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Coupled Fixed Point for (α, ψ)-Contractive in Partially Ordered Metric Spaces Using Compatible Mappings

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Abstract

In this paper, first we introduce notions of (α, ψ)-contractive and (α)-admissible for a pair of map and prove a coupled coincidence point theorem for compatible mappings using these notions. Our work extends and generalizes the results of Mursaleen *et al.* [1]. At the end, we will provide an example in support of our result.

Keywords

Coupled Coincidence Point, α-ψ-Contractive Mapping, Compatible Mappings

1. Introduction

Fixed point theorems give the conditions under which maps have solutions.

Fixed point theory is a beautiful mixture of Analysis, Topology and Geometry. Fixed points Theory has been playing a vital role in the study of nonlinear phenomena. In particular, fixed point techniques have been applied in diverse fields as Biology, Chemistry, and Economics, Engineering, Game theory and Physics. The usefulness of the concrete applications has increased enormously due to the development of accurate techniques for computing fixed points.

The fixed point theory has many important applications in numerical methods like Newton-Raphson Method and establishing Picard's Existence Theorem regarding existence and uniqueness of solution of first order differential equation, existence of solution of integral equations and a system of linear equations. The credit of making the concept of fixed point theory useful and popular goes to polish mathematician Stefan Banach. In 1922, Banachproved a fixed point theorem, which ensures the existence and uniqueness of a fixed point under appropriate conditions. This result of Banach is known as Banach fixed point theoremor contraction mapping prin-

ciple, "Let x be any non empty set and (X,d) be a completemetric space If T is mapping of X into itself satisfying $d(Tx,Ty) \le kd(x,y)$ for each $x, y \in X$ where $0 \le k < 1$, then T has a unique fixed point in X". This principle provides a technique for solving a variety of applied problems in Mathematical sciences and Engineering and guarantees the existence and uniqueness of fixed points of certain self maps of metric spaces and provides a constructive method to find out fixed points. Now the question arise what type of problems have the fixed point. The fixed point problems can be elaborated in the following manner:

1) What functions/maps have a fixed point?
2) How do we determine the fixed point?
3) Is the fixed point unique?

Currently, fixed point theory has been receiving much attention on in partially ordered metric spaces; that is, metric spaces endowed with a partial ordering. Turinici [2] extending the Banach contraction principle in the setting of partially ordered sets and laid the foundation a new trend in fixed point theory. Ran and Reurings [3] developed some applications of Turinici's theorem to matrix equations and established some results in this direction. The results were further extended by Nieto and Rodŕguez-Ĺpez [4] [5] for non-decreasing mappings. Bhaskar and Lakshmikantham [6] [7] introduced the new notion of coupled fixed points for the mappings satisfying the mixed monotone property in partially ordered spaces and discussed the existence and uniqueness of a solution for a periodic boundary value problem. Later on, Lakshmikantham and Ciríc [8] proved coupled coincidence and coupled common fixed point theorems for nonlinear contractive mappings in partially ordered complete metric spaces.

Choudhury and Kundu [9], proved the coupled coincidence result for compatible mappings in the settings of partially ordered metric space. Recently, Samet *et al.* [10] [11] have introduced the notion of α-ψ-contractive and α-admissible mapping and proved fixed point theorems for such mappings in complete metric spaces. For more results regarding coupled fixed points in various metric spaces one can refer to [12]-[23].

In this paper, we will generalize the results of Mursaleen *et al.* [1] for α-ψ-contractive and α-admissible mappings using compatible mappings under α-ψ-contractions and α-admissible conditions.

2. Mathematical Preliminaries

In order to obtain our results we need to consider the followings.

Definition 2.1. [6]. Let $(X, \le)$ be a partially ordered set and $X \times X \to X$ be a mapping. Then a map F is said to have the mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and is monotone non-increasing in y; that is, for any $x, y \in X$,

$x_1, x_2 \in X, x_1 \le x_2$ implies $F(x_1, y) \le F(x_2, y)$ and
$y_1, y_2 \in X, y_1 \le y_2$ implies $F(x, y_1) \ge F(x, y_2)$.

