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Design and Analysis of Some Third Order Explicit Almost Runge-Kutta Methods

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Abstract

In this paper, we propose two new explicit Almost Runge-Kutta (ARK) methods, ARK3 (a three stage third order method, *i.e.*, $s = p = 3$) and ARK34 (a four-stage third-order method, *i.e.*, $s = 4$, $p = 3$), for the numerical solution of initial value problems (IVPs). The methods are derived through the application of order and stability conditions normally associated with Runge-Kutta methods; the derived methods are further tested for consistency and stability, a necessary requirement for convergence of any numerical scheme; they are shown to satisfy the criteria for both consistency and stability; hence their convergence is guaranteed. Numerical experiments carried out further justified the efficiency of the methods.

Keywords

Almost Runge-Kutta, Stability, Consistency, Convergence, Order Conditions, Rooted Trees

1. Introduction

According to [1] the s-stage Runge-Kutta method for solving the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0, \tag{1}$$

is defined by

$$y_{n+1} = y_n + h\sum_{i=1}^{s} b_i k_i, \tag{2}$$

where

$$k_i = f\left(x_n + c_i h, y_n + h\sum_{j=1}^{s} a_{ij} k_j\right), \quad i = 1, 2, \cdots, s, \tag{3}$$

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and

$$c_i = \sum_{j=1}^{s} a_{ij}, \quad i = 1, 2, 3, \cdots, s. \tag{4}$$

Alternative forms of the above equations are:

$$y_{n+1} = y_n + h\sum_{i=1}^{s} b_i f\left(x_n + c_i h, Y_i\right), \tag{5}$$

where

$$Y_i = y_n + h\sum_{j=1}^{s} a_{ij} f\left(x_n + c_j h, Y_j\right), \ i = 1, \cdots, s. \tag{6}$$

The two forms of Equations (2) and (5) are equivalent by making the interpretation

$$k_i = f\left(x_n + c_i h, Y_i\right), \ i = 1, \cdots, s \tag{7}$$

where Y_i is the inner stages that tend to estimate the solutions at some points; s is the number of stages and c_i is the points where the function f is computed for a step. ARK methods are a special class of RK methods that arose out of the quest to develop efficient and accurate methods that have advantages over the traditional methods by retaining the simple stability function of RK methods, allowing minimal information to be passed between steps and adjusting stepsize easily. Since the introduction of ARK methods in by [2], other researchers who have made their input toward the development of this method include [3]-[7].

2. Materials and Methods

2.1. Method ARK3 ($s = p = 3$)

The general third order three stages Almost Runge-Kutta scheme is of the form:

$$\left[\begin{array}{c|c} A & U \\ \hline B & V \end{array}\right] = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & c_1 & \frac{1}{2}c_1^2 \\ a_{21} & 0 & 0 & 1 & c_2 - a_{21} & \frac{1}{2}c_2^2 - a_{21}c_1 \\ b_1 & b_2 & 0 & 1 & b_0 & 0 \\ \hline b_1 & b_2 & 0 & 1 & b_0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \beta_1 & \beta_2 & \beta_3 & 0 & \beta_0 & 0 \end{array}\right]. \tag{8}$$

We represent the abscissa vector $c = \left[c_1, c_2, 1\right]^{\mathrm{T}}$, $b^{\mathrm{T}} = \left[b_1, b_2, 0\right]$, $\beta^{\mathrm{T}} = \left[\beta_1, \beta_2, \beta_3\right]$.

The order conditions for order three ARK schemes are derived through the standard rooted tree approach used for Runge-Kutta methods [8].

$$\left.\begin{array}{l} b_0 + b^{\mathrm{T}}e = 1 \\ b^{\mathrm{T}}c = \frac{1}{2} \\ b^{\mathrm{T}}c^2 = \frac{1}{3} \end{array}\right\} \Rightarrow \left.\begin{array}{r} b_0 + b_1 + b_2 = 1 \\ b_1c_1 + b_2c_2 = \frac{1}{2} \\ b_1c_1^2 + b_2c_2^2 = \frac{1}{3} \end{array}\right\}. \tag{9}$$

The conditions of Runge-Kutta stability for 3rd order, 3 stages are:

$$\beta^{\mathrm{T}}\left(I + \beta_3 A\right) = \beta_3 e_3^{\mathrm{T}} \tag{10}$$

$$\left(1+\frac{1}{2}\beta_3 c_1\right)b^{\mathrm{T}}Ac=\frac{1}{6} \tag{11}$$

$$c_1=-\frac{2exp_3(-\beta_3)}{\beta_3 exp_2(-\beta_3)} \tag{12}$$

where $exp_n(x)=1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}$.

