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Study of the Convergence of the Increments of Gaussian Process

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Abstract

Let $\{X(t); t \geq 0\}$ be a Gaussian process with stationary increments $E\{X(t+s)-X(t)\}^2 = \sigma^2(s)$. Let $a_t\,(t \geq 0)$ be a nondecreasing function of t with $0 \leq a_t \leq t$. This paper aims to study the almost sure behaviour of $\limsup_{k\to\infty} \sup_{0\leq s\leq a_{t_k}} \beta_{(t_k,\alpha)} \left| X(t_k+s) - X(t_k) \right|$ where

$$\beta_{(t_k,\alpha)} = \left[2\sigma^2\left(a_{t_k}\right)\left(\log\left(t_k/a_{t_k}\right) + \alpha \log\log t_k + (1-\alpha)\log\log a_{t_k}\right)\right]^{-1/2}$$

with $0 \leq \alpha \leq 1$ and $\{t_k\}$ is an increasing sequence diverging to ∞.

Keywords

Wiener Process, Gaussian Process, Law of the Iterated Logarithm, Regularly Varying Function

1. Introduction

Let $\{W(t); t \geq 0\}$ be a standard Wiener process. Suppose that $a_t\,(t \geq 0)$ is a nondecreasing function of t such that $0 < a_t \leq t$ with a_t/t is nonincreasing and $\{t_k\}$ is an increasing sequence diverging to ∞. In [1] the following results are established.

i) If $\limsup_{k\to\infty} \left(t_{k+1} - t_k\right)/a_{t_k} < 1$, then

$$\limsup_{k\to\infty} \sup_{0\leq s\leq a_{t_k}} \lambda_{(t_k,\alpha)} \left| W(t_k+s) - W(t_k) \right| = 1 \qquad a.s. \tag{1}$$

and

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$$\limsup_{k\to\infty} \lambda_{(t_k,\alpha)} \left| W\left(t_k + a_{t_k}\right) - W\left(t_k\right) \right| = 1 \qquad a.s. \tag{2}$$

where $0 \le \alpha \le 1$ and

$$\lambda_{(t_k,\alpha)} = \left[2 a_{t_k} \left(\log\left(t_k / a_{t_k}\right) + \alpha \log\log t_k + (1-\alpha) \log\log a_{t_k} \right) \right]^{-1/2}.$$

ii) If $\liminf_{k\to\infty} \left(t_{k+1} - t_k\right)/a_{t_k} > 1$, then

$$\limsup_{k\to\infty} \lambda_{(t_k,\alpha)} \left| W\left(t_k + a_{t_k}\right) - W\left(t_k\right) \right| = \limsup_{k\to\infty} \sup_{0\le s\le a_{t_k}} \lambda_{(t_k,\alpha)} \left| W\left(t_k + s\right) - W\left(t_k\right) \right| = \varepsilon^* \qquad a.s.,$$

where $0 \le \alpha \le 1$, $\varepsilon^* = \inf\left\{ \gamma > 0 : \sum_k \left(g_\alpha\left(t_k\right)\right)^{-\gamma^2} < \infty \right\}$ and $g_\alpha\left(t_k\right) = t_k \left(\log t_k\right)^\alpha \left(\log a_{t_k}\right)^{1-\alpha} / a_{t_k}$.

In this paper the limit theorems on increments of a Wiener process due to [1] are developed to the case of a Gaussian process. This can be considered also as an extension of the results to Gaussian processes obtained in [2]. Throughout this paper, we shall always assume the following statements: Let $\{X(t); t \ge 0\}$ be an almost surely continuous Gaussian process with $X(0) = 0$, $E\{X(t)\} = 0$ and $E\{X(t+s) - X(t)\}^2 = \sigma^2(s)$, where $\sigma(s)$ is a function of $s \ge 0$. Further we assume that $\sigma(t)$, $t \ge 0$, is a nondecreasing continuous concave, regularly varying function at exponent $\tau (0 < \tau < 1)$ at ∞ (e.g., if $\{X(t); t \ge 0\}$ is a standard Wiener process, then $\sigma(t) = \sqrt{t}$).