Definition 2.2. [6]. An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \to X$ if

$F(x, y) = x$ and $F(y, x) = y$.

Definition 2.3. [8]. Let (X, d) be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$ be two mappings. We say F has the mixed g-monotone property if F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument, that is, for any $x, y \in X$

$x_1, x_2 \in X$; $g(x_1) \le g(x_2)$ implies $F(x_1, y) \le F(x_2, y)$ and
$y_1, y_2 \in X$; $g(y_1) \le g(y_2)$ implies $F(x, y_1) \ge F(x, y_2)$.

Definition 2.4. [8]. An element $(x, y) \in X \times X$ is called a coupled coincidence point of mappings $F : X \times X \to X$ and $g : X \to X$ if

$F(x, y) = g(x)$ and $F(y, x) = g(y)$.

Choudhury *et al.* [9] introduced the notion of compatible maps in partially ordered metric spaces as follows:

Definition 2.5. [9]. The mappings F and g where $F : X \times X \to X$ and $g : X \to X$ be are said to be compatible if

$$\lim_{n \to \infty} d\left(g\left(F\left(x_n, y_n\right)\right), F\left(g\left(x_n\right), g\left(y_n\right)\right)\right) = 0 \text{ and}$$

$$\lim_{n \to \infty} d\left(g\left(F\left(y_n, x_n\right)\right), F\left(g\left(y_n\right), g\left(x_n\right)\right)\right) = 0$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X, such that

$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$ for all $x, y \in X$ are satisfied.

In order to obtain our results we need to consider the followings.

Definition 2.6. [1]. Denote by Ψ the family of non-decreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that

$\sum_{n=0}^{\infty} \psi_n(t) < t$ for all $t > 0$, where ψ_n is the nth iterate of ψ satisfying

1) $\psi^{-1}(\{0\}) = \{0\}$.

2) $\psi(t) < t$ for all $t > 0$ and

3) $\lim_{r \to t+} \psi(r) < t$ for all $t > 0$.

Lemma 2.7. [1]. If $\psi : [0, \infty] \to [0, \infty]$ is non-decreasing and right continuous, then

$\psi_n(t) \to 0$ as $n \to \infty$ for all $t \geq 0$ if and only if $\psi(t) < t$ for all $t > 0$.

Definition 2.8. [1]. Let (X, d) be a partially ordered metric space and $F : X \times X \to X$ then F is said to be α-contractive if there exist two functions

$\alpha : X_2 \times X_2 \to [0, +\infty)$ and $\psi \in \Psi$ such that

$$\alpha((x, y), (u, v)) d(F(x, y), F(u, v)) \leq \psi\left(\frac{d(x, u) + d(y, v)}{2}\right) \text{ for } x, y, u, v \in X \text{ with } x \geq u \text{ and } y \leq v.$$

Definition 2.9. [1]. Let $F : X \times X \to X$ and $\alpha : X_2 \times X_2 \to [0, +\infty)$ be two mappings. Then F is said to be (α)-admissible if

$\alpha((x, y), (u, v)) \geq 1$ implies $\alpha((F(x, y), F(y, x)), (F(u, v), F(v, u))) \geq 1$, for all $x, y, u, v \in X$.

Now, we will introduce our notions:

Definition 2.10. Let (X, d) be a partially ordered metric space and $F : X \times X \to X$ and $g : X \to X$ be two mappings. Then the maps F *and g are* said to be (α, ψ)-*contractive* if there exist two functions

$\alpha : X^2 \times X^2 \to [0, +\infty)$ and $\psi \in \Psi$ such that

$$\alpha((g(x), g(y)), (g(u), g(v))) d(F(x, y), F(u, v)) \leq \psi\left(\frac{d(g(x), g(u)) + d(g(y), g(v))}{2}\right) \text{ for } x, y, u, v \in X$$

with $g(x) \geq g(u)$ and $g(y) \leq g(v)$.