Acquiring order 2 estimation with respect to 2nd scaled derivative for the 3rd outgoing solution, we need:

$$\beta^{\mathrm{T}}e+\beta_0=0. \tag{13}$$

$$\beta^{\mathrm{T}}c=1. \tag{14a}$$

From Equation (12) we have,

$$c_1=-\frac{2\left(1-\beta_3+\frac{1}{2}\beta_3^2-\frac{1}{6}\beta_3^3\right)}{\beta_3\left(1-\beta_3+\frac{1}{2}\beta_3^2\right)}. \tag{14b}$$

Solving Equation (9) we obtain

$$b_1=\frac{3c_2-2}{6c_1(c_2-c_1)}. \tag{15}$$

$$b_2=-\frac{3c_1-2}{6c_2(c_2-c_1)}. \tag{16}$$

$$b_0=\frac{6c_1c_2-3c_1-3c_2+2}{6c_1c_2}. \tag{17}$$

And from Equation (11), we obtain

$$a_{21}=\frac{1}{3b_2c_1(2+\beta_3c_1)}. \tag{18}$$

Evaluating both sides of Equation (10) we obtain

$$\left(\beta_1-a_{21}b_2\beta_3^3+b_1\beta_3^2,\beta_2+b_2\beta_3^2,\beta_3\right)=\left(0,0,\beta_3\right). \tag{19}$$

This implies that

$$\left.\begin{aligned}&\beta_1-a_{21}b_2\beta_3^3+b_1\beta_3^2=0\\&\beta_2+b_2\beta_3^2=0\\&\beta_3=\beta_3\end{aligned}\right\}. \tag{20}$$

Thus Equation (13) becomes

$$\beta_0=-\beta_1-\beta_2-\beta_3. \tag{21}$$

Two free parameters, β_3 and c_2 are required for an order three scheme. Thus $c^{\mathrm{T}}=\left[\frac{8}{15},\frac{1}{2},1\right]$; and after calculating the members of the U matrix we obtain the a scheme for method $s=p=3$.

$$\left[\begin{array}{c|c} A & U \\ \hline B & V \end{array}\right] = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & \frac{8}{15} & \frac{32}{225} \\ -\frac{25}{576} & 0 & 0 & 1 & \frac{313}{576} & \frac{4}{27} \\ \frac{75}{16} & -4 & 0 & 1 & \frac{5}{16} & 0 \\ \hline \frac{75}{16} & -4 & 0 & 1 & \frac{5}{16} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{75}{2} & 36 & 3 & 0 & -\frac{3}{2} & 0 \end{array}\right]. \tag{22}$$

2.2. Method ARK34 ($s = 4$, $p = 3$)

The third order four stages scheme has the general form:

$$\left[\begin{array}{c|c} A & U \\ \hline B & V \end{array}\right] = \left[\begin{array}{cccc|ccc} 0 & 0 & 0 & 0 & 1 & c_1 & \frac{1}{2}c_1^2 \\ a_{21} & 0 & 0 & 0 & 1 & c_2 - a_{21} & \frac{1}{2}c_2^2 - a_{21}c_1 \\ a_{31} & a_{32} & 0 & 0 & 1 & c_3 - a_{31} - a_{32} & \frac{1}{2}c_3^2 - a_{31}c_1 - a_{31}c_2 \\ b_1 & b_2 & b_3 & 0 & 1 & b_0 & 0 \\ \hline b_1 & b_2 & b_3 & b_4 & 1 & b_0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & 0 & \beta_0 & 0 \end{array}\right]. \tag{23}$$

Its stability function is expressed as

$$R(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + Kz^4. \tag{24}$$

The order conditions are derived using the standard rooted tree approach used for Runge-Kutta methods [8].