Let $a_t (t > 0)$ be a nondecreasing function of t with $0 < a_t \le t$. For large t, let us denote

$$\beta_{(t_k,\alpha)} = \left[2\sigma^2\left(a_{t_k}\right)\left(\log h_\alpha(t)\right) \right]^{-1/2}$$

where $0 \le \alpha \le 1$ and $h_\alpha(t) = t(\log t)^\alpha (\log a_t)^{1-\alpha} / a_t$ is an increasing function of t.

We define two continuous parameter processes $Y_1(t)$ and $Y_2(t)$ by

$$Y_1(t) = \sup_{0\le s\le a_t} \left| X(t+s) - X(t) \right|$$

and

$$Y_2(t) = \left| X\left(t + a_t\right) - X(t) \right|.$$

2. Main Results

In this section we provide the following two theorems which are the main results. We concern here with the development of the limit theorems of a Wiener process to the case of a Gaussian process under consideration the above given assumptions.

Theorem 1. Let $a_t (t > 0)$ be a nondecreasing function of t where $0 < a_t \le t$ with the nonincreasing function a_t / t and let $\{t_k\}$ be any increasing sequence diverging to ∞ such that

$$\limsup_{k\to\infty} \left(t_{k+1} - t_k\right)/a_{t_k} < 1, \tag{3}$$

then

$$\limsup_{k\to\infty} \beta_{(t_k,\alpha)} Y_1\left(t_k\right) = 1 \qquad a.s. \tag{4}$$

and

$$\limsup_{k\to\infty} \beta_{(t_k,\alpha)} Y_2\left(t_k\right) = 1 \qquad a.s., \tag{5}$$

where $\beta_{(t_k,\alpha)} = \left[2\sigma^2\left(a_{t_k}\right)\left(\log h_\alpha(t)\right) \right]^{-1/2}$.

We note that $\beta_{(t_k,\alpha)} \ge \lambda_{(t_k,\alpha)}$ for large k in case of the Wiener process. It is interesting to compare (1) and (2) with (4) and (5) respectively.

Theorem 2. Let $a_t\,(t>0)$ be a nondecreasing function of t where $0<a_t\le t$ with the nonincreasing function a_t/t and let $\{t_k\}$ be an increasing sequence diverging to ∞ such that

$$\liminf_{k\to\infty}\left(t_{k+1}-t_k\right)/a_{t_k}>1, \tag{6}$$

then

$$\limsup_{k\to\infty}\beta_{(t_k,\alpha)}Y_1(t_k)=\varepsilon^{**}\qquad a.s. \tag{7}$$

and

$$\limsup_{k\to\infty}\beta_{(t_k,\alpha)}Y_2(t_k)=\varepsilon^{**}\qquad a.s., \tag{8}$$

where $0\le\alpha\le1$ and $\varepsilon^{**}=\inf\left\{\gamma>0:\ \sum_k\left(h_\alpha(t_k)\right)^{-\gamma^2}<\infty\right\}$.

3. Proofs

In order to prove Theorems 1 and 2, we need to give the following lemmas.

Lemma 1. (See [3]). For any small $\varepsilon'>0$ there exists a positive $C_{\varepsilon'}$ depending on ε' such that for all $u>0$

$$P\left\{\sup_{0\le s\le m}\left|\frac{X(t+s)-X(t)}{\sigma(m)}\right|>u\right\}\le C_{\varepsilon'}\,u\,\mathrm{e}^{-u^2/(2+\varepsilon')},$$

where m is any large number and $\{X(t);\,t\ge0\}$ is defined above.