Definition 2.11. Let $F : X \times X \to X$, $g : X \to X$ and $\alpha : X_2 \times X_2 \to [0, +\infty)$ be mappings.

Then F *and g are* said to be (α)-*admissible* if

$$\alpha((g(x), g(y)), (g(u), g(v))) \geq 1 \text{ implies } \alpha((F(x, y), F(y, x)), (F(u, v), F(v, u))) \geq 1, \text{ for all}$$

$x, y, u, v \in X$.

3. Main Results

Recently, Mursaleen *et al.* [1] proved the following coupled fixed point theorem with α-ψ-contractive conditions in partial ordered metric spaces:

Theorem 3.1 [1] Let $(X, \leq)$ be a partially ordered set and there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be mapping and suppose F has mixed monotone property. Suppose there exists $\psi \in \Psi$ and $\alpha : X^2 \times X^2 \to [0, +\infty]$

Such that for $x, y, u, v \in X$, the following holds:

$$\alpha((x, y), (u, v)) d(F(x, y), F(u, v)) \leq \psi\left(\frac{d(x, u) + d(y, v)}{2}\right) \text{ for } x, y, u, v \in X, \text{ with } x \geq u \text{ and } y \leq v.$$

Suppose also that

1) F is (α)-admissible.

2) There exists $x_o, y_o \in X$ such that

$$\alpha((x_o, y_o), (F(x_o, y_o), F(y_o, x_o))) \geq 1 \text{ and } \alpha((y_o, x_o), (F(y_o, x_o), F(x_o, y_o))) \geq 1.$$

3) F is continuous.

If there exists $x_o, y_o \in X$ such that $x_o \leq F(x_o, y_o)$ and $y_o \geq F(y_o, x_o)$.

Then F has a coupled fixed point, that is, there exist, $x, y \in X$ such that $F(x, y) = x$ and $F(y, x) = y$.

Now we are ready to prove our results for compatible mappings.

Theorem 3.2 Let $(X,\leq)$ be a partially ordered set and there exists a metric d on X such that (X,d) is a complete metric space. Let $F : X \times X \to X$ be mapping and $g : X \to X$ be another mapping. Suppose F has g-mixed monotone property and there exists $\psi \in \Psi$ and $\alpha : X^2 \times X^2 \to [0,+\infty]$

$$\alpha\big((g(x),g(y)),(g(u),g(v))\big)d\big(F(x,y),F(u,v)\big) \leq \psi\left(\frac{d\big(g(x),g(u)\big)+d\big(g(y),g(v)\big)}{2}\right) \tag{3.3}$$

For all $x,y,u,v \in X$ with $g(x) \geq g(u)$ and $g(y) \geq g(v)$.
Suppose also that
1) F and g are (α)-admissible.
2) There exists $x_o, y_o \in X$ such that

$$\alpha\big((g(x_o),g(y_o)),(F(x_o,y_o),F(y_o,x_o))\big) \geq 1 \text{ and } \alpha\big(g(y_o),g(x_o)),(F(y_o,x_o),F(x_o,y_o))\big) \geq 1$$

3) $F(X \times X) \subseteq g(X)$, g is continuous and F and g are compatible in X.
4) F is continuous.
If there exists $x_o, y_o \in X$ such that $g(x_o) \leq F(x_o,y_o)$ and $g(y_o) \geq F(y_o,x_o)$.
Then F and g has coupled coincidence point that is there exist, $x,y \in X$ such that

$$F(x,y) = g(x) \text{ and } F(y,x) = g(y).$$

Proof: Let $x_o, y_o \in X$ be such that

$$\alpha\big((g(x_o),g(y_o)),(F(x_o,y_o),F(y_o,x_o))\big) \geq 1 \text{ and } \alpha\big(g(y_o),g(x_o)),(F(y_o,x_o),F(x_o,y_o))\big) \geq 1.$$

and $g(x_o) \leq F(x_0,y_o) = g(x_1)$ and $g(y_o) \geq F(y_0,x_o) = g(y_1)$.
Let $x_2, y_2 \in X$ be such that, $F(x_1,y_1) = g(x_2)$ and $F(y_1,x_1) = g(y_2)$.
Continuing this process, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X as follows:

$$g(x_{n+1}) = F(x_n,y_n) \text{ and } g(y_{n+1}) = F(y_n,x_n) \text{ for all, } n \geq 0.$$

