

Numerical Modelling and Simulation of Sand Dune Formation in an Incompressible Out-Flow

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Abstract

In this paper, we are concerned with computation of a mathematical model of sand dune formation in a water of surface to incompressible out-flows in two space dimensions by using Chebyshev projection scheme. The mathematical model is formulate by coupling Navier-Stokes equations for the incompressible out-flows in 2D fluid domain and Prigozhin's equation which describes the dynamic of sand dune in strong parameterized domain in such a way which is a subset of the fluid domain. In order to verify consistency of our approach, a relevant test problem is considered which will be compared with the numerical results given by our method.

Keywords

Sand Dune Formation, Navier-Stokes Equations, Incompressible Out-Flows, Chebyshev Projection Scheme

1. Introduction

The sandbank is a real physical phenomenon that constitutes a threat for our environment through the occupation of the roads, the arable earths and especially the waters of surfaces, as it is the case of the Niger stream. The main goal of this paper is to compute numerically the height of sand dune in a water of surface to the incompressible out-flows (streams, lakes, seas, ...). For this, we formulate a mathematical model which couples the Navier-Stokes equations for the incompressible out-flows in two space dimensions and Prigozhin's equation that describes the sand dune dynamic [1]-[3]. The numerical approach that we develop to solve this model is made in three stages. The first stage aims to approach the Navier-Stokes equations by using Chebyshev projection

scheme, following $\mathbb{P}_{N,M} - \mathbb{P}_{N-2,M-2}$ method [4]-[7], the second stage is dedicated to the determination of the mass density of the sand grains transported by the out-flows, and we compute the dune height in the third stage.

The outline of this paper is as follows. In Section 2, we give the problem formulation and description of parameters. In Section 3, the numerical scheme which will be used in this paper is presented. In Section 4, some numerical simulations of the solution and temporal errors evolution are presented. We end this paper with a conclusion and the perspectives in Section 5.

2. The Problem Formulation

Let Ω be a bounded open subset with regular boundary $\partial\Omega$ in \mathbb{R}^2 in which flows out a fluid to incompressible out-flows with a velocity u and a pressure p [6]. We suppose a sand dune isolated and completely immersed in Ω and occupying a strong subdomain Γ of Ω . Let denote by m and h , respectively the mass density of the sand grains transported by the out-flows and the dune height. While supposing that the mass density is transported by a flux $-\frac{u}{\|u\|}m$, we propose the following mathematical model to describe the interaction between the out-flow of the fluid and the dynamics of the dune in two space dimensions given by:

$$\begin{cases} \partial_t u - \frac{1}{Re} \nabla^2 u + \nabla p + (u \cdot \nabla)u = f(t, x, y) & \text{in }]0; T[\times \Omega & (1) \\ \nabla \cdot u = 0 & \text{in }]0; T[\times \Omega & (2) \\ u(0, x, y) = u_0(x, y) & \text{in } \Omega & (3) \\ u(t, x, y) = g(t, x, y) & \text{in }]0; T[\times \partial\Omega & (4) \\ \partial_t m - \frac{u}{\|u\|} \cdot \nabla m = \Psi(t, x, y) & \text{in }]0; T[\times \Omega & (5) \\ \partial_t h - \nabla \cdot (m \nabla h) = \Phi(t, x, y) & \text{in }]0; T[\times \Gamma & (6) \\ \|\nabla h\| \leq 1, m(\|\nabla h\| - 1) = 0 & \text{in }]0; T[\times \Gamma & (7) \\ h(0, x, y) = h_0(x, y) & \text{in } \Gamma & (8) \end{cases}$$

where

- $u(t, x, y) = (w(t, x, y), v(t, x, y))$ is the vector velocity, w is the component following the x -axis and v the y -axis one;
- $p(t, x, y)$ is the pressure;
- $f(t, x, y)$, $\Psi(t, x, y)$ and $\Phi(t, x, y)$ are the source term;
- $m(t, x, y)$ is the mass density;
- $h(t, x, y)$ is the height of sand dune;
- Re is the Reynolds number;
- T is a given positive time-parameter.

The out-flow of the fluid is modelling by Equations (1)-(4). The transportation of the sand grains under the effect of averaged velocity is modelling by Equation (5). The dynamics of the sand dune is modelling by Equations (6)-(8).

To ensure the regularity of the solution we suppose that the functions f , u_0 and Ψ are square integrable on Ω while functions Φ and h_0 are square integrable on Γ . We also suppose that the boundary conditions given by Equation (4) verify the said condition of debit:

$$\int_{\partial\Omega} g \cdot n d\gamma = 0,$$

and the initial data u_0 must verify:

$$u_0 \cdot n = g \cdot n \text{ in } \partial\Omega \text{ and } \nabla \cdot u_0 = 0 \text{ in } \Omega,$$

where n is the unit vector normal to the boundary of the domain Ω .

3. Numerical Schemes

3.1. Temporal Discretisation

For a given positive integer r , we consider a time step discretisation $\Delta t = \frac{T}{r}$, with $T \geq 1$. Then, we define the knots of the interval $[0; T]$ given by $t_n = n\Delta t$, with $n \in \{0; \dots; r\}$.

For a given continuous function $\varphi(t, x, y)$, we approximate φ at the knots t_n by: $\varphi(t_n, x, y) \approx \varphi^n(x, y)$. In order to approach in time Equations (1)-(8), we used second-order backward Euler scheme which is given by:

$$\partial_t \varphi(t_{n+1}, x, y) \approx \frac{3\varphi^{n+1}(x, y) - 4\varphi^n(x, y) + \varphi^{n-1}(x, y)}{2\Delta t}, \text{ for } \varphi \in \{u, m, h\}. \quad (9)$$

While doing an extrapolation of order 1 of the pressure at the time of the prediction stage and while approaching the convection term $(u \cdot \nabla)u$ by a numerical scheme of Adams-Bashforth type, the basic principle of the projection methods in [8] [9] applied to Equations (1)-(4), allows us to get:

- **prediction stage:**

$$a\tilde{u}^{n+1} - \frac{1}{Re} \nabla^2 \tilde{u}^{n+1} = \hat{f}^{n+1} \quad (10)$$

$$\tilde{u}'_{/\partial\Omega} = g^{n+1}, \quad (11)$$

where \tilde{u} denotes the predicted velocity, $a = \frac{3}{2\Delta t}$ and

$$\hat{f}^{n+1} = f^{n+1} - \nabla p^n + \frac{2}{\Delta t} u^n - \frac{1}{2\Delta t} u^{n-1} - 2(u^n \cdot \nabla)u^n + (u^{n-1} \cdot \nabla)u^{n-1},$$

- **projection stage:**

$$au^{n+1} + \nabla p^{n+1} = \tilde{f}^{n+1}, \quad (12)$$

$$\nabla \cdot u^{n+1} = 0, \quad (13)$$

$$u^{n+1} \cdot n_{/\partial\Omega} = g^{n+1} \cdot n, \quad (14)$$

where $\tilde{f}^{n+1} = a\tilde{u}^{n+1} + \nabla p^n$ for $n \in \{0; 1; \dots; r\}$, ($r \geq 1$). This last stage corresponds to a Darcy problem [10] that is as well as the Stokes problem of type saddle point.