Lemma 2. (See [4]) Let $\{X(t);\,t\in T\}$ and $\{Y(t);\,t\in T\}$ be centered Gaussian processes such that $EX^2(t)=EY^2(t)$ for all $t\in T$ and $E\{X(t)X(s)\}\le E\{Y(t)Y(s)\}$ for all $s,t\in T$. Then for any real number u

$$P\left\{\sup_{t\in T}X(t)\le u\right\}\le P\left\{\sup_{t\in T}Y(t)\le u\right\}.$$

Proof of Theorem 1. Firstly, we prove that

$$\limsup_{k\to\infty}\beta_{(t_k,\alpha)}\left|Y_1(t_k)\right|\le1\qquad a.s. \tag{9}$$

For any $\{t_k\}$ with the condition (3), we define an increasing sequence $\{u_k\}$ by

$$0<u_k<t_k\le u_{k+1}\text{ and }a_{u_k}<t_{k+1}-t_k,\ k\ge1.$$

For instance, let $t_k=k^\beta$ for some $\beta\ge1$,

$$u_k=\left(\frac{k}{k+1}\right)^\beta t_k\text{ and }a_{t_k}=\left(\frac{k+1}{k+2}\right)^\beta t_k.$$

The condition (3) is satisfied, and for large k, $u_k<t_k\le u_{k+1}$ and $a_{u_k}<a_{t_k}<t_k$. By Lemma 1, we have, for any small $\varepsilon>0$,

$$\begin{aligned}P\left\{\beta_{(t_k,\alpha)}Y_1(u_k)\le1+\varepsilon\right\}&=P\left\{\sup_{0\le s\le a_{u_k}}\frac{X(u_k+s)-X(u_k)}{\sigma\left(a_{u_k}\right)}\le(1+\varepsilon)\left(2\log h_\alpha(t)\right)^{1/2}\right\}\\&\ge1-2C_\varepsilon\left(h_\alpha(u_k)\right)^{-2(1+\varepsilon)^2/(2+\varepsilon)}\ge\exp\left(-C'\left(h_\alpha(u_k)\right)^{-1}\right)\\&\ge\exp\left(-C'\Big/\left(\left(\log u_k\right)^\alpha\left(\log a_{u_k}\right)^{1-\alpha}\right)\right)\\&\ge\exp\left(-C'\left(\log u_k/\log a_{u_k}\right)^{1-\alpha}\left(1/\log u_k\right)\right)\\&\ge\exp\left(-C'\left(\log u_k/\log a_{u_k}\right)\left(1/\log u_k\right)\right)\\&\ge\exp\left(-C'\left(\log a_{u_k}\right)^{-1}\right)\end{aligned} \tag{10}$$

where k is large enough and C' is a constant. By the definition of a_{u_k}, $S=\sum_k \exp\left(-C'\left(\log a_{u_k}\right)^{-1}\right)=\infty$.

We shall follow the similar proof process as in [5]. Set

$$S=\sum_k \exp\left(-C'\left(\log a_{u_{2k-1}}\right)^{-1}\right)+\sum_k \exp\left(-C'\left(\log a_{u_{2k}}\right)^{-1}\right)=S_1+S_2.$$

Since $\{a_{u_k}\}$ is an increasing sequence, the fact that $S=\infty$ implies $S_1=S_2=\infty$. Consider the odd subsequence $\{t_{2k-1}\}$ of $\{t_k\}$ and define the sequence of events $\{A_k\}$ in the following form

$$A_k=\left\{\beta_{(t_{2k-1},\,\alpha)}Y_1\left(t_{2k-1}\right)\le 1+\varepsilon\right\}.$$

By (10), for large k we have

$$P\left(A_k\right)\ge \exp\left(-C''\left(\log a_{t_{2k-1}}\right)^{-1}\right)$$

where C'' is a constant. From the fact $u_{2k-1}<t_{2k-1}\le u_{2k}$, it is clear that

$$P\left(A_k\right)\ge \exp\left(-C''\left(\log a_{u_{2k-1}}\right)^{-1}\right).$$

Since $S_1=\infty$, we get $\sum_k P\left(A_k\right)=\infty$. Also,

$$t_{2k-1}+a_{t_{2k-1}}\le u_{2k}+a_{u_{2k}}<t_{2k}+a_{u_{2k}}=t_{2k+1}. \tag{11}$$