Now we will show that

$$g(x_n) \leq g(x_{n+1}) \text{ and } g(y_n) \geq g(y_{n+1}) \text{ for all, } n \geq 0. \tag{3.4}$$

For $n = 0$, since, $g(x_o) \leq F(x_o,y_o)$ and $g(y_o) \geq F(y_o,x_o)$
and as $g(x_1) = F(x_o,y_o)$ and $g(y_1) = F(y_o,x_o)$
We have, $g(x_o) \leq g(x_1)$ and $g(y_o) \geq g(y_1)$.
Thus (3.4) holds for $n = 0$.
Now suppose that (3.4) holds for some fixed $n \geq 0$.
Then, since $g(x_n) \leq g(x_{n+1})$ and $g(y_n) \geq g(y_{n+1})$.
Therefore, by g-mixed monotone property of F, we have

$$g(x_{n+2}) = F(x_{n+1},y_{n+1}) \geq F(x_n,y_{n+1}) \geq F(x_n,y_n) = g(x_{n+1}) \text{ and}$$

$$g(y_{n+2}) = F(y_{n+1},x_{n+1}) \leq F(y_n,x_{n+1}) \leq F(y_n,x_n) = g(y_{n+1}).$$

From above, we conclude that

$$g(x_{n+1}) \leq g(x_{n+2}) \text{ and } g(y_{n+1}) \geq g(y_{n+2}).$$

Thus, by mathematical induction, we conclude that (3.4) holds for all $n \geq 0$.
If following holds for some $n \in N$,

$$\big(g(x_{n+1}),g(y_{n+1})\big) = \big(g(x_n),g(y_n)\big)$$

Then obviously, $F(x_n,y_n) = g(x_n)$ and $F(y_n,x_n) = g(y_n)$, *i.e.*, F has coupled coincidence point.
Now, we assume that $(x_{n+1},y_{n+1}) \neq (x_n,y_n)$ for all, $n \geq n(\epsilon)$.
Since, F and g α-admissible, we have

$$\alpha\big(\big(g(x_o),g(y_o)\big),\big(g(x_1),g(y_1)\big)\big)=\alpha\big(g(x_o),g(y_o)\big),\ \big(F(x_0,y_o),F(y_o,x_o)\big)\geq 1,$$

implies, $\alpha\big(\big(F(x_0,y_o),F(y_o,x_o)\big),\big(F(x_1,y_1),F(y_1,x_1)\big)\big)=\alpha\big(\big(g(x_1),g(y_1)\big),\big(g(x_2),g(y_2)\big)\big)\geq 1.$

Thus by mathematical induction, we have

$$\alpha\big(\big(g(x_n),g(y_n)\big),\big(g(x_{n+1}),g(y_{n+1})\big)\big)\geq 1 \tag{3.5}$$

Similarly, we have

$$\alpha\big(\big(g(y_n),g(x_n)\big),\big(g(y_{n+1}),g(x_{n+1})\big)\big)\geq 1 \text{ for all, } n\in N. \tag{3.6}$$

From (3.3) and conditions 1) and 2) of hypothesis, we get

$$
\begin{aligned}
d\big(g(x_n),g(x_{n+1})\big) &= d\big(F(x_{n-1},y_{n-1}),F(x_n,y_n)\big)\\
&\leq \alpha\big(\big(g(x_{n-1}),g(y_{n-1})\big),\big(g(x_n),g(y_n)\big)\big)d\big(F(x_{n-1},y_{n-1}),F(x_n,y_n)\big)\\
&\leq \psi\left(\frac{d\big(g(x_{n-1}),g(x_n)\big)+d\big(g(y_{n-1}),g(y_n)\big)}{2}\right).
\end{aligned}
\tag{3.7}
$$