Thus, when one does a spatial discretisation of this problem by using a Chebyshev spectral method, so that the resulting discrete problem is well posed, it is necessary that the discrete spaces of velocity and pressure verify a compatibility condition inf-sup of Brezzi [11].

To answer this question of compatibility condition, we use the spectral method $\mathbb{P}_{N,M} - \mathbb{P}_{N-2,M-2}$ by using only one grid defined by the usual Chebyshev-Gauss-Lobatto [12] [13].

3.2. Spatial Discretisation

In this section we present the basic principle of the method $\mathbb{P}_{N,M} - \mathbb{P}_{N-2,M-2}$.

So for a given positive integers N and M we denote by $\mathbb{P}_N(\Lambda)$ and $\mathbb{P}_{N-2}(\Lambda)$ sets of orthogonal polynomials of degree less than or equal to N and $N-2$, respectively, where Λ is an open subset such that $\Omega = \Lambda \times \Lambda$. Let denote by $\mathbb{P}_{N,M}(\Omega) = \mathbb{P}_N(\Lambda) \otimes \mathbb{P}_M(\Lambda)$, the set of polynomials defined on Ω of degree N according to the variable x and degree M according to the variable y , and $\mathbb{P}_{N-2,M-2}(\Omega) = \mathbb{P}_{N-2}(\Lambda) \otimes \mathbb{P}_{M-2}(\Lambda)$, the set of polynomials defined on Ω of degree $N-2$ according to the variable x and degree $M-2$ according to the variable y .

The $\mathbb{P}_{N,M} - \mathbb{P}_{N-2,M-2}$ method consists in approaching the pressure by orthogonal polynomials of degree less than two units as those approaching the velocity while considering only one grid.

In this paper, we consider Chebyshev polynomials and choose the Chebyshev-Gauss-Lobatto mesh defined by:

$$x_i = \cos\left(\frac{i\pi}{N}\right) \text{ and } y_j = \cos\left(\frac{j\pi}{M}\right), \text{ for } i = 0, 1, \dots, N \text{ and } j = 0, 1, \dots, M.$$

Then, we consider the velocity at $(N+1) \times (M+1)$ points of Ω and the pressure at $(N-1) \times (M-1)$ points inside of Ω . Therefore, compatibility between the spaces of approximation of the velocity and the pressure is assured and the condition inf-sup is satisfied [14] [15].

Let us making the following space approximation for $n \in \{0; 1; \dots; r\}$

$$\begin{cases} w^{n+1}(x_i; y_j) \approx w_{i,j}^{n+1}, \\ v^{n+1}(x_i; y_j) \approx v_{i,j}^{n+1}, \\ i = 0, \dots, N; j = 0, \dots, M; \end{cases}$$

$$p^{n+1}(x_i; y_j) \approx p_{i,j}^{n+1}, i = 1, \dots, N-1; j = 1, \dots, M-1.$$

We approach the first and secondary operators of derivation of $\varphi \in \{w, v\}$ in $\mathbb{P}_{N,M}(\Omega)$ [6] [13] [16] by:

$$\begin{aligned} \frac{\partial \varphi^{n+1}(x_i; y_j)}{\partial x} &= \sum_{k=0}^N d_{i,k}^{N,1} \varphi_{k,j}^{n+1}, & \frac{\partial \varphi^{n+1}(x_i; y_j)}{\partial y} &= \sum_{l=0}^M d_{j,l}^{M,1} \varphi_{i,l}^{n+1} \\ \frac{\partial^2 \varphi^{n+1}(x_i; y_j)}{\partial x^2} &= \sum_{k=0}^N d_{i,k}^{N,2} \varphi_{k,j}^{n+1}, & \frac{\partial^2 \varphi^{n+1}(x_i; y_j)}{\partial y^2} &= \sum_{l=0}^M d_{j,l}^{M,2} \varphi_{i,l}^{n+1}; \end{aligned}$$

where $(d_{i,k}^{N,1})_{0 \leq i \leq N; 0 \leq k \leq N}$ and $(d_{i,k}^{N,2})_{0 \leq i \leq N; 0 \leq k \leq N}$ are coefficients of the Chebyshev differentiation matrixes of order 1 D_N and order 2 D_N^2 , respectively in $\mathbb{P}_N(\Lambda)$. We approach the first operators of derivation of pressure p in $\mathbb{P}_{N-2, M-2}(\Omega)$ [17] by:

$$\begin{aligned} \frac{\partial p^{n+1}(x_i; y_j)}{\partial x} &= \sum_{k=1}^{N-1} \tilde{d}_{i,k}^{N,1} p_{k,j}^{n+1} \\ \frac{\partial p^{n+1}(x_i; y_j)}{\partial y} &= \sum_{l=1}^{M-1} \tilde{d}_{j,l}^{M,1} p_{i,l}^{n+1}, \end{aligned}$$

where $(\tilde{d}_{i,k}^{N,1})_{1 \leq i \leq N-1; 1 \leq k \leq N-1}$ are coefficients of the Chebyshev differentiation matrix of order 1 \tilde{D}_N in $\mathbb{P}_{N-2}(\Lambda)$, given by the following relation:

$$\begin{cases} \tilde{d}_{i,j}^{N,1} = \frac{-\sin^2\left(\frac{\pi j}{N}\right)}{2\sin^2\left(\frac{\pi i}{N}\right)} \frac{(-1)^{i+j}}{\sin\frac{\pi(i+j)}{2N} \sin\frac{\pi(i-j)}{2N}} & \text{for } i \neq j \\ \tilde{d}_{i,i}^{N,1} = \frac{3}{2} \frac{\cos\left(\frac{\pi i}{N}\right)}{\sin^2\left(\frac{\pi i}{N}\right)} & \text{for } 1 \leq i \leq N-1. \end{cases}$$

Let us consider the following approximation spaces:

$$X_{N,M}(\Omega) = \{w \in \mathbb{P}_{N,M}(\Omega) / w_{\partial\Omega} = g^1\}$$

$$Y_{N,M}(\Omega) = \{v \in \mathbb{P}_{N,M}(\Omega) / v_{\partial\Omega} = g^2\},$$

where g^1 and g^2 are the first and second component of g , respectively.

