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A Remarkable Chord Iterative Method for Roots of Uncertain Multiplicity

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Abstract

In this note we at first briefly review iterative methods for effectively approaching a root of an unknown multiplicity. We describe a first order, then a second order estimate for the multiplicity index *m* of the approached root. Next we present a second order, two-step method for iteratively nearing a root of an unknown multiplicity. Subsequently, we introduce a novel chord, or a two-step method, not requiring beforehand knowledge of the multiplicity index *m* of the sought root, nor requiring higher order derivatives of the equilibrium function, which is quadratically convergent for any $m \le 4$, and then reverts to superlinear.

Keywords

Iterative Methods, Unknown Root Multiplicity, Two-Step Methods

1. Introduction

The multiplicity index m of root $x = a$, $f(a) = 0$ of equilibrium function $f(x)$ may be a well latent property of the root, not cursorily revealed, nor readily available, yet this multiplicity can profoundly affect the behavior of the iterative approach [1]-[3] to the root. In this note, we briefly review the iterative methods [4]-[8] for approaching a root of an unknown multiplicity, and present a first oder [9] as well as a second order estimate for the multiplicity index m of the approached root. Then we present a novel chord, or a two-step method, not requiring beforehand knowledge of m, nor requiring the higher derivatives of the equilibrium function, which is quadratically convergent for any $m \le 4$, and then reverts to superlinear.

2. Assumed Nature of the Equilibrium Function

We assume that near root $a, f(a) = 0$, function $f(x)$ has the power series representation

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$$f(x)=(x-a)^m\left(A+B(x-a)+C(x-a)^2+\cdots\right),\ A\neq 0,\ m\geq 1 \tag{1}$$

where m is the multiplicity index of root a, and where $A,B,C,$ etc. are, for $m=1$, the coefficients

$$A=f'(a), B=\frac{1}{2!}f''(a), C=\frac{1}{3!}f'''(a) \tag{2}$$

and so on.

3. The Newton-Raphson Method

The Newton-Raphson method

$$x_1=x_0-u_0,\ u=\frac{f(x)}{f'(x)} \tag{3}$$

is quadratic

$$x_1-a=\frac{B}{A}(x_0-a)^2+\frac{2(B^2+AC)}{A^2}(x_0-a)^3+O\left((x_0-a)^4\right) \tag{4}$$

if $m=1$. However, if $m>1$, the method declines to mere linear

$$x_1-a=\frac{m-1}{m}(x_0-a)+\frac{B}{m^2A}(x_0-a)^2+O\left((x_0-a)^3\right). \tag{5}$$

See also [10].

4. Extrapolation to the Limit

Let $x_0, x_1=x_0-u_0, x_2=x_1-u_1$ be already near root a. Then, if $m=1$

$$x_1-a=\frac{B}{A}(x_0-a)^2 \text{ and } x_2-a=\frac{B}{A}(x_1-a)^2 \tag{6}$$

nearly. Eliminating B/A from the two equations we obtain

$$(-2x_0+3x_1-x_2)a^2+\left(x_0^2-3x_1^2+2x_0x_2\right)a+\left(x_1^3-x_0^2x_2\right)=0 \tag{7}$$

which we solve for an approximate a, as

$$x_3=a=x_0-\frac{3+\sqrt{1+4\rho}}{2(2-\rho)}u_0 \tag{8}$$

where

$$\rho=u_1/u_0=\frac{B}{A}(x_0-a)+O\left((x_0-a)^2\right). \tag{9}$$

The square root in Equation (8) may be approximated as

$$\sqrt{1+4\rho}=1+2\rho-2\rho^2+4\rho^3-10\rho^4+28\rho^5-84\rho^6\pm\cdots \tag{10}$$

and for this extrapolated x_3 of Equation (8) we have

$$x_3-a=\frac{2B^2\left(B^2-AC\right)}{A^4}(x_0-a)^5+O\left((x_0-a)^6\right). \tag{11}$$

For example, for $f(x)=x+x^2+x^3$, and starting with $x_0=0.2$, we compute $x_1=0.0368$, $x_2=0.0135$; and then from Equation (8), $x_3=0.000112$. Another such cycle starting with $x_0=x_3$ produces a next $x_3=-1.36\times 10^{-20}$.