Similarly, we have

$$
\begin{aligned}
d\big(g(y_n),g(y_{n+1})\big) &= d\big(F(y_{n-1},x_{n-1}),F(y_n,x_n)\big)\\
&\leq \alpha\big(\big(g(y_{n-1}),g(x_{n-1})\big),\big(g(y_n),g(x_n)\big)\big)d\big(F(y_{n-1},x_{n-1}),F(y_n,x_n)\big)\\
&\leq \psi\left(\frac{d\big(g(y_{n-1}),g(y_n)\big)+d\big(g(x_{n-1}),g(x_n)\big)}{2}\right)
\end{aligned}
\tag{3.8}
$$

On adding (3.7) and (3.8), we get

$$\frac{d\big(g(x_n),g(x_{n+1})\big)+d\big(g(y_n),g(y_{n+1})\big)}{2}\leq \psi\left(\frac{d\big(g(x_{n-1})g(x_n)\big)+d\big(g(y_{n-1}),g(y_n)\big)}{2}\right)$$

Repeating the above process, we get

$$\frac{d\big(g(x_n),g(x_{n+1})\big)+d\big(g(y_n),g(y_{n+1})\big)}{2}\leq \psi^n\left(\frac{d\big(g(x_o)g(x_1)\big)+d\big(g(y_0),g(y_1)\big)}{2}\right)\text{ for }n\in N.$$

For $\varepsilon>0$ there exists $n(\varepsilon)\in N$ such that

$$\sum_{n\geq n(\varepsilon)}\psi^n\left(\frac{d\big(g(x_o),g(x_1)\big)+d\big(g(y_o),g(y_1)\big)}{2}\right)<\frac{\varepsilon}{2}.$$

Let $n,m\in N$ be such that $m>n>n(\varepsilon)$, then by using the triangle inequality, we have

$$
\begin{aligned}
&\frac{d\big(g(x_n),g(x_m)\big)+d\big(g(y_n),g(y_m)\big)}{2}\\
&\leq \sum_{k=n}^{m-1}\frac{d\big(g(x_k),g(x_{k+1})\big)+d\big(g(y_n),g(y_{n+1})\big)}{2}\\
&\leq \sum_{k=n}^{m-1}\psi^k\left(\frac{d\big(g(x_o),g(x_1)\big)+d\big(g(y_o),g(y_1)\big)}{2}\right)\\
&\leq \sum_{n\geq n(\in)}\psi^n\left(\frac{d\big(g(x_o),g(x_1)\big)+d\big(g(y_o),g(y_1)\big)}{2}\right)<\frac{\varepsilon}{2}.
\end{aligned}
$$

that is; $d\big(g(x_n),g(x_m)\big)+d\big(g(y_n),g(y_n)\big)<\varepsilon$

Since, $d\big(g(x_n),g(x_m)\big)\le d\big(g(x_n),g(x_m)\big)+d\big(g(y_n),g(y_m)\big)<\varepsilon$ and

$$d\big(g(y_n),g(y_m)\big)\le d\big(g(x_n),g(x_m)\big)+d\big(g(y_n),g(y_m)\big)<\varepsilon.$$

Hence, $g(x_n)$ and $g(y_n)$ are Cauchy sequences in (X,d).
Since, (X,d) is complete, therefore, $g(x_n)$ and $g(y_n)$ are convergent in (X,d).
There exists, $x,y\in X$ such that

$$\lim_{n\to\infty}F(x_n,y_n)=\lim_{n\to\infty}g(x_n)=x \text{ and}$$

$$\lim_{n\to\infty}F(y_n,x_n)=\lim_{n\to\infty}g(y_n)=y$$

Since, F and g are compatible mappings; therefore, we have

$$\lim_{n\to\infty}d\big(g(F(x_n,y_n)),F(g(x_n),g(y_n))\big)=0 \tag{3.9}$$

$$\lim_{n\to\infty}d\big(g(F(y_n,x_n)),F(g(y_n),g(x_n))\big)=0 \tag{3.10}$$

Next we will show that $g(x)=F(x,y)$ and $g(y)=F(y,x)$.
For all $n\ge 0$, we have

$$d\big(g(x),F(g(x_n),g(y_n))\big)\le d\big(g(x),g(F(x_n,y_n))\big)+d\big(g(F(x_n,y_n)),F(g(x_n),g(y_n))\big).$$