We define by:

$$\begin{cases} \hat{f}^{n+1}(x_i; y_j) = (\hat{f}^{n+1,1}(x_i; y_j); \hat{f}^{n+1,2}(x_i; y_j)), \\ \hat{g}^{n+1}(x_i; y_j) = (\hat{g}^{n+1,1}(x_i; y_j); \hat{g}^{n+1,2}(x_i; y_j)), \\ i = 1, \dots, N-1; j = 1, \dots, M-1. \end{cases}$$

Then the prediction stage (9)-(10) decomposes itself in two-Helmholtz problems for each components of the predicted velocity with Dirichlet boundary conditions:

$$a\tilde{w}^{n+1}(x_i; y_j) - \frac{1}{Re} \left(\frac{\partial^2 \tilde{w}^{n+1}(x_i; y_j)}{\partial x^2} + \frac{\partial^2 \tilde{w}^{n+1}(x_i; y_j)}{\partial y^2} \right) = \hat{f}^{n+1,1}(x_i; y_j), \quad (15)$$

$$\tilde{w}_{/\partial\Omega}^{n+1} = \hat{g}^{n+1,1} \quad (16)$$

$$a\tilde{v}^{n+1}(x_i; y_j) - \frac{1}{Re} \left(\frac{\partial^2 \tilde{v}^{n+1}(x_i; y_j)}{\partial x^2} + \frac{\partial^2 \tilde{v}^{n+1}(x_i; y_j)}{\partial y^2} \right) = \hat{f}^{n+1,2}(x_i; y_j), \quad (17)$$

$$\tilde{v}_{/\partial\Omega}^{n+1} = \hat{g}^{n+1,2}. \quad (18)$$

The Chebyshev collocation approximation of Helmholtz problems (15) and (17) is given by:

$$\begin{aligned} a\tilde{w}_{i,j}^{n+1} - \frac{1}{Re} \left(\sum_{k=1}^{N-1} d_{i,k}^{N,2} \tilde{w}_{k,j}^{n+1} + \sum_{l=1}^{M-1} d_{j,l}^{M,2} \tilde{w}_{i,l}^{n+1} \right) \\ = \hat{f}_{i,j}^{n+1,1} + \frac{1}{Re} (d_{i,0}^{N,2} \tilde{w}_{0,j}^{n+1} + d_{i,N}^{N,2} \tilde{w}_{N,j}^{n+1} + d_{j,0}^{M,2} \tilde{w}_{i,0}^{n+1} + d_{j,M}^{M,2} \tilde{w}_{i,M}^{n+1}); \end{aligned} \quad (19)$$

and

$$\begin{aligned} a\tilde{v}_{i,j}^{n+1} - \frac{1}{Re} \left(\sum_{k=1}^{N-1} d_{i,k}^{N,2} \tilde{v}_{k,j}^{n+1} + \sum_{l=1}^{M-1} d_{j,l}^{M,2} \tilde{v}_{i,l}^{n+1} \right) \\ = \hat{f}_{i,j}^{n+1,2} + \frac{1}{Re} (d_{i,0}^{N,2} \tilde{v}_{0,j}^{n+1} + d_{i,N}^{N,2} \tilde{v}_{N,j}^{n+1} + d_{j,0}^{M,2} \tilde{v}_{i,0}^{n+1} + d_{j,M}^{M,2} \tilde{v}_{i,M}^{n+1}). \end{aligned} \quad (20)$$

Multiplying these equations by $-Re$, we obtain the following relations given by:

$$\left(\sum_{k=1}^{N-1} d_{i,k}^{N,2} \tilde{w}_{k,j}^{n+1} + \sum_{l=1}^{M-1} d_{j,l}^{M,2} \tilde{w}_{i,l}^{n+1} \right) - \sigma \tilde{w}_{i,j}^{n+1} = G_{i,j}^1 \quad (21)$$

and

$$\left(\sum_{k=1}^{N-1} d_{i,k}^{N,2} \tilde{v}_{k,j}^{n+1} + \sum_{l=1}^{M-1} d_{j,l}^{M,2} \tilde{v}_{i,l}^{n+1} \right) - \sigma \tilde{v}_{i,j}^{n+1} = G_{i,j}^2 \quad (22)$$

where

$$\begin{aligned} \sigma = aRe, \quad G_{i,j}^1 = -Re\hat{f}_{i,j}^{n+1,1} - (d_{i,0}^{N,2} \tilde{w}_{0,j} + d_{i,N}^{N,2} \tilde{w}_{N,j} + d_{j,0}^{M,2} \tilde{w}_{i,0} + d_{j,M}^{M,2} \tilde{w}_{i,M}); \\ G_{i,j}^2 = -Re\hat{f}_{i,j}^{n+1,2} - (d_{i,0}^{N,2} \tilde{v}_{0,j} + d_{i,N}^{N,2} \tilde{v}_{N,j} + d_{j,0}^{M,2} \tilde{v}_{i,0} + d_{j,M}^{M,2} \tilde{v}_{i,M}); \end{aligned}$$

Let us denote by:

$$\begin{aligned} \tilde{W} &= (\tilde{w}_{i,j}^{n+1})_{1 \leq i \leq N-1; 1 \leq j \leq M-1}, \\ \tilde{V} &= (\tilde{v}_{i,j}^{n+1})_{1 \leq i \leq N-1; 1 \leq j \leq M-1}, \\ G^1 &= (G_{i,j}^1)_{1 \leq i \leq N-1; 1 \leq j \leq M-1}, \\ G^2 &= (G_{i,j}^2)_{1 \leq i \leq N-1; 1 \leq j \leq M-1} \end{aligned}$$

Then, we can rewrite Equations (21) and (22) by:

$$\overline{D_N^2} \tilde{W} + \tilde{W} (\overline{D_M^2})^t - \sigma \tilde{W} = G^1, \tag{23}$$

and

$$\overline{D_N^2} \tilde{V} + \tilde{V} (\overline{D_M^2})^t - \sigma \tilde{V} = G^2, \tag{24}$$

$\overline{D_N^2}$ is a matrix $(N-1)(M-1)$ obtained by suppressing the first and last lines, the first and last columns of the matrix D_N^2 .

Systems (23) and (24) are solving by using diagonalisation method [10].