$$b^{\mathrm{T}}c = \frac{1}{2}. \tag{25}$$

$$b^{\mathrm{T}}c^2 = \frac{1}{3}. \tag{26}$$

$$b_0 = 1 - b^{\mathrm{T}}e. \tag{27}$$

$$\beta_0 = -\beta^{\mathrm{T}}e \tag{28}$$

$$\beta_4 e_4^{\mathrm{T}}(I + \theta A) = \left(I + \varnothing A + \beta_4\theta A^2\right). \tag{29}$$

$$K\left(\frac{1}{2}\beta_4 c_1\theta\alpha_3 - \alpha_4\right) = \left(1 + \frac{1}{2}\beta_4 c_1\right)\left(1 + \alpha_1 + \frac{\alpha_2}{2!} + \frac{\alpha_3}{3!}\right). \tag{30}$$

The α_i values are obtained by expanding

$$\frac{1 + (\varnothing - \beta_4)z}{1 + \varnothing z + \beta_4\theta z^2} = \sum_{i=0}^{\infty}\alpha_i z^i. \tag{31}$$

Also,

$$b^{\mathrm{T}}Ac-\frac{1}{6}=\theta\left(b^{\mathrm{T}}A^2c-K\right). \tag{32}$$

$$\beta_4\left(\frac{1}{2}b^{\mathrm{T}}Ac^2-K\right)=\left(\beta_4-\varnothing\right)\left(b^{\mathrm{T}}A^2c-K\right). \tag{33}$$

There is also the additional condition

$$b^{\mathrm{T}}c^3=L \tag{34}$$

$c_1,c_2,c_3,\beta_4,\varnothing,\theta$ and L will be assumed to be the free parameters, where $L-\frac{1}{4}$ is the error coefficient comparable to the bushy tree. From Equations (25)-(27) together with Equation (34) we have

$$\left.\begin{aligned} b_1c_1+b_2c_2+b_3c_3&=\frac{1}{2}\\ b_1c_1^2+b_2c_2^2+b_3c_3^2&=\frac{1}{3}\\ b_1c_1^3+b_2c_2^3+b_3c_3^3&=L\\ b_0+b_1+b_2+b_3&=1\end{aligned}\right\}. \tag{35}$$

Thus

$$b_1=\frac{3c_2c_3+6L-2c_2-2c_3}{6\left(c_3-c_1\right)\left(c_2-c_1\right)c_1}. \tag{36}$$

$$b_2=-\frac{Bc_1c_3+6L2c_1-2c_3}{6c_2\left(c_2-c_1\right)\left(c_3-c_2\right)}. \tag{37}$$

$$b_3=\frac{3c_1c_2 6L-2c_1-2c_2}{6c_3\left(c_1c_2-c_1c_3-c_2c_3+c_3^2\right)}. \tag{38}$$

$$b_0=-\frac{1}{6}\left(\frac{-6c_1c_2c_3+3c_1c_2+3c_1c_3+3c_2c_3+6L-2c_1-2c_2-2c_3}{c_1c_2c_3}\right). \tag{39}$$

From Equation (30) we obtain

$$K=\frac{\left(1+\frac{1}{2}\beta_4c_1\right)\left(1+\alpha_1+\frac{\alpha_2}{2!}+\frac{\alpha_3}{3!}\right)}{\frac{1}{2}\beta_4c_1\theta\alpha_3-\alpha_4}. \tag{40}$$

Evaluating the stability matrix of a four stage third order method, we arrive at

$$\mathrm{Tr}\left(BA^3U\right)=K. \tag{41}$$

where Tr is the trace of a matrix and

$$BA^3U=\begin{bmatrix} A^3b^{\mathrm{T}}e & A^3\beta^{\mathrm{T}}\left(c-Ae\right) & A^3b^{\mathrm{T}}\left(\frac{1}{2}c^2-Ac\right)\\ A^3e_s^{\mathrm{T}} & A^3e_s^{\mathrm{T}}\left(c-Ae\right) & A^3e_s^{\mathrm{T}}\left(\frac{1}{2}c^2-Ac\right)\\ A^3\beta^{\mathrm{T}}e & A^3\beta^{\mathrm{T}}\left(c-Ae\right) & A^3\beta^{\mathrm{T}}\left(\frac{1}{2}c^2-Ac\right)\end{bmatrix}. \tag{42}$$

Hence,

$$\mathrm{Tr}\left(BA^3U\right) = b^{\mathrm{T}}A^3e + e_s^{\mathrm{T}}A^3\left(c - Ae\right) + \beta^{\mathrm{T}}A^3\left(\frac{1}{2}c^2 - Ac\right). \tag{43}$$

$$b^{\mathrm{T}}A^3e + e_s^{\mathrm{T}}A^3\left(c - Ae\right) + \beta^{\mathrm{T}}A^3\left(\frac{1}{2}c^2 - Ac\right) = K. \tag{44}$$

