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Global Attractors and Dimension Estimation of the 2D Generalized MHD System with Extra Force

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Abstract

In this paper, firstly, some priori estimates are obtained for the existence and uniqueness of solutions of a two dimensional generalized magnetohydrodynamic (MHD) system. Then the existence of the global attractor is proved. Finally, the upper bound estimation of the Hausdorff and fractal dimension of attractor is got.

Keywords

MHD System, Existence, Global Attractor, Dimension Estimation

1. Introduction

In this paper, we study the following magnetohydrodynamic system:

$$\begin{cases} \frac{\partial u}{\partial t}+(u\cdot\nabla)u-(v\cdot\nabla)v+\gamma A^{2\alpha}u=f(x) \\ \frac{\partial v}{\partial t}+(u\cdot\nabla)v-(v\cdot\nabla)u+\eta A^{2\beta}v=g(x) \\ \nabla u=\nabla v=0 \\ (u,v)(x,0)=(u_0,v_0)(x) \\ u(x,t)\big|_{\partial\Omega}=v(x,t)\big|_{\partial\Omega}=0. \end{cases} \tag{1.1}$$

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here $\Omega \subset R^2$ is bounded set, $\partial\Omega$ is the bound of Ω, where u is the velocity vector field, v is the magnetic vector field, $\gamma,\eta > 0, \alpha,\beta > \frac{n}{2}$ are the kinematic viscosity and diffusivity constants respectively. $A = (-\Delta)$. Let $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}, \|\cdot\|_\infty = \|\cdot\|_{L^\infty(\Omega)}$.

When $\alpha = \beta = 1$, problem (1.1) reduces to the MHD equations. In particular, if $\gamma = \eta = 0$, problem (1.1) becomes the ideal MHD equations. It is therefore reasonable to call (1.1) a system of generalized MHD equations, or simply GMHD. Moreover, it has similar scaling properties and energy estimate as the Navier-Stokes and MHD equations.

The solvability of the MHD system was investigated in the beginning of 1960s. In particular in [1]-[4] the global existence of weak solutions and local in time well-posedness was proved for various initial boundary value problems. However, similar to the situation with the Navier-Stokes equations, the problem of the global smooth solvability for the MHD equations is still open.

Analogously to the case of the Navier-Stokes system (see [5]-[8]) we introduce the concept of suitable weak solutions. We prove the existence of the global attractor (see [9]) and getting the upper bound estimation of the Hausdorff and fractal dimension of attractor for the MHD system.

2. The Priori Estimate of Solution of Problem (1.1)

Lemma 1. Assume $(u_0,v_0) \in L^2(\Omega)\times L^2(\Omega), (f(x),g(x)) \in L^2(\Omega)\times L^2(\Omega)$, so the smooth solution $(u(x,t),v(x,t))$ of problem (1.1) satisfies

$$\left(\|u\|^2 + \|v\|^2\right) \le \left(\|u_0\|^2 + \|v_0\|^2\right)\mathrm{e}^{-at} + \frac{1}{a^2}\left(\|f\|^2 + \|g\|^2\right).$$

Proof. We multiply u with both sides of the first equation of problem (1.1) and obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + (u\nabla u,u) - (v\nabla v,u) + \gamma\left\|A^\alpha u\right\|^2 = (f(x),u), \tag{2.1}$$

We multiply v with both sides of the second equation of problem (1.1) and obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 + (u\nabla v,v) - (v\nabla u,v) + \eta\left\|A^\beta v\right\|^2 = (g(x),v), \tag{2.2}$$

According to $b(u,u,v) = -b(u,v,u)$, we obtain

$$b(u,u,u) = b(u,v,v) = 0,\ b(v,v,u) = -b(v,u,v), \tag{2.3}$$

According to (2.1) + (2.2), so we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2 + \|v\|^2\right) + \gamma\left\|A^\alpha u\right\|^2 + \eta\left\|A^\beta v\right\|^2 = (f(x),u) + (g(x),v), \tag{2.4}$$

According to Poincare and Young inequality, we obtain

$$\left\|A^\alpha u\right\|^2 \ge \lambda_1^{2\alpha}\|u\|^2, \left\|A^\beta v\right\|^2 \ge \lambda_1^{2\beta}\|v\|^2, \tag{2.5}$$

$$\left|(f(x),u)\right| \le \|u\|\|f\| \le \frac{\gamma\lambda_1^{2\alpha}}{4}\|u\|^2 + \frac{1}{\gamma\lambda_1^{2\alpha}}\|f\|^2, \tag{2.6}$$

$$\left|(g(x),v)\right| \le \|v\|\|g\| \le \frac{\eta\lambda_1^{2\beta}}{4}\|v\|^2 + \frac{1}{\eta\lambda_1^{2\beta}}\|g\|^2, \tag{2.7}$$