5. Always Quadratic Newton-Raphson Method

The modified Newton-Raphson method

$$x_1 = x_0 - mu_0 = x_0 - m\frac{f_0}{f_0'} \tag{12}$$

converges quadratically to a root of any multiplicity m

$$x_1 - a = \frac{1}{m}\frac{B}{A}(x_0-a)^2 - \frac{1}{m^2}\left((m+1)\frac{B^2}{A^2} - 2m\frac{C}{A}\right)(x_0-a)^3 + O\left((x_0-a)^4\right). \tag{13}$$

But for this we need to know m.

By Equation (1) we readily deduce that, for any x

$$\mu = \frac{1}{u'} = \frac{f'^2}{f'^2 - ff''} = m + \frac{2B}{A}(x-a) + \frac{B^2 - 3B^2m + 6ACm}{A^2 m}(x-a)^2 + O\left((x-a)^3\right) \tag{14}$$

obtained at the price of a second derivative. For finite-difference approximations of the needed derivatives see [11]-[13]. Using μ in Equation (14) for m in Equation (12) we obtain the method

$$x_1 = x_0 - \frac{f_0 f_0'}{f_0'^2 - f_0 f_0''} \tag{15}$$

which is quadratic for any, provided, m

$$x_1 - a = -\frac{1}{m}\frac{B}{A}(x_0-a)^2 + O\left((x_0-a)^3\right). \tag{16}$$

The method of Equation (15) is also obtained by applying Newton's method not to f, but rather to $u = f/f'$. For $f(x) = x^m(3+x)$, we obtain by the method of Equation (15) that requires not only f' but also f'', starting with $x_0 = 1$.

For $m = 1$

$$x = \{1, -0.176, -0.012, -4.6\times10^{-5}, -6.98\times10^{-10}, -1.63\times10^{-19}\}. \tag{17a}$$

For $m = 7$

$$x = \{1, -0.027, -3.4\times10^{-5}, -5.6\times10^{-11}, -1.47\times10^{-22}\}. \tag{17b}$$

Equation (15) may be written as

$$x_1 = x_0 - \frac{1}{1-z_0}\frac{f_0}{f_0'},\ z_0 = \frac{f_0 f_0''}{f_0'^2} \tag{18}$$

and it is of interest to know that

$$z_0 = \frac{f_0 f_0''}{f_0'^2} = \frac{m-1}{m} + \frac{2}{m^2}\frac{B}{A}(x-a) + O\left((x-a)^2\right). \tag{19}$$

For the price of a third derivative we may have the quadratic approximation

$$\mu = \frac{u'}{u'^2 - uu''} = \frac{f'^2\left(f'^2 - ff''\right)}{f'^4 - ff'^2 f'' - f^2 f''^2 + f^2 ff'''} = \frac{m + B^2 + 3B^2 m - 6ACm}{A^2 m}(x_0-a)^2 + \cdots \tag{20}$$

6. An Erroneous *m*

The method

$$x_1 = x_0 - m(1+\epsilon)u_0 \tag{21}$$

produces the superlinear

$$x_1 - a = -\epsilon(x_0 - a) + \frac{B(1+\epsilon)}{Am}(x_0 - a)^2 + O\left((x_0 - a)^3\right) \tag{22}$$

and if $\epsilon > 0$, convergence is alternating.

7. Estimation of the Leading Term

We readily have that

$$-\frac{1}{2}m^2 u'' = -\frac{1}{2}m^2 \frac{-f'^2 f'' + 2ff''^2 - ff'f'''}{f'^3} = \frac{B}{A} - \left(\frac{3(1+m)}{m}\left(\frac{B}{A}\right)^2 - 6\frac{C}{A}\right)(x-a). \tag{23}$$

For example, for $f = x + 10x^2$, we compute using Equation (23) the B/A approximations as depending on the chosen x

$$\{x, B/A\} = \{10^{-2}, 5.79\}, \{10^{-3}, 9.42\}, \{10^{-4}, 9.94\}, \{10^{-5}, 9.994\}. \tag{24}$$