And it follows that:

$$b^{\mathrm{T}}A^2c + \frac{1}{2}\beta^{\mathrm{T}}A^3c^2 - \beta^{\mathrm{T}}A^4c = K. \tag{45}$$

Since $A^4 = 0$ we obtain

$$b^{\mathrm{T}}A^2c\left(1 + \frac{1}{2}\beta_4 c_1\right) = K. \tag{46}$$

We introduce $K_1 = b^{\mathrm{T}}A^2c$, $K_2 = b^{\mathrm{T}}Ac$ and $K_3 = b^{\mathrm{T}}Ac^2$. Thus from Equation (46) we arrived at

$$K_1 = \frac{K}{1 + \frac{1}{2}c_1\beta_4}. \tag{47}$$

And from Equations (32) and (33) we obtain respectively

$$K_2 = \frac{1}{6} + \theta\left(K_1 - K\right). \tag{48}$$

$$K_3 = 2K - 2\left(\frac{\varnothing}{\beta_4} - 1\right)\left(K_1 - K\right). \tag{49}$$

Further simplification produces the following results

$$a_{21} = \frac{c_2 * \left(c_2 - c_1\right) * K_1}{c_1 * \left(K_3 - c_1 * K_2\right)}. \tag{50}$$

$$a_{32} = \frac{K_1}{b_3 a_{21} c_1}. \tag{51}$$

$$a_{31} = \frac{K_3 - b_2 a_{21} c_1^2 - b_3 a_{32} c_1^2}{b_3 c_1^2}. \tag{52}$$

Setting $\varnothing = \beta_4 + \theta$ and substituted this into Equation (29), we obtain

$$\beta_4 e_4^{\mathrm{T}}\left(I + \theta A\right) = \beta^{\mathrm{T}}\left(I + \theta A + \beta_4\theta A^2\right). \tag{53}$$

$$\beta_4 e_4^{\mathrm{T}}\left(I + \theta A\right) = \beta^{\mathrm{T}}\left(I + \beta_4 A\right)\left(I + \theta A\right). \tag{54}$$

$$\beta_4 e_4^{\mathrm{T}} = \frac{\beta^{\mathrm{T}}\left(I + \beta_4 A\right)\left(I + \theta A\right)}{\left(I + \theta A\right)}. \tag{55}$$

$$\beta_4 e_4^{\mathrm{T}} = \beta^{\mathrm{T}}\left(I + \beta_4 A\right). \tag{56}$$

Thus

$$\beta_1 = a_{31}b_3\beta_4^3 + a_{21}b_2\beta_4^3 - a_{21}a_{32}b_3\beta_4^4 - b_1\beta_4^2. \tag{57}$$

$$\beta_2 = a_{32}b_3\beta_4^2 - b_2\beta_4^2. \tag{58}$$

$$\beta_3 = -b_3\beta_4^2. \tag{59}$$

$$\beta_4 = \beta_4. \tag{60}$$

And the proposed ARK34 with $c^{\mathrm{T}}=\left[\frac{1}{4},\frac{1}{2},1,1\right]$ is

$$\left[\begin{array}{c|c} A & U \\ \hline B & V \end{array}\right]=\left[\begin{array}{cccc|ccc} 0 & 0 & 0 & 0 & 1 & \frac{1}{4} & \frac{1}{32} \\ \frac{8}{9} & 0 & 0 & 0 & 1 & -\frac{7}{18} & -\frac{7}{72} \\ -\frac{2}{3} & \frac{1}{2} & 0 & 0 & 1 & \frac{7}{6} & \frac{5}{12} \\ 0 & \frac{2}{3} & \frac{1}{6} & 0 & 1 & \frac{1}{6} & 0 \\ \hline 0 & \frac{2}{3} & \frac{1}{6} & 0 & 1 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{8}{9} & 0 & -\frac{2}{3} & 2 & 0 & -\frac{4}{9} & 0 \end{array}\right]. \tag{61}$$

3. Convergence Analysis

For the method ARK3 represented by Equation (24), the matrix

$$V=\begin{bmatrix} 1 & \frac{5}{16} & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{3}{2} & 0 \end{bmatrix}, \tag{62}$$

must have bounded powers for the method to be stable.