Setting

$$A'_k=\left\{\sup_{0\le s\le a_{t_{2k-1}}}\beta_{(t_{2k-1},\,\alpha)}\left(X\left(t_{2k-1}+s\right)-X\left(t_{2k-1}\right)\right)\le 1+\varepsilon\right\}$$

and

$$A''_k=\left\{\sup_{0\le s\le a_{t_{2k-1}}}\beta_{(t_{2k-1},\,\alpha)}\left(X\left(t_{2k-1}+s\right)-X\left(t_{2k-1}\right)\right)\ge -1-\varepsilon\right\},$$

we have

$$\sum_k P\left(A'_k\right)=\sum_k P\left(A''_k\right)=\infty.$$

Let

$$X_1=\sup_{0\le s\le a_{t_{2k-1}}}\left(X\left(t_{2k-1}+s\right)-X\left(t_{2k-1}\right)\right)=\left(X\left(t_{2k-1}+s_1\right)-X\left(t_{2k-1}\right)\right),$$

and

$$X_2=\sup_{0\le s\le a_{t_{2k+1}}}\left(X\left(t_{2k+1}+s\right)-X\left(t_{2k+1}\right)\right)=\left(X\left(t_{2k+1}+s_2\right)-X\left(t_{2k+1}\right)\right).$$

Then, by (11) and the concavity of $\sigma^2(t)$ we find that

$$\begin{aligned}Cov\left(X_1,X_2\right)&=E\left\{X\left(t_{2k+1}+s_2\right)X\left(t_{2k-1}+s_1\right)\right\}-E\left\{X\left(t_{2k+1}+s_2\right)X\left(t_{2k-1}\right)\right\}\\&\quad-E\left\{X\left(t_{2k+1}\right)X\left(t_{2k-1}+s_1\right)\right\}+E\left\{X\left(t_{2k+1}\right)X\left(t_{2k-1}\right)\right\}\\&=1/2\left\{\sigma^2\left(t_{2k+1}-t_{2k-1}+s_2\right)-\sigma^2\left(t_{2k+1}-t_{2k-1}+s_2-s_1\right)\right\}\\&\quad-1/2\left\{\sigma^2\left(t_{2k+1}-t_{2k-1}\right)-\sigma^2\left(t_{2k+1}-t_{2k-1}-s_1\right)\right\}.\end{aligned}$$

This implies that $Cov\left(X_1,X_2\right)\le 0$. Using Lemma 2, we obtain

$$P\left(A'_k\cap A'_l\right)\le P\left(A'_k\right)P\left(A'_l\right)\ \text{and}\ P\left(A''_k\cap A''_l\right)\le P\left(A''_k\right)P\left(A''_l\right)$$

where $k\ne l$. It follows from the Borel-Cantelli lemma that

$$-1-\varepsilon \le \limsup_{k\to\infty} \sup_{0\le s\le a_{t_{2k-1}}} \beta_{(t_{2k-1},\,\alpha)}\left(X\left(t_{2k-1}+s\right)-X\left(t_{2k-1}\right)\right)\le 1+\varepsilon, \quad a.s.$$

Also, the same result for the even subsequence $\{t_{2k}\}$ of $\{t_k\}$ is easily obtained. Therefore we have (9).
To finish the proof of Theorem 1, we need to prove

$$\limsup_{k\to\infty} \beta_{(t_k,\,\alpha)} Y_2\left(t_k\right)\ge 1 \qquad a.s. \tag{12}$$

The proof of (12) is similar to the provided proof in [1]. Thus the proof of Theorem 1 is complete.

