = (1 + C \Delta t_n) \tilde{\mathbb{E}}_{t_n}^{x^n} \left[ |e_y^{n+1}| \right],$$

$b = C\Delta t_n |e_y^n| + C(\Delta t_n)^3$ , and  $\epsilon = \Delta t_n$ , we deduce,

$$\begin{aligned} |e_y^n|^p &\leq \left( (1 + C\Delta t_n) \tilde{\mathbb{E}}_{t_n}^{x^n} \left[ |e_y^{n+1}|^p \right] \right)^p (1 + \Delta t_n (2^{p-1} - 1)) + (C\Delta t_n |e_y^n| + C(\Delta t_n)^3)^p \left( 1 + \frac{2^{p-1} - 1}{(\Delta t_n)^{p-1}} \right) \\ &\leq (1 + C\Delta t_n)^{p+1} \tilde{\mathbb{E}}_{t_n}^{x^n} \left[ |e_y^{n+1}|^p \right] + (C\Delta t_n)^p \left[ |e_y^n| + (\Delta t_n)^2 \right]^p \left( 1 + \frac{C}{(\Delta t_n)^{p-1}} \right), \end{aligned} \tag{38}$$

which by the inequality  $(a + b)^p \leq 2^{p-1}(a^p + b^p)$  gives

$$|e_y^n|^p \leq (1 + C\Delta t_n) \tilde{\mathbb{E}}_{t_n}^{x^n} \left[ |e_y^{n+1}|^p \right] + C\Delta t_n \left[ |e_y^n|^p \right] + C(\Delta t_n)^{2p+1}. \tag{39}$$

Taking the mathematical expectation on both sides of (35), for sufficiently small  $\Delta t_n$  we have

$$\begin{aligned} |e_y^n|^p &\leq \frac{(1 + Ch) \tilde{\mathbb{E}}_{t_n}^{x^n} \left[ |e_y^{n+1}|^p \right]}{(1 - Ch)} + \frac{Ch^{2p+1}}{(1 - Ch)} \\ &\leq \left( \frac{1 + Ch}{1 - Ch} \right)^{N-n} \tilde{\mathbb{E}}_{t_n}^{x^n} \left[ |e_y^N|^p \right] + \frac{Ch^{2p+1}}{1 - Ch} \sum_{i=0}^{N-n-1} \left( \frac{1 + Ch}{1 - Ch} \right)^i, \end{aligned} \tag{40}$$

for  $n = N - 1, \dots, 0$ . The terminal condition  $\mathbb{E} \left[ |e_y^N|^p \right] = \mathbb{E} \left[ |y_T - y^N|^p \right] \leq C(\Delta t_n)^{2p}$ , the time step constraint (28) and the inequality

$$\frac{Ch^{2p+1}}{1 - Ch} \sum_{i=0}^{N-n-1} \left( \frac{1 + Ch}{1 - Ch} \right)^i \leq \frac{h^{2p}}{2} (e^{2c_0CT} - 1), \tag{41}$$

lead to  $\mathbb{E} \left[ |e_y^n|^p \right] \leq Ch^{2p}$  for  $n = N - 1, \dots, 0$ . The proof is completed.

Then we turn to estimating the error  $e_z^n = z_{t_n}^{t_n, x^n} - z^n$ .

**Theorem 2.** Let  $(x_t^{t_n, x^n}, y_t^{t_n, x^n}, z_t^{t_n, x^n}, \Gamma_t^{t_n, x^n})$  be the solution of the FBSDE(1) and  $(x^{n+1}, y^n, z^n, \Gamma^n)$  be the solution of Scheme 1. Assume  $\mathbb{E} \left[ |e_y^N|^p \right] = \mathbb{E} \left[ |y_T - y^N|^p \right] \leq C(\Delta t_n)^{2p}$ . Then for sufficiently small time step  $\Delta t_n$ , we have

$$\max_{0 \leq n \leq N} \mathbb{E} \left[ |z_{t_n}^{t_n, x^n} - z^n|^p \right] \leq C(\Delta t_n)^p$$

for  $1 \leq p \leq \infty$ , where  $C$  is a constant depending on  $T$ .