Let us denote by Λ_x and Λ_y the diagonal matrixes whose entries are the eigenvalues $\lambda_{x,i}$, $i = 1, \dots, N-1$, and $\lambda_{y,j}$, $j = 1, \dots, M-1$, of the matrixes D_N^2 and D_M^2 , respectively, so that

$$\overline{D_N^2} = Q_1 \Lambda_x Q_1^{-1} \tag{25}$$

and

$$\overline{D_M^2} = Q_2 \Lambda_y Q_2^{-1}, \tag{26}$$

where Q_1 and Q_2 are matrixes defined by the eigenvectors.

Multiplying the Equation (23) on the left by Q_1^{-1} , we obtain:

$$Q_1^{-1} \overline{D_N^2} \tilde{W} + Q_1^{-1} \tilde{W} (\overline{D_M^2})^t - \sigma Q_1^{-1} \tilde{W} = Q_1^{-1} G^1, \tag{27}$$

we deduce that:

$$Q_1^{-1} \overline{D_N^2} Q_1 (Q_1^{-1}) \tilde{W} + Q_1^{-1} \tilde{W} (\overline{D_M^2})^t - \sigma Q_1^{-1} \tilde{W} = Q_1^{-1} G^1. \tag{28}$$

Let us denote by $\overline{W} = Q_1^{-1} \tilde{W}$ and $\overline{G}^1 = Q_1^{-1} G^1$, the Equation (28) can be rewrite as:

$$Q_1^{-1} \overline{D_N^2} Q_1 \overline{W} + \overline{W} (\overline{D_M^2})^t - \sigma \overline{W} = \overline{G}^1, \tag{29}$$

From (25) and (26), we deduce:

$$\Lambda_x \overline{W} + \overline{W} (\overline{D_M^2})^t - \sigma \overline{W} = \overline{G}^1, \tag{30}$$

and multiplying this equation on the right by $(Q_2^t)^{-1}$, we obtain:

$$\Lambda_x \overline{W} (Q_2^t)^{-1} + \overline{W} (\overline{D_M^2})^t (Q_2^t)^{-1} - \sigma \overline{W} (Q_2^t)^{-1} = \overline{G}^1 (Q_2^t)^{-1}, \tag{31}$$

so that, we deduce the following equation:

$$\Lambda_x \overline{W} (Q_2^t)^{-1} + \overline{W} ((Q_2^t)^{-1}) Q_2^t (\overline{D_M^2})^t (Q_2^t)^{-1} - \sigma \overline{W} (Q_2^t)^{-1} = \overline{G}^1 (Q_2^t)^{-1}. \tag{32}$$

Denoting by $\hat{W} = \overline{W} (Q_2^t)^{-1}$ and $\widehat{G}^1 = \overline{G}^1 (Q_2^t)^{-1}$, Equation (24) becomes:

$$\Lambda_x \hat{W} + \hat{W} Q_2^t (\overline{D_M^2})^t (Q_2^t)^{-1} - \sigma \hat{W} = \widehat{G}^1, \tag{33}$$

using relation (26), we obtain:

$$\Lambda_x \hat{W} + \hat{W} \Lambda_y - \sigma \hat{W} = \widehat{G}^1. \tag{34}$$

Then, we deduce:

$$\hat{W}_{i,j} = \frac{\widehat{G}^1_{i,j}}{\lambda_{x,i} + \lambda_{y,j} - \sigma}, i = 1, \dots, N-1; j = 1, \dots, M-1. \tag{35}$$

We compute completely \overline{W} and \tilde{W} , by using the following algorithm:

- 1) Compute $G^1 = Q_1^{-1}G^1$.
- 2) Compute $\widehat{G}^1 = \overline{G}^1(Q_2^t)^{-1}$.
- 3) Compute \widehat{W} from (27).
- 4) Compute $\overline{W} = \widehat{W}Q_2^t$.
- 5) Compute $\tilde{W} = Q_1\overline{W}$.

When applying the same algorithm to Equation (24), we can compute completely \tilde{V} .

In order to make the projection stage, we define:

$$\tilde{f}^{n+1}(x_i; y_j) = (\tilde{f}^{n+1,1}(x_i; y_j); \tilde{f}^{n+1,2}(x_i; y_j)), 1 \leq i \leq N-1; 1 \leq j \leq M-1$$

then we can rewrite Equations (12)-(13) by:

$$aw^{n+1}(x_i; y_j) + \frac{\partial p^{n+1}(x_i; y_j)}{\partial x} = \tilde{f}^{n+1,1}(x_i; y_j), \quad (36)$$

$$av^{n+1}(x_i; y_j) + \frac{\partial p^{n+1}(x_i; y_j)}{\partial y} = \tilde{f}^{n+1,2}(x_i; y_j), \quad (37)$$

$$\frac{\partial w^{n+1}(x_i; y_j)}{\partial x} + \frac{\partial v^{n+1}(x_i; y_j)}{\partial y} = 0, \quad (38)$$

with boundary conditions:

$$w^{n+1}(x_0; y_j) = g^{n+1,1}(x_0; y_j), \text{ and } w^{n+1}(x_N; y_j) = g^{n+1,1}(x_N; y_j), \text{ for } j = 1, \dots, M-1;$$

$$v^{n+1}(x_i; y_0) = g^{n+1,2}(x_i; y_0), \text{ and } v^{n+1}(x_i; y_N) = g^{n+1,2}(x_i; y_N), \text{ for } i = 1, \dots, N-1.$$

So, while noting:

$$\tilde{f}^{n+1,1}(x_i; y_j) = \tilde{f}_{i,j}^{n+1,1},$$

$$\tilde{f}^{n+1,2}(x_i; y_j) = \tilde{f}_{i,j}^{n+1,2},$$

$$g^{n+1,1}(x_i; y_j) = g_{i,j}^{n+1,1},$$

$$g^{n+1,2}(x_i; y_j) = g_{i,j}^{n+1,2}$$

then by using spectral method $\mathbb{P}_{N,M} - \mathbb{P}_{N-2,M-2}$, we obtain the spatial discretisation of Equations (36)-(38) as following :

$$aw_{i,j}^{n+1} + \sum_{k=1}^{N-1} \tilde{d}_{i,k}^{N,1} p_{k,j}^{n+1} = \tilde{f}_{i,j}^{n+1,1}, \quad (39)$$

$$av_{i,j}^{n+1} + \sum_{l=1}^{M-1} \tilde{d}_{j,l}^{N,1} p_{i,l}^{n+1} = \tilde{f}_{i,j}^{n+1,2}, \quad (40)$$

$$\sum_{k=1}^{N-1} \tilde{d}_{i,k}^{N,1} w_{k,j}^{n+1} + \sum_{l=1}^{M-1} \tilde{d}_{j,l}^{N,1} v_{i,l}^{n+1} = s_{ij}^{n+1}, \quad (41)$$

where $s_{ij}^{n+1} = -[d_{i0}w_{0j} + d_{iN}w_{Nj} + d_{j0}v_{i0} + d_{jN}v_{iN}]$, $i = 1, \dots, N-1; j = 1, \dots, M-1$, with boundary conditions :

$$w_{0,j}^{n+1} = g_{0,j}^{n+1,1}, \text{ and } w_{N,j}^{n+1} = g_{N,j}^{n+1,1}, \text{ for } j = 1, \dots, M-1;$$

$$v_{i,0}^{n+1} = g_{i,0}^{n+1,2}, \text{ and } v_{i,M}^{n+1} = g_{i,M}^{n+1,2}, \text{ for } i = 1, \dots, N-1,$$