8. An Elementary Discrete Two-Step Newton Method for Roots of Any Multiplicity

If

$$x_0, x_1 = x_0 - u_0, x_2 = x_1 - u_1, u = \frac{f}{f'} \tag{25}$$

are already close to root a of multiplicity $m > 1$, then according to Equation (5)

$$x_1 - a = \left(1 - \frac{1}{m}\right)(x_0 - a), \text{ and } x_2 - a = \left(1 - \frac{1}{m}\right)(x_1 - a) \tag{26}$$

nearly, from which we extract the approximation

$$a = \frac{x_1^2 - x_0 x_2}{-x_0 + 2x_1 - x_2} = x_0 - \frac{u_0}{u_0 - u_1}u_0 = x_1 - \frac{u_0}{u_0 - u_1}u_1. \tag{27}$$

Setting a back into Equation (26) yields

$$\mu = \frac{x_1 - x_0}{u_1 - u_0} = \frac{1}{1-\rho}, \rho = \frac{u_1}{u_0} \tag{28}$$

and the two-step method

$$\mu_0 = \mu_0, u_0 = \frac{f_0}{f_0'}, x_1 = x_0 - \mu_0 u_0, u_1 = \frac{f_1}{f_1'}, \mu_1 = \frac{x_1 - x_0}{u_1 - u_0} = \frac{1}{1-\rho}, \rho = \frac{u_1}{u_0}, x_2 = x_1 - \mu_1 u_1 \tag{29}$$

where μ in Equation (28) is seen to be but the finite-difference approximation of $\mu = \mathrm{d}x/\mathrm{d}u$ in Equation (14).

For example, for $f(x) = x^3 + x^4$, and starting with $x_0 = 1, \mu_0 = 1$, we compute by Equation (29), the successive approximations

$$x_0 = 1, \mu_0 = 1, x_1 = 0.71, \mu_1 = 3.72, x_2 = -6.4 \times 10^{-2} \tag{30a}$$

$$x_0 = -0.064, \mu_0 = 3.72, x_1 = 0.018, \mu_1 = 2.95, x_2 = 4 \times 10^{-4} \tag{30b}$$

$$x_0 = 4 \times 10^{-4}, \mu_0 = 2.95, x_1 = 6.9 \times 10^{-6}, \mu_1 = 3.0004, x_2 = -9.3 \times 10^{-10} \tag{30c}$$

$$x_0 = -9.3\times10^{-10},\ \mu_0 = 3.0004,\ x_1 = 1.26\times10^{-13},\ \mu_1 = 3,\ x_2 = 3.9\times10^{-23}. \tag{30d}$$

Generally, starting with

$$\mu_0 = m + \epsilon_1,\ x_0 = a + \epsilon_2 \tag{31}$$

we have from the method of Equation (29) that

$$\mu_1 = m + \frac{B}{A}\left(1 - \frac{\epsilon_1}{m}\right)\epsilon_2 + O\left(\epsilon_2^2\right),\ x_2 = a + \frac{B}{Am^2}\epsilon_1\epsilon_2^2 + O\left(\epsilon_2^3\right). \tag{32}$$

The repeated classical Newton's method, $x_1 = x_0 - f_0/f_0', x_2 = x_1 - f_1/f_1'$, we recall, is only linear if $m > 1$

$$x_2 - a = \left(1 - \frac{1}{m}\right)^2 (x_0 - a) + \frac{(2m-1)(m-1)}{m^4}\frac{B}{A}(x_0 - a)^2 + O\left((x_0 - a)^3\right). \tag{33}$$

See also [14] [15].

9. Derivation of the Chord Method

It is a rational two step method of the form

$$x_2 = x_1 + (x_1 - x_0)\frac{f_1 + Pf_0}{Qf_1 + Rf_0},\ x_1 = x_0 + k\frac{f_0}{f_0'},\ f_0 = f(x_0),\ f_1 = f(x_1). \tag{34}$$

With

$$P = \frac{6 + 11k + 6k^2 + k^3}{-6 + 4k},\ Q = \frac{9 - 2k}{-3 + 2k},\ R = \frac{18 + 14k + 5k^2 + k^3}{6 - 4k} \tag{35}$$

the method is quadratic for $m = 1, m = 2$ and $m = 3$. In fact;