From (2.5)-(2.7), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2 + \|v\|^2\right) + \frac{\gamma}{2}\left\|A^\alpha u\right\|^2 + \frac{\gamma\lambda_1^{2\alpha}}{2}\|u\|^2 + \frac{\eta}{2}\left\|A^\beta v\right\|^2 + \frac{\eta\lambda_1^{2\beta}}{2}\|v\|^2\\ &\le \frac{\gamma\lambda_1^{2\alpha}}{4}\|u\|^2 + \frac{1}{\gamma\lambda_1^{2\alpha}}\|f\|^2 + \frac{\eta\lambda_1^{2\beta}}{4}\|v\|^2 + \frac{1}{\eta\lambda_1^{2\beta}}\|g\|^2,\end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2+\|v\|^2\right)+\frac{\gamma\lambda_1^{2\alpha}}{2}\|u\|^2+\frac{\eta\lambda_1^{2\beta}}{2}\|v\|^2\le\frac{2}{\gamma\lambda_1^{2\alpha}}\|f\|^2+\frac{2}{\eta\lambda_2^{2\beta}}\|g\|^2,$$

Let $a=\min\left\{\frac{\gamma\lambda_1^{2\alpha}}{2},\frac{\eta\lambda_1^{2\beta}}{2}\right\}$, according that we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2+\|v\|^2\right)+a\left(\|u\|^2+\|v\|^2\right)\le\frac{1}{a}\left(\|f\|^2+\|g\|^2\right).$$

Using the Gronwall's inequality, the Lemma 1 is proved. □

Lemma 2. Under the condition of Lemma 1, and $(u_0,v_0)\in H^{2\alpha}(\Omega)\times H^{2\beta}(\Omega)$, $(f(x),g(x))\in L^2(\Omega)\times L^2(\Omega)$, $\alpha>\frac{n}{2}$, $\beta>\frac{n}{2}$, so the solution $(A^\alpha u,A^\beta v)$ of problem (1.1) satisfies

$$\left(\|A^\alpha u\|^2+\|A^\beta v\|^2\right)\le\left(\|A^\alpha u_0\|^2+\|A^\beta v_0\|^2\right)\mathrm{e}^{-at}+\frac{1}{a^2}\left(\|A^\alpha f\|^2+\|A^\beta g\|^2\right)+\frac{2}{a}C_{10}.$$

Proof. For the problem (1.1) multiply the first equation by $A^{2\alpha}u$ with both sides, for the problem (1.1) multiply the second equation by $A^{2\beta}v$ with both sides and obtain

$$\begin{cases}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A^\alpha u\|^2+\left(u\nabla u,A^{2\alpha}u\right)-\left(v\nabla v,A^{2\alpha}u\right)+\gamma\|A^{2\alpha}u\|^2=\left(f,A^{2\alpha}u\right),\\ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A^\beta v\|+\left(u\nabla v,A^{2\beta}v\right)-\left(v\nabla u,A^{2\beta}v\right)+\eta\|A^{2\beta}v\|^2=\left(g,A^{2\beta}v\right).\end{cases}\tag{2.8}$$

$$\left|\left(u\nabla u,A^{2\alpha}u\right)\right|\le\int_\Omega|u\|\nabla u\|A^{2\alpha}u|\mathrm{d}x\le\|u\|_{L^4}\|\nabla u\|_{L^4}\|A^{2\alpha}u\|,$$

According to the Sobolev's interpolation inequalities,

$$\|u\|_{L^4}\le C_0\|\Delta u\|^{\frac{n}{16\alpha}}\|u\|^{1-\frac{n}{16\alpha}},\quad\|\nabla u\|_{L^4}\le C_1\|\Delta u\|^{\frac{4+n}{16\alpha}}\|u\|^{1-\frac{4+n}{16\alpha}},\tag{2.9}$$

$$\|v\|_{L^4}\le C_2\|\Delta v\|^{\frac{n}{16\beta}}\|v\|^{1-\frac{n}{16\beta}},\quad\|\nabla v\|_{L^4}\le C_3\|\Delta v\|^{\frac{4+n}{16\beta}}\|v\|^{1-\frac{4+n}{16\beta}},\tag{2.10}$$

According to (2.9)-(2.10), we have

$$\begin{aligned}\|u\|_{L^4}\|\nabla u\|_{L^4}\|A^{2\alpha}u\|&\le\|\Delta u\|^{\frac{2+n}{8\alpha}}\|u\|^{2-\frac{2+n}{8\alpha}}\|A^{2\alpha}u\|\le C_4\|\Delta u\|^{\frac{2+n}{8\alpha}}\|A^{2\alpha}u\|\\&\le C_5\|A^{2\alpha}u\|^{1+\frac{2+n}{8\alpha}}\le\frac{\gamma}{16}\|A^{2\alpha}u\|^2+C_6,\end{aligned}\tag{2.11}$$

Here

$$C_6\ge C_5^{\frac{8\alpha+2+n}{8\alpha-2-n}},$$

In a similar way, we can obtain

$$\begin{aligned}\left|\left(v\nabla v,A^{2\alpha}u\right)\right|&\le\|v\|_{L^4}\|\nabla v\|_{L^4}\|A^{2\alpha}u\|\le\|\Delta v\|^{\frac{2+n}{8\beta}}\|v\|^{2-\frac{2+n}{8\beta}}\|A^{2\alpha}u\|\\&\le C_7\|\Delta v\|^{\frac{2+n}{8\beta}}\|A^{2\alpha}u\|\le C_8\|A^{2\alpha}u\|\|A^{2\beta}v\|^{\frac{2+n}{8\beta}}\\&