Let us denote by:

$$\begin{aligned}
 S &= \left(s_{ij}^{n+1} \right)_{1 \leq i \leq N-1; 1 \leq j \leq M-1}, \\
 W &= \left(w_{i,j}^{n+1} \right)_{1 \leq i \leq N-1; 1 \leq j \leq M-1}, \\
 V &= \left(v_{i,j}^{n+1} \right)_{1 \leq i \leq N-1; 1 \leq j \leq M-1}, \\
 P &= \left(p_{i,j}^{n+1} \right)_{1 \leq i \leq N-1; 1 \leq j \leq M-1}, \\
 \tilde{F}^1 &= \left(\tilde{f}_{i,j}^{n+1,1} \right)_{1 \leq i \leq N-1; 1 \leq j \leq M-1}, \\
 \tilde{F}^2 &= \left(\tilde{f}_{i,j}^{n+1,2} \right)_{1 \leq i \leq N-1; 1 \leq j \leq M-1}.
 \end{aligned}$$

Then, we obtain the following matrix formulation for Equations (39)-(41), given by:

$$aW + \widetilde{D}_N P = \tilde{F}^1, \tag{42}$$

$$aV + P \left(\widetilde{D}_M \right)^t = \tilde{F}^2, \tag{43}$$

$$\overline{D}_N W + V \left(\overline{D}_M \right)^t = S, \tag{44}$$

where \overline{D}_N is a matrix obtaining by suppressing the first and last lines, the first and last columns of the Chebyshev matrix of derivation D_N .

Reformulating Equations (42), (43) and (44), we deduce :

$$W = \frac{1}{a} \left(\tilde{F}^1 - \widetilde{D}_N P \right), \tag{45}$$

$$V = \frac{1}{a} \left(\tilde{F}^2 - P \left(\widetilde{D}_M \right)^t \right), \tag{46}$$

$$B_N P + P \left(B_M \right)^t = \tilde{S} \tag{47}$$

where

$$B_N = \overline{D}_N \widetilde{D}_N,$$

$$B_M = \overline{D}_M \widetilde{D}_M,$$

$$\tilde{S} = \overline{D}_N \tilde{F}^1 + \tilde{F}^2 \left(\overline{D}_M \right)^t - aS.$$

We solve Equation (47) by using the same strategy using for solving Equation (21) and (22). Then we determine completely the P matrix for the pressure and deduce the matrixes W and V containing the values of the first and the second components of velocity, respectively from Equations (45) and (46).

Let us denote by $m_{i,j}^{n+1} \approx m^{n+1}(x_i; y_j)$ the approximation of the masse density at the mesh (x_i, y_j) for $i \in \{1; \dots; N-1\}$, $j \in \{1; \dots; M-1\}$ and $n \in \{0; 1; \dots; r\}$. While approaching the first derivation of the density m in $\mathbb{P}_{N-2, M-2}(\Omega)$, and using relation (9), Equation (5) give:

$$\frac{3m_{i,j}^{n+1} - 4m_{i,j}^n + m_{i,j}^{n-1}}{2\Delta t} - \frac{1}{\|U^{n+1}\|} \left(w_{i,j}^{n+1} \sum_{k=1}^{N-1} \tilde{d}_{i,k}^{N,1} m_{k,j}^{n+1} + v_{i,j}^{n+1} \sum_{l=1}^{M-1} \tilde{d}_{j,l}^{M,1} m_{i,l}^{n+1} \right) = \Psi_{i,j}^{n+1}. \tag{48}$$

We denote by m^{n+1} the vector given by:

$$m^{n+1} = \left(m_{1,1}^{n+1}, \dots; m_{1,M-1}^{n+1}; m_{2,1}^{n+1}, \dots; m_{2,M-1}^{n+1}; \dots; m_{N-1,1}^{n+1}, \dots; m_{N-1,M-1}^{n+1} \right)^t.$$

We can rewrite Equation (48) by:

$$(3(I_{N-1} \otimes I_{M-1}) - J_1^{n+1} - J_2^{n+1})m^{n+1} = 4m^n - m^{n-1} + K^{n+1}, \quad (49)$$

where:

$$\begin{aligned} K^{n+1} &= 2\Delta t \left(\Psi_{1,1}^{n+1}, \dots, \Psi_{1,M-1}^{n+1}, \Psi_{2,1}^{n+1}, \dots, \Psi_{2,M-1}^{n+1}, \dots, \Psi_{N-1,1}^{n+1}, \dots, \Psi_{N-1,M-1}^{n+1} \right)^t; \\ \underline{W}^{n+1} &= \left(w_{1,1}^{n+1}, \dots, w_{1,M-1}^{n+1}, w_{2,1}^{n+1}, \dots, w_{2,M-1}^{n+1}, \dots, w_{N-1,1}^{n+1}, \dots, w_{N-1,M-1}^{n+1} \right)^t; \\ \underline{V}^{n+1} &= \left(v_{1,1}^{n+1}, \dots, v_{1,M-1}^{n+1}, v_{2,1}^{n+1}, \dots, v_{2,M-1}^{n+1}, \dots, v_{N-1,1}^{n+1}, \dots, v_{N-1,M-1}^{n+1} \right)^t; \\ J_1^{n+1} &= \frac{2\Delta t}{\|U^{n+1}\|} \left(\text{diag}(\underline{W}^{n+1}) \mathbb{I} \right) \cdot * \left(\widetilde{D}_N \otimes I_{N-1} \right); \\ J_2^{n+1} &= \frac{2\Delta t}{\|U^{n+1}\|} \left(\text{diag}(\underline{V}^{n+1}) \mathbb{I} \right) \cdot * \left(I_{M-1} \otimes \widetilde{D}_M \right), \end{aligned}$$

where I_{N-1} is the identity matrix of order $(N-1)$ and \mathbb{I} the matrix of order $(N-1)(M-1)$ of entries equal to 1 and $U = (W, V)$ denotes the velocity of the out-flow.