For $m = 1$

$$x_2 - a = -\frac{B}{A}\frac{9 + 7k}{9 + k}(x_0 - a)^2 + O\left((x_0 - a)^3\right). \tag{36a}$$

For $m = 2$

$$x_2 - a = -\frac{B}{2A}\frac{9 + 7k + k^2}{9 + 4k}(x_0 - a)^2 + O\left((x_0 - a)^3\right). \tag{36b}$$

For $m = 3$

$$x_2 - a = -\frac{B}{3A}\frac{81 + 63k + 14k^2}{81 + 45k + 4k^2}(x_0 - a)^2 + O\left((x_0 - a)^3\right). \tag{36c}$$

For $m = 4$ the method produces

$$x_2 - a = \frac{(k-2)k^2}{576 + 352k + 46k^2 + 4k^3}(x_0 - a) + O\left((x_0 - a)^2\right) \tag{37}$$

and for $k = 2$ the method is quadratic for $m = 4$ as well.

According to Equation (36a), if $m = 1, k = -9/7$, then the method is higher than quadratic.

10. The Method is Further Superlinear

For $k = 2$ we have:

For $m = 1$

$$x_2 - a = -\frac{23B}{11A}(x_0 - a)^2 + \frac{914B^2 - 1628AC}{121A^2}(x_0 - a)^3 + O\left((x_0 - a)^4\right). \tag{38a}$$

For $m=2$

$$x_2-a=-\frac{27B}{34A}(x_0-a)^2+\frac{1277B^2-2414AC}{578A^2}(x_0-a)^3+O\left((x_0-a)^4\right). \tag{38b}$$

For $m=3$

$$x_2-a=-\frac{263B}{561A}(x_0-a)^2+\frac{370334B^2-715836AC}{314721A^2}(x_0-a)^3+O\left((x_0-a)^4\right). \tag{38c}$$

For $m=4$

$$x_2-a=-\frac{245B}{748A}(x_0-a)^2+\frac{435571B^2-851224AC}{559504A^2}(x_0-a)^3+O\left((x_0-a)^4\right). \tag{38d}$$

For $m=5$

$$x_2-a=\frac{1}{9871.9}(x_0-a)-\frac{1}{4}\frac{B}{A}(x_0-a)^2+O(x_0-a)^3. \tag{38e}$$

For $m=7$

$$x_2-a=\frac{1}{1657}(x_0-a)-\frac{B}{5.95A}(x_0-a)^2+O(x_0-a)^3. \tag{38f}$$

For $m=9$

$$x_2-a=\frac{1}{718}(x_0-a)-\frac{B}{7.94A}(x_0-a)^2+O(x_0-a)^3. \tag{38g}$$

For $m=11$

$$x_2-a=\frac{1}{423.3}(x_0-a)-\frac{B}{9.97A}(x_0-a)^2+O(x_0-a)^3. \tag{38h}$$

For $m=17$

$$x_2-a=\frac{1}{171.4}(x_0-a)-\frac{B}{16.2A}(x_0-a)^2+O(x_0-a)^3. \tag{38k}$$

For $m=27$

$$x_2-a=\frac{1}{81}(x_0-a)-\frac{B}{27A}(x_0-a)^2+O(x_0-a)^3. \tag{38l}$$

11. Lowering the Value of k

We leave k in $x_1=x_0+kf_0/f_0'$ of Equation (34), free, and have by power series expansion, for multiplicity index $m=5$, for $f(x)$ in Equation (1), that

$$x_2-a=\frac{2k^2}{5}\frac{-125+55k+4k^2}{5625+3625k+550k^2+82k^3+4k^4}(x_0-a)+O\left((x_0-a)^2\right). \tag{39}$$

The linear term in the above expansion is annulled with

$$-125+55k+4k^2=0,\ k=1.9859043. \tag{40}$$

We do this for higher values of m and find that

$$\{m,k\}=\{4,2\},\{5,1.9859043\},\{7,1.9689621\},\{9,1.9591333\},\{11,1.9527133\}. \tag{41}$$

We try $k=1.95$, and get

For $m=1$

$$x_2-a=-8.9\times10^{-16}\left(x_0-a\right)-\frac{2.07B}{A}\left(x_0-a\right)^2+O\left(\left(x_0-a\right)^3\right). \tag{42a}$$