The characteristic polynomial of V is given as

$$\rho(\lambda)=\det(\lambda I_3-V)=|\lambda I_3-V|. \tag{63}$$

$$\rho(\lambda)=\begin{vmatrix} \lambda-1 & -\frac{5}{16} & 0 \\ 0 & \lambda & 0 \\ 0 & 1 & \lambda \end{vmatrix}=\lambda^3-\lambda^2. \tag{64}$$

Thus $(\lambda_1,\lambda_2,\lambda_3)=(1,0,0)$.

Applying Cayley-Hamilton theorem to matrix V

$$\rho(V)=V^3-V^2=0. \tag{65}$$

$$\begin{bmatrix} 1 & \frac{5}{16} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}-\begin{bmatrix} 1 & \frac{5}{16} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{66}$$

This implies that

$$V^3=V^2. \tag{67}$$

Similarly,

$$V^4-V^2=0, V^5-V^2=0,\cdots,V^n=V^2 \tag{68}$$

for every n greater than 2. It implies V^n is bounded, which shows that the method is stable. It is known that methods of order at least one are always consistent; hence the method is consistent since the order of the method is $p=3>1$. Therefore, Hence the proposed scheme ARK3 is convergent due to the fact that it is both stable and consistent.

Similarly, for the ARK34 method of Equation (61), the matrix

$$V=\begin{bmatrix}1 & \frac{1}{6} & 0\\ 0 & 0 & 0\\ 0 & -\frac{4}{9} & 0\end{bmatrix}. \tag{69}$$

$$\rho(\lambda)=\det(\lambda I_n-V)=\det(\lambda I_3-V)=\lambda^3-\lambda^2. \tag{70}$$

And the eigenvalues are evaluated to be $(\lambda_1,\lambda_2,\lambda_3)=(1,0,0)$.

Thus,

$$\rho(V)=V^3-V^2=0. \tag{71}$$

And similarly, it implies that $V^3=V^2, V^4-V^2=0, V^5-V^2=0,\cdots,V^n=V^2$, for every n greater than 2. It indicates that V^n is bounded which shows that the method is stable. Also, the method is consistent since it is of order 3, *i.e.*, $p=3>1$. Hence the proposed scheme (ARK34) is convergent due to the fact that it is both stable and consistent.

4. Numerical Examples

Considering the problem below:

$$\left.\begin{array}{l} y(x)'=\dfrac{y(x)}{4}\left(1-\dfrac{y(x)}{20}\right),\quad y(0)=1\\ \text{steplength equals 0.1},\quad x\in[0,2]\\ \text{Analytical solution}: y_E(x)=\dfrac{20}{1+19e^{\frac{-1}{4}x}}\end{array}\right\}. \tag{72}$$

Source: Rattenbury [3].

Problem (72) is solved using the proposed ARK34 method. The results are obtained and compared with similar ARK34 methods of [3] and [5] respectively and presented in **Figure 1**.

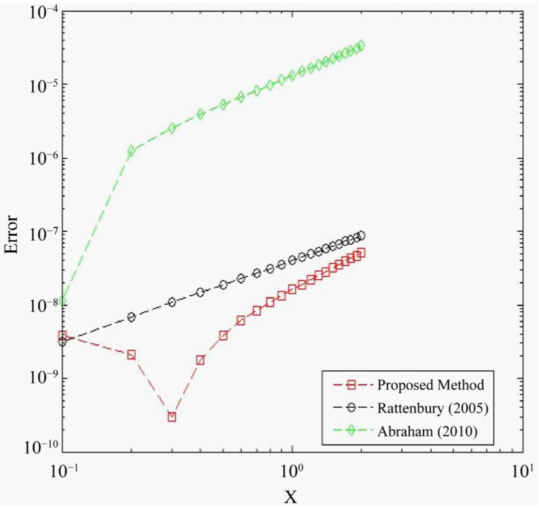

Figure 1. Comparison of ARK34 with other methods (h = 0.1).

From **Figure 1** it is evident that our Proposed ARK34 method performed better than the methods of [3] and [5] since it exhibits lesser error than the errors of the existing methods.

5. Conclusion

Two ARK methods are proposed, ARK3 $(s = p = 3)$ and ARK34 $(s = 4, p = 3)$. The methods have been proven to be consistent and stable, thereby guaranteeing their convergence. This is further illustrated by comparing the performance of one of the methods with other methods of similar order. The proposed method ARK34 is shown to perform better than the existing methods.

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