And while denoting by $A^{n+1} = 3(I_{N-1} \otimes I_{M-1}) - J_1^{n+1} - J_2^{n+1}$, we obtain:

$$m^{n+1} = (A^{n+1})^{-1} (4m^n - m^{n-1} + K^{n+1}) \quad \text{for } n \in \{1; \dots; r\}. \quad (50)$$

To make the approximation of Equations (6)-(8), we suppose that the strong domain occupied by sand dune is parameterized by $\Gamma_\mu = \left] \frac{-1}{\mu}; \frac{1}{\mu} \right[$, with $\mu > 1$ so that this domain is contained in Ω .

What brings us to consider another grid to approach the dune height by using new grid $(xx_i; yy_j)$, defined by: $xx_i = \frac{1}{\mu} \cos\left(\frac{i\pi}{N}\right)$ and $yy_j = \frac{1}{\mu} \cos\left(\frac{j\pi}{N}\right)$, for $i = 1, \dots, N-1$ and $j = 1, \dots, M-1$.

Let us denote by $h_{i,j}^{n+1} \approx h^{n+1}(xx_i; yy_j)$ the approximation of the dune height at the mesh $(xx_i; yy_j)$ for $i \in \{1; \dots; N-1\}$, $j \in \{1; \dots; M-1\}$ and $n \in \{0; 1; \dots; r\}$. While approaching the first derivation of the dune height h in $\mathbb{P}_{N-2, M-2}(\Omega)$, and using relation (9), Equation (6) give:

$$\frac{3h_{i,j}^{n+1} - 4h_{i,j}^n + h_{i,j}^{n-1}}{2\Delta t} - \left(\sum_{k=1}^{N-1} \tilde{d}_{i,k}^{N,1} m_{k,j}^{n+1} \right) \left(\sum_{k=1}^{N-1} \tilde{d}_{i,k}^{N,1} h_{k,j}^{n+1} \right) - m_{i,j}^{n+1} \sum_{k=1}^{N-1} \tilde{d}_{i,k}^{N,2} h_{k,j}^{n+1} \quad (51)$$

$$- \left(\sum_{l=1}^{M-1} \tilde{d}_{j,l}^{M,1} m_{i,l}^{n+1} \right) \left(\sum_{l=1}^{M-1} \tilde{d}_{j,l}^{M,1} h_{i,l}^{n+1} \right) - m_{i,j}^{n+1} \sum_{l=1}^{M-1} \tilde{d}_{j,l}^{M,2} h_{i,l}^{n+1} = \phi_{i,j}^{n+1} \quad (52)$$

Denoting by H^{n+1} the vector given by:

$$H^{n+1} = \left(h_{1,1}^{n+1}, \dots, h_{1,M-1}^{n+1}, h_{2,1}^{n+1}, \dots, h_{2,M-1}^{n+1}, \dots, h_{N-1,1}^{n+1}, \dots, h_{N-1,M-1}^{n+1} \right)^t,$$

we obtain the following matrix formulation:

$$(3(I_{N-1} \otimes I_{M-1}) - A_1^{n+1} - A_2^{n+1} - A_3^{n+1} - A_4^{n+1})H^{n+1} = 4H^n - H^{n-1} + R^{n+1} \quad (53)$$

where:

$$\begin{aligned} R^{n+1} &= 2\Delta t \left(\Phi_{1,1}^{n+1}, \dots, \Phi_{1,M-1}^{n+1}, \Phi_{2,1}^{n+1}, \dots, \Phi_{2,M-1}^{n+1}, \dots, \Phi_{N-1,1}^{n+1}, \dots, \Phi_{N-1,M-1}^{n+1} \right)^t; \\ A_1^{n+1} &= 2\Delta t \left(\text{diag} \left[\left(\widetilde{D}_N \otimes I_{N-1} \right) m^{n+1} \right] \mathbb{I} \right) \cdot * \left(\widetilde{D}_N \otimes I_{N-1} \right); \\ A_2^{n+1} &= 2\Delta t \left[\text{diag} \left(m^{n+1} \right) \mathbb{I} \right] \cdot * \left(\left(\widetilde{D}_N \right)^2 \otimes I_{N-1} \right); \end{aligned}$$

$$A_3^{n+1} = 2\Delta t \left(\text{diag} \left[\left(I_{M-1} \otimes \widetilde{D}_M \right) m^{n+1} \right] \mathbb{I} \right) * \left(I_{M-1} \otimes \widetilde{D}_M \right);$$

$$A_4^{n+1} = 2\Delta t \left[\text{diag} \left(m^{n+1} \right) \mathbb{I} \right] * \left(I_{M-1} \otimes \left(\widetilde{D}_M \right)^2 \right)$$

and while denoting by :

$C^{n+1} = \left(3(I_{N-1} \otimes I_{M-1}) - A_1^{n+1} - A_2^{n+1} - A_3^{n+1} - A_4^{n+1} \right)$, we can rewrite Equation (53):

$$H^{n+1} = \left(C^{n+1} \right)^{-1} \left(4H^n - H^{n-1} + R^{n+1} \right), \text{ for } n \in \{1; \dots; r\}. \tag{54}$$

4. Numerical Result

For the numerical simulation, we consider an experimental solution on the one hand for the Navier-Stokes equations and other for the mass density and the dune height.

For example:

$$w(t, x, y) = \cos(5t) \sin\left(\frac{\pi x}{2}\right) \cos\left(\frac{\pi y}{2}\right),$$

$$v(t, x, y) = \cos(5t) \cos\left(\frac{\pi x}{2}\right) \sin\left(\frac{\pi y}{2}\right),$$

$$p(t, x, y) = \frac{\cos^2(5t)}{4} \left[\cos(\pi x) \cos(\pi y) \right] + 10(x + y) \cos(5t),$$

$$m(t, x, y) = (x^2 + y^2) \exp(-t) \text{ and}$$

$$h(t, x, y) = (1 - x^2)(1 - y^2) \exp(-t).$$

We take $Re = 100$, $T = 1$ and $16 \leq N = M \leq 30$ for the cases tests. While noting φ_N ($\varphi \in \{w, v, m, h\}$) the calculated fields, we give the evolution of the temporal error $E_\varphi(t, \Delta t) = \left\| \varphi_N^n - \varphi(t) \right\|_{L_2}$ during the time. The integration in time of this error is initialized while taking the fields to the instants $-\Delta t, 0$ equals to the exact solution to the same time level.