For $m=2$

$$x_2-a=-6.6\times10^{-16}\left(x_0-a\right)-\frac{0.787B}{A}\left(x_0-a\right)^2+O\left(\left(x_0-a\right)^3\right). \tag{42b}$$

For $m=3$

$$x_2-a=-4.4\times10^{-16}\left(x_0-a\right)-\frac{0.466B}{A}\left(x_0-a\right)^2+O\left(\left(x_0-a\right)^3\right). \tag{42c}$$

For $m=4$

$$x_2-a=-17712\left(x_0-a\right)-\frac{0.326B}{A}\left(x_0-a\right)^2+O\left(\left(x_0-a\right)^3\right). \tag{42d}$$

For $m=5$

$$x_2-a=-13999\left(x_0-a\right)-\frac{0.249B}{A}\left(x_0-a\right)^2+O\left(\left(x_0-a\right)^3\right). \tag{42e}$$

For $m=7$

$$x_2-a=-12799\left(x_0-a\right)-\frac{0.168B}{A}\left(x_0-a\right)^2+O\left(\left(x_0-a\right)^3\right). \tag{42f}$$

For $m=9$

$$x_2-a=-13315\left(x_0-a\right)-\frac{0.127B}{A}\left(x_0-a\right)^2+O\left(\left(x_0-a\right)^3\right). \tag{42g}$$

For $m=11$

$$x_2-a=-17608\left(x_0-a\right)-\frac{0.101B}{A}\left(x_0-a\right)^2+O\left(\left(x_0-a\right)^3\right). \tag{42h}$$

For $m=17$

$$x_2-a=11312\left(x_0-a\right)-\frac{0.063B}{A}\left(x_0-a\right)^2+O\left(\left(x_0-a\right)^3\right). \tag{42k}$$

For $m=27$

$$x_2-a=1358\left(x_0-a\right)-\frac{0.038B}{A}\left(x_0-a\right)^2+O\left(\left(x_0-a\right)^3\right). \tag{42l}$$

For $m=37$

$$x_2-a=1198\left(x_0-a\right)-\frac{0.027B}{A}\left(x_0-a\right)^2+O\left(\left(x_0-a\right)^3\right). \tag{42m}$$

The general form of the linear part of x_2-a in Equations (42) is of the form $c(m)(x_0-a)$ with a constant $c(m)$ that is small if multiplicity index *m* is not much above 5. For instance, $c(11)=-1/7608$, meaning that at each iteration the error x_2-a is reduced by this factor. Such convergence behavior we term superlinear. More concretely, for $f(x)=x^m(3+x)$, we obtain by the above method, using $k=1.95$, starting with $x_0=1$.

For $m=1$

$$x=\left\{1,-0.26,-0.066,-7.2\times10^{-6},-3.6\times10^{-11},-9\times10^{-22}\right\} \tag{43a}$$

For $m = 3$

$$x = \{1, -0.76, -9.6\times10^{-4}, -1.44\times10^{-7}, -3.2\times10^{-15}, 7.9\times10^{-31}\} \tag{43b}$$

For $m = 7$

$$x = \{1, -0.03, -4.1\times10^{-5}, 1.47\times10^{-8}, -5.2\times10^{-12}, 1.88\times10^{-15}, -6.7\times10^{-19}, 2.4\times10^{-22}\}. \tag{43c}$$

12. Conclusions

The paper is predicated on the realistic assumption that the multiplicity index *m* of the iteratively targeted root is unknown. We conclude that if one prefers to shun second order derivatives, then the quadratic two-step method of Equation (29), that provides also ever better approximations for the multiplicity index *m* of the approached root, is a practically appealing alternative.

Otherwise, one may use the rational two-step method of Equation (34) with a constant *k* that is only slightly less than 2. Thus stating the method becomes superlinear, albeit, of a reduced speed of convergence for highly elevated root multiplicities. For the sake of brevity, the present paper does not explore the possibility of estimating the multiplicity index *m* of the sought root by the method of Equation (29), then applying this estimate to the choice of an optimal *k* in the method of Equations (34) and (35).

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