Taking limit $n\to\infty$ in the above inequality by continuity of F and g and from (3.9) we get

$$d\big(g(x),F(x,y)\big)=0.$$

Similarly, we have $d\big(g(y),F(y,x)\big)=0$.
Thus
$g(x)=F(x,y)$ and $g(y)=F(y,x)$.
Hence, we have proved that F and g has coupled coincidence point.
Now, we will replace continuity of F in the theorem 3.2 by a condition on sequences.

Theorem 3.3. Let $(X,\le)$ be a partially ordered set and there exists a metric d on X such that (X,d) is a complete metric space. Let $F:X\times X\to X$ and $g:X\to X$ be maps and F has g-mixed monotone property. Suppose there exists $\psi\in\Psi$ *and* $\alpha:X^2\times X^2\to[0,+\infty]$ such that for $x,y,u,v\in X$, the following holds:
1) Inequality (3.3) and conditions 1), 2) and 3) hold.
2) if $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\alpha\big((g(x_n),g(y_n)),(g(x_{n+1}),g(y_{n+1}))\big)\ge 1 \text{ and } \alpha\big((g(y_n),g(x_n)),(g(y_{n+1}),g(x_{n+1}))\big)\ge 1$$

for all n and $\lim_{n\to\infty}g(x_n)=x$ and $\lim_{n\to\infty}g(y_n)=y$, for all $x,y\in X$, then

$$\alpha\big((g(x_n),g(y_n)),(g(x),g(y))\big)\ge 1 \text{ and } \alpha\big((g(y_n),g(x_n)),(g(y),g(x))\big)\ge 1.$$

If there exists $x_o,y_o\in X$ such that $g(x_o)\le F(x_o,y_o)$ and $g(y_o)\ge F(y_o,x_o)$.
Then F and g has coupled coincidence point, that is, there exist, $x,y\in X$ such that

$$F(x,y)=g(x) \text{ and } F(y,x)=g(y).$$

Proof. Proceeding along the same lines as in the proof of Theorem 3.2, we know that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences in the complete metric space (X,d). Then there exists $x,y\in X$ such that $\lim_{n\to\infty}g(x_n)=x$ and $\lim_{n\to\infty}g(y_n)=y$

$$\alpha\big((g(x_n),g(y_n)),(g(x),g(y))\big)\ge 1 \tag{3.11}$$

$$\text{Similarly, } \alpha\big((g(y_n),g(x_n)),(g(y),g(x))\big)\ge 1 \tag{3.12}$$

Using the triangle inequality, (3.11) and the property of $\psi(t) < t$ for all $t > 0$, we get

$$d(F(x,y), g(x)) \le d(F(x,y), F(x_n, y_n)) + d(g(x_{n+1}), g(x))$$

$$\le \alpha((g(x_n), g(y_n)), (g(x), g(y))) d(F(x_n, y_n), F(x,y)) + d(g(x_{n+1}), g(x))$$

$$\le \psi\left(\frac{d(g(x), g(x_n)) + d(g(y), g(y_n))}{2}\right) + d(g(x_{n+1}), g(x))$$

$$< \frac{d(g(x_n), g(x)) + d(g(y_n), g(y))}{2} + d(g(x_{n+1}), g(x)).$$

Similarly, on using (3.12), we have

$$d(F(y,x), g(y)) \le d(F(y,x), F(y_n, x_n)) + d(g(y_{n+1}), g(y))$$

$$\le \alpha((g(y_n), g(x_n)), (g(y), g(x))) d(F(y_n, x_n), F(y,x)) + d(g(y_{n+1}), g(y))$$

$$\le \psi\left(\frac{d(g(y), g(y_n)) + d(g(x), g(x_n))}{2}\right) + d(g(y_{n+1}), g(y))$$

$$< \frac{d(g(y_n), g(y)) + d(g(x_n), g(x))}{2} + d(g(y_{n+1}), g(y)).$$

Proceeding limit $n \to \infty$ in above two inequalities, we get

$$d(F(x,y), g(x)) = 0 \quad \text{and} \quad d(F(y,x), g(y)) = 0.$$

Thus, $F(x,y) = g(x)$ and $F(y,x) = g(y)$.