We represent temporal errors according to the first components $E_w(t, \Delta t)$ (Figure 1(a)) and the second

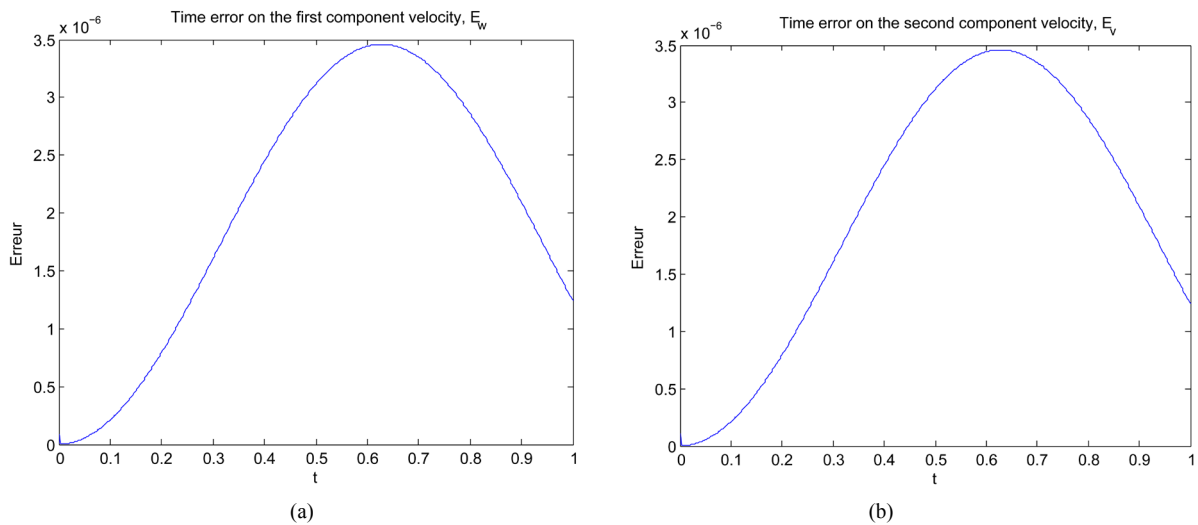


Figure 1. Temporal evolution of the errors in time on the first component of the velocity, $E_w(t, \Delta t)$ (a), and the second component $E_v(t, \Delta t)$ (b), for a step of time $\Delta t = 2 \times 10^{-3}, N = M = 20$.

components $E_v(t, \Delta t)$ (Figure 1(b)) of the velocity by using the following parameters of discretisation $\Delta t = 2 \times 10^{-3}$ and $N = M = 20$. One notices a likeness between the two figures, that shows the precision of the second order in time by the numerical scheme used. Also, these errors don't depend on the chosen of spatial discretisation.

We also represent the temporal errors for the mass density of sand grains $E_m(t, \Delta t)$ (Figure 2(a)) and the dune height $E_h(t, \Delta t)$ (Figure 2(b)) by using $\Delta t = 2 \times 10^{-3}$, $\mu = 150$ and $N = M = 20$. These also confirm the precision of the second order in time by the numerical scheme used.

The profile of the dune height is represented at $t = 0.095$, for $\Delta t = 5 \times 10^{-3}$ by using $\mu = 10$ (Figure 3), $\mu = 100$ (Figure 4), and $\mu = 150$ (Figure 5). The experimental height on the left and the approach height on the right. One notices that the simulations made for $\mu = 150$ give a better approximation of the dune height that those achieved for $\mu = 10$ and $\mu = 100$. That permits us to conclude that for a higher value of parameter μ we obtain a good approximation for a dune height in the strong domain Γ_μ . Also, these figures show a likeness between the numerical solution and the experimental solution for each value of parameter $\mu \in \{10; 100; 150\}$. That permits us to conclude the consistency of our approach.

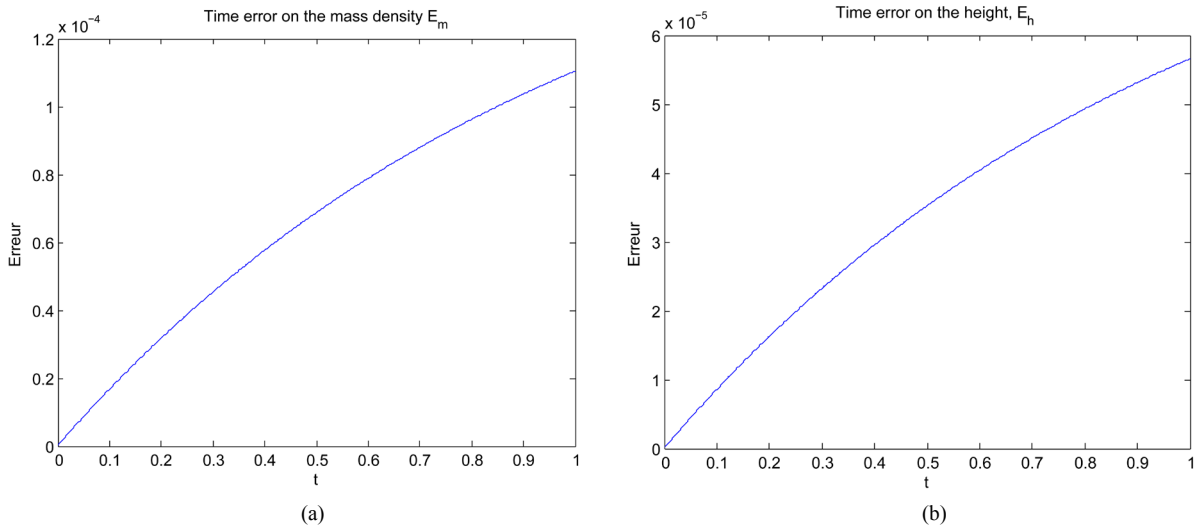


Figure 2. Temporal evolution of the errors in time on the mass density of sand grains, $E_m(t, \Delta t)$ (a), and on the dune height, $E_h(t, \Delta t)$ (b), for a step of time $\Delta t = 2 \times 10^{-3}$, $N = M = 20$ and $\mu = 150$.