Proof of Theorem 2. Firstly, we prove that

$$\limsup_{k\to\infty} \beta_{(t_k,\,\alpha)} Y_1\left(t_k\right)\le \varepsilon^{**} \qquad a.s. \tag{13}$$

According to Lemma 1, we have

$$\begin{aligned} P\left\{\beta_{(t_k,\,\alpha)}Y_1\left(t_k\right)\ge \varepsilon^{**}+\varepsilon\right\} &= P\left\{\sup_{0\le s\le a_{t_k}} \frac{X\left(t_k+s\right)-X\left(t_k\right)}{\sigma\left(a_{t_k}\right)}\ge\left(\varepsilon^{**}+\varepsilon\right)\left(2\log h_\alpha\left(t_k\right)\right)^{1/2}\right\} \\ &\le 2C_\varepsilon\left(h_\alpha\left(t_k\right)\right)^{-2\left(\varepsilon^{**}+\varepsilon\right)^2/(2+\varepsilon)} \\ &\le 2C_\varepsilon\left(h_\alpha\left(t_k\right)\right)^{-2\left(\varepsilon^{**}+\varepsilon_1\right)^2} \end{aligned}$$

provided k is large enough, where $\varepsilon>0$ and $0<\varepsilon_1<\varepsilon^{3/2}$.
From the definition of ε^{**}, it follows that

$$\sum_k P\left\{\beta_{(t_k,\,\alpha)} Y_1\left(t_k\right)\ge \varepsilon^{**}+\varepsilon\right\}<\infty .$$

Thus, (13) is immediate by using Borel Cantelli lemma.
To finish the proof of Theorem 2 we need to prove

$$\limsup_{k\to\infty} \beta_{(t_k,\,\alpha)}\left(X\left(t_k+a_{t_k}\right)-X\left(a_{t_k}\right)\right)\ge \varepsilon^{**}, \quad a.s. \tag{14}$$

Let

$$\Phi(u)=\frac{1}{\sqrt{2\pi}}\int_u^{+\infty} \mathrm{e}^{-x^2/2}\mathrm{d}x, \quad u\ge 0 .$$

Using the well known probability inequality

$$\frac{1}{\sqrt{2\pi}(u+1)}\mathrm{e}^{-u^2/2}\le\Phi(u)\le\frac{4}{3}\frac{1}{\sqrt{2\pi}(u+1)}\mathrm{e}^{-u^2/2}, \quad u\ge 0$$

(see [6]), one can find positive constants C and K such that, for all $k\ge K$,

$$\begin{aligned} P\left(B_k\right) &= P\left\{\frac{X\left(t_k+a_{t_k}\right)-X\left(a_{t_k}\right)}{\sigma\left(a_{t_k}\right)}\ge\left(\varepsilon^{**}-\varepsilon\right)\left(2\log h_\alpha\left(t_k\right)\right)^{1/2}\right\} \\ &\ge \frac{1}{\sqrt{2\pi}}\left\{\left(\varepsilon^{**}-\varepsilon\right)\left(2\log h_\alpha\left(t_k\right)\right)^{1/2}+1\right\}^{-1}\left(h_\alpha\left(t_k\right)\right)^{-\left(\varepsilon^{**}-\varepsilon\right)^2} \\ &\ge C\left(h_\alpha\left(t_k\right)\right)^{-\left(\varepsilon^{**}-\varepsilon'\right)^2} \end{aligned}$$

where $0<\varepsilon'<\varepsilon<\varepsilon^{**}$ and $B_k=\left\{\beta_{(t_k,\,\alpha)}\left(X\left(t_k+a_{t_k}\right)-X\left(a_{t_k}\right)\right)\ge\left(\varepsilon^{**}-\varepsilon\right)\right\}$. By the definition of ε^{**}, we have $\sum_k P\left(B_k\right)=\infty$.

The condition (6) implies that there exists $K>0$ such that $t_{k+1}\ge t_k+a_{t_k}$ for all $k\ge K$. So, using Lemma 2 and the concavity of $\sigma^2(t)$, we obtain

$$P(B_k \cap B_l) \le P(B_k)P(B_l),$$

where $k \ne l$ and Borel-Cantelli lemma implies (14). If $\varepsilon^{**} = 0$, then Theorem 2 is immediate. Thus the proof of Theorem 2 is complete.