Remark. On putting $g = I$, identity map, we get the required result of Mursaleen *et al*. [14].

Example 3.4. Let $X = [0,1]$. Then $(X, \le)$ is a partially ordered set with the natural ordering of real numbers. Let

$d(x,y) = |x - y|$ for $x, y \in [0,1]$.

Then (X, d) is a complete metric space.

Let $g : X \to X$ be defined as $g(x) = x^2$, for all $x \in X$.

Let $F : X \times X \to X$ be defined as $F(x,y) = \dfrac{x^2 - y^2}{3} + \dfrac{2}{3}$

Let $\psi : [0,1) \to [0,1)$ be defined as $\psi(t) = \dfrac{2t}{3}$ for $t \in [0,1)$.

Let, $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that,
$\lim_{n \to \infty} F(x_n, y_n) = a$, $\lim_{n \to \infty} g(x_n) = a$, $\lim_{n \to \infty} F(y_n, x_n) = b$, $\lim_{n \to \infty} g(y_n) = b$.

Then obviously, $a = \dfrac{2}{3}$ and $b = \dfrac{2}{3}$

Now, for all $n \ge 0$, $g(x_n) = x_n^2$, $g(y_n) = y_n^2$

$$F(x_n, y_n) = \frac{x_n^2 - y_n^2}{3} + \frac{2}{3} \quad \text{and} \quad F(y_n, x_n) = \frac{y_n^2 - x_n^2}{3} + \frac{2}{3}$$

Then it follows that,

$$\lim_{n \to \infty} d(g(F(x_n, y_n)), F(g(x_n), g(y_n))) = 0 \quad \text{and} \quad \lim_{n \to \infty} d(g(F(y_n, x_n)), F(g(y_n), g(x_n))) = 0.$$

Hence, the mappings F and g are compatible in X.

Consider a mapping $\alpha : X_2 \times X_2 \to [0, +\infty)$ be such that

$$\alpha((g(x), g(y)), (g(u), g(v))) = \begin{cases} 1 & \text{if } x \ge y \text{ and } u \ge v \\ 0 & \text{otherwise} \end{cases}$$

$$d\left(F\left(x,y\right),F\left(u,v\right)\right)=\left|\frac{x^2-y^2}{3}+\frac{2}{3}-\left(\frac{u^2-v^2}{3}+\frac{2}{3}\right)\right|$$

$$=\frac{1}{3}\left|\left(x^2-u^2\right)+(v^2-y^2)\right|\le\frac{1}{3}\left(\left|\left(x^2-u^2\right)\right|+\left|\left(v^2-y^2\right)\right|\right)$$

$$\le\frac{1}{3}\left(d\left(g\left(x\right),g\left(u\right)\right)+d\left(g\left(y\right),g\left(v\right)\right)\right).$$

Thus (3.3) holds for $\psi\left(t\right)=\frac{2t}{3}$ for all $t>0$, and we also see that $F\left(X\times X\right)\subseteq g\left(X\right)$ and F satisfies g-mixed monotone property. Let $x_0=0.6$ and $y_0=0.9$. Then $g\left(x_0\right)=\left(0.6\right)^2=0.36\le0.51=F\left(x_0,y_0\right)$ and $g\left(y_0\right)=\left(0.9\right)^2=0.81\ge0.81=F\left(y_0,x_0\right)$. Thus, all the conditions of theorem 3.2 are satisfied. Here $\left(\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}}\right)$ is a coupled coincidence point of *g* and *F* in *X*.

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