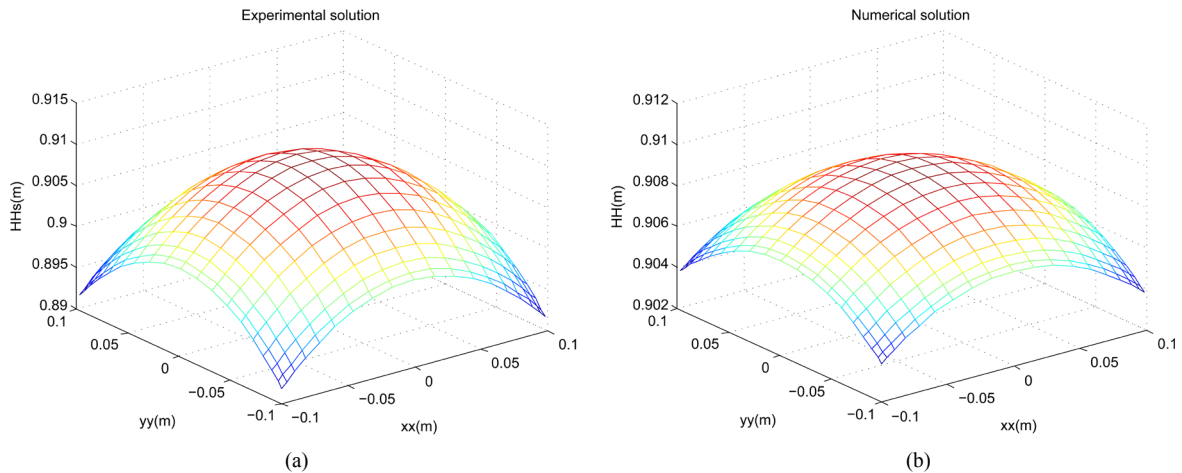


Figure 3. Profile of the experimental and approached of the dune height in the space at $t = 0.095$, for a time step $\Delta t = 5 \times 10^{-3}$, $N = M = 20$ and $\mu = 10$.

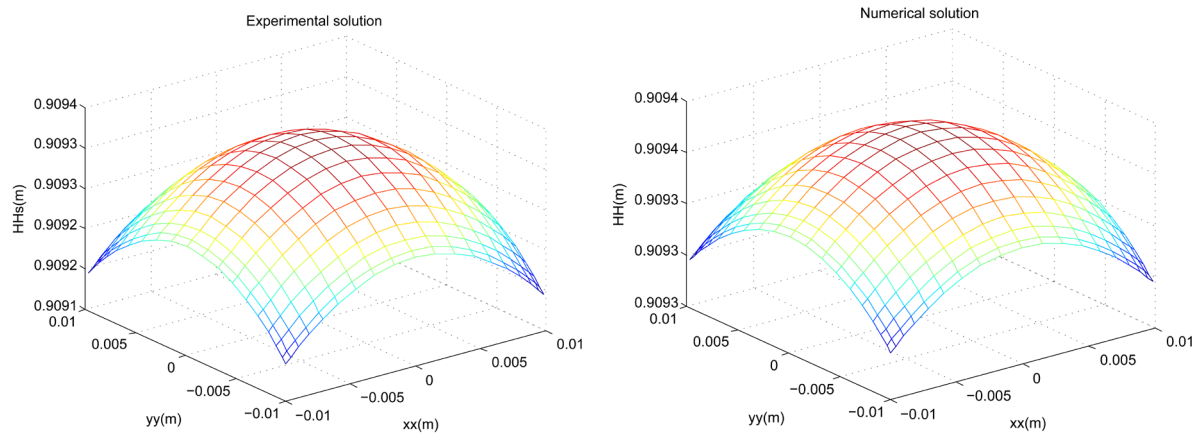


Figure 4. Profile of the experimental and approached of the dune height in the space at $t = 0.095$, for a time step $\Delta t = 5 \times 10^{-3}$, $N = M = 20$ and $\mu = 100$.

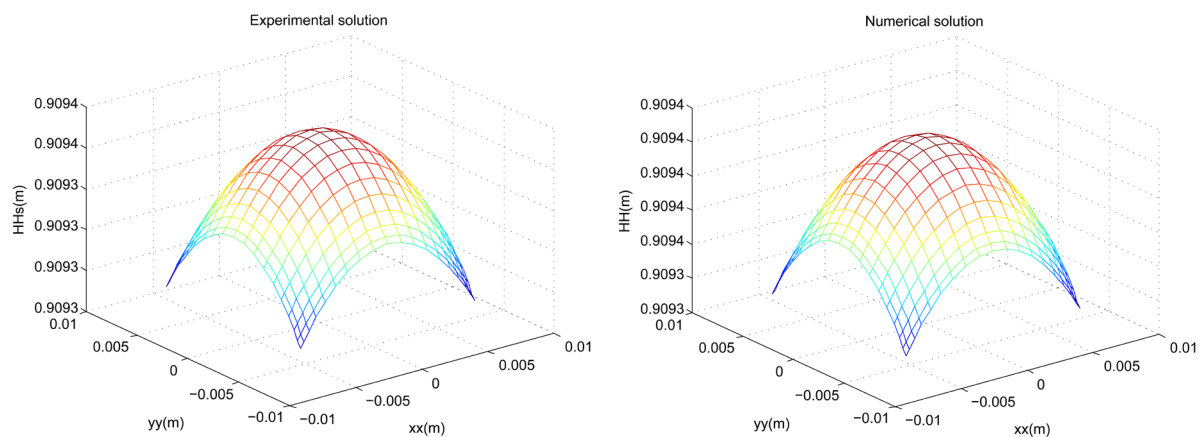


Figure 5. Profile of the experimental and approached of the dune height in the space at $t = 0.095$, for a time step $\Delta t = 5 \times 10^{-3}$, $N = M = 20$ and $\mu = 150$.

5. Conclusions and Perspectives

We have solved numerically a mathematical model of sand dune formation in a surface water to incompressible out-flows in two space dimensions. This model couples the Navier-Stokes equations governing the incompressible out-flows in two-dimension of space and the Prigozhin equation that describes the evolution of a sand dune in a surface water. One of the difficulties of this approach resides in the treatment of the pressure which appears only in Navier-Stokes equations as Lagrange multiplier. We used a Chebyshev projection scheme following a spectral approach $\mathbb{P}_{N,M} - \mathbb{P}_{N-2,M-2}$ to solve the Navier-Stokes equations, which permitted us to ignore the boundary conditions on the pressure. And, as we don't have any boundary condition on the mass density and the dune height, we have expressed the first and secondary operator derivations in $\mathbb{P}_{N-2,M-2}(\Omega)$ for the mass density and in $\mathbb{P}_{N-2,M-2}(\Gamma)$ for the dune height. It is evident from the gotten results that the smaller the strong domain Γ occupied by the dune is, the better the approximation of the dune height is.

In our future works, we count to pass in dimension 3 of space and to put a optimal control in place to determine the optimal height of sand dune in a surface water, from which other dunes can be formed in the fluid.

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