4. Some Results for Partial Sums of Stationary Gaussian Sequence

In this section we obtain similar results as Theorems 1 and 2 for the case of partial sums of a stationary Gaussian sequence. Let $\{X_n\}$ be a stationary Gaussian sequence with $X_0 = 0$, $E\{X_1\} = 0$, $E\{X_1^2\} = 1$ and $E\{X_1 X_{1+n}\} \le 0$ for all $n = 1, 2, \cdots$. We define $S(n) = \sum_{i=1}^n X_i$ with $S(0) = 0$ and set $\sigma^2(n) = E\{S^2(n)\}$. Assume that $\sigma(n)$ can be extended to a continuous function $\sigma(t)$ with $t > 0$ which is nondecreasing and regularly varying with exponent $\tau(0 < \tau < 1)$ at ∞. Suppose that $\{a_n\}$ is a nondecreasing sequence of positive integers such that $0 \le a_n \le n$. For large *n*, we define

$$\beta_{(n,\alpha)} = \left[2\sigma^2(a_n)(\log h_\alpha(n))\right]^{-1/2},$$

where $0 \le \alpha \le 1$ and $h_\alpha(n) = n(\log n)^\alpha (\log a_n)^{1-\alpha} / a_n$ is an increasing function of *n* and also we define discrete time parameter processes by

$$Y_1(n_k) = \max_{0 \le j \le a_{n_k}} \left| S(n_k + j) - S(n_k) \right|$$

and

$$Y_1(n_k) = \max_{0 \le j \le a_{n_k}} \left| S(n_k + a_{n_k}) - S(n_k) \right|,$$

respectively, where $\{n_k\}$ is an increasing sequence of positive integers diverging to ∞. By the same way as in the proofs of Theorems 1 and 2, we obtain the following results.

Theorem 3. Under the above statements of $\{X_n\}$, $\beta_{(n,\alpha)}$ and $Y_i(n_k)$, $i = 1, 2$, for $0 \le \alpha \le 1$ we have the following:

i) If $\limsup_{k \to \infty} (n_{k+1} - n_k) / a_{n_k} < 1$, then

$$\limsup_{k \to \infty} \beta_{(n_k,\alpha)} Y_i(n_k) = 1 \qquad a.s., \quad i = 1, 2.$$

ii) If $\liminf_{k \to \infty} (n_{k+1} - n_k) / a_{n_k} > 1$, then

$$\limsup_{k \to \infty} \beta_{(n_k,\alpha)} Y_i(n_k) \le \varepsilon^{**} \qquad a.s., \quad i = 1, 2,$$

where

$$\varepsilon^{**} = \inf\left\{\gamma > 0 : \sum_k \left(h_\alpha(n_k)\right)^{-\gamma^2} < \infty\right\}.$$

Example. Let $\{X(t); 0 \le t < \infty\}$ be a fractional Brownian motion with the covariance function $E\{X(t)X(s)\} = \left\{|t|^\tau + |s|^\tau - |t-s|^\tau\right\} / 2$, $0 < \tau < 1$. Then

$$E\{X(t) - S(s)\}^2 = |t-s|^\tau.$$

Define random variables

$$X_0 = 0,$$

$$X_n = X_{(n)} - X_{(n-1)}, \quad n = 1, 2, \cdots,$$

$$S(n) = \sum_{i=1}^n X_i \quad \text{and} \quad S(0) = 0.$$

Then

$$\sigma^2(n) = E\{S^2(n)\} = E\{X^2(n)\} = n^\tau$$

and $\{X_n; n = 1, 2, \cdots\}$ is a stationary Gaussian sequence with $E\{X_1\} = 0$, $E\{X_1^2\} = 1$ and $E\{(X_1 X_{1+n})\} \le 0$ for all $n = 1, 2, \cdots$. So we have Theorem 3.

In particular if $\tau = 1/2$, then $\{X_n; n = 1, 2, \cdots\}$ is an i.i.d. Gaussian sequence with $E\{X_1\} = 0$ and $E\{X_1^2\} = 1$.

5. Conclusion

In this paper, we developed some limit theorems on increments of a Wiener process to the case of a Gaussian process. Moreover, we obtained similar results of these limit theorems for the case of partial sums of a stationary Gaussian sequence. Some obtained results can be considered as extensions of some previous given results to Gaussian processes.

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