*Proof.* Let  $e_z^n = z_{t_n}^{t_n, x^n} - z^n$ . From (19) and (25), we get

$$\Delta t_n e_z^n = \mathbb{E}_{t_n}^{x^n} \left[ e_y^{n+1} \Delta W_{t_{n+1}} \right] + \sum_{j=1}^2 R_{zj}^n.$$

By Lemma 2, the inequality  $(a + b)^p \leq 2^{p-1}(a^p + b^p)$  and Hölder's inequality, we deduce

$$\begin{aligned} (\Delta t_n |e_z^n|)^p &\leq 2^{p-1} \left| \mathbb{E}_{t_n}^{x^n} \left[ e_y^{n+1} \Delta W_{t_{n+1}} \right] \right|^p + 2^{p-1} \sum_{j=1}^2 |R_{zj}^n|^p \\ &\leq C_p \left( \mathbb{E}_{t_n}^{x^n} \left[ |e_y^{n+1}|^2 \right]^{\frac{p}{2}} (\Delta t_n)^{\frac{p}{2}} + (\Delta t_n)^{2p} \right) \\ &\leq C_p \left( \mathbb{E}_{t_n}^{x^n} \left[ |e_y^{n+1}|^2 \right]^{\frac{p}{2}} h^{\frac{p}{2}} + h^{2p} \right), \end{aligned} \tag{42}$$

where  $C_p$  is a positive number which depends on  $p$  and the constant  $C$  in Lemma 2. Taking the mathematical

expectation on both sides of the Equation (38) gives

$$(\Delta t_n)^p \mathbb{E} \left[ |e_z^n|^p \right] \leq C_p \left( \mathbb{E} \left[ |e_y^{n+1}|^2 \right]^{\frac{p}{2}} h^{\frac{p}{2}} + h^{2p} \right), \tag{43}$$

by using Theorem 1 and constraint (28), leads to  $\mathbb{E} \left[ |e_z^n|^p \right] \leq Ch^p$  for  $n = N - 1, \dots, 1, 0$ . The proof is completed.

At last, we estimate the error  $e_\Gamma^n = \Gamma_{t_n}^{t_n, x^n} - \Gamma^n$ .

**Theorem 3.** Let  $(x_t^{t_n, x^n}, y_t^{t_n, x^n}, z_t^{t_n, x^n}, \Gamma_{t_n}^{t_n, x^n})$  be the solution of the FBSDE(1) and  $(x^{n+1}, y^n, z^n, \Gamma^n)$  be the solution of Scheme 1. Assume  $\mathbb{E} \left[ |e_y^N|^p \right] = \mathbb{E} \left[ |y_T - y^N|^p \right] \leq C(\Delta t_n)^{2p}$ . Then for sufficiently small time step  $\Delta t_n$ , we have

$$\max_{0 \leq n \leq N} \mathbb{E} \left[ \left| \Gamma_{t_n}^{t_n, x^n} - \Gamma^n \right|^p \right] \leq C(\Delta t_n)^p \tag{44}$$

for  $1 \leq p \leq \infty$ , where  $C$  is a constant depending on  $T$ .

*Proof.* Let  $e_\Gamma^n = \Gamma_{t_n}^{t_n, x^n} - \Gamma^n$ . From (26) and (31), we get

$$\Delta t_n e_\Gamma^n = \mathbb{E}_{t_n}^{x^n} \left[ e_y^{n+1} \Delta \tilde{N}_{t_{n+1}} \right] + \sum_{j=1}^2 R_{\Gamma_j}^n. \tag{45}$$

By Lemma 2, the inequality  $(a + b)^p \leq 2^{p-1} (a^p + b^p)$  and Hölder's inequality, we deduce

$$\begin{aligned} (\Delta t_n |e_\Gamma^n|)^p &\leq 2^{p-1} \left| \mathbb{E}_{t_n}^{x^n} \left[ e_y^{n+1} \Delta \tilde{N}_{t_{n+1}} \right] \right|^p + 2^{p-1} |R_\Gamma^n|^p \\ &\leq C_p \left( \mathbb{E}_{t_n}^{x^n} \left[ |e_y^{n+1}|^2 \right]^{\frac{p}{2}} (\Delta t_n)^{\frac{p}{2}} + (\Delta t_n)^{2p} \right) \\ &\leq C_p \left( \mathbb{E}_{t_n}^{x^n} \left[ |e_y^{n+1}|^2 \right]^{\frac{p}{2}} h^{\frac{p}{2}} + h^{2p} \right), \end{aligned} \tag{46}$$

where  $C_p$  is a positive number which depends on  $p$  and the constant  $C$  in Lemma 2. Taking the mathematical expectation on both sides of the Equation (42) gives

$$(\Delta t_n)^p \mathbb{E} \left[ |e_\Gamma^n|^p \right] \leq C_p \left( \mathbb{E} \left[ |e_y^{n+1}|^2 \right]^{\frac{p}{2}} h^{\frac{p}{2}} + h^{2p} \right), \tag{47}$$

by using Theorem 1 and constraint (28), leads to  $\mathbb{E} \left[ |e_\Gamma^n|^p \right] \leq Ch^p$  for  $n = N - 1, \dots, 1, 0$ . The proof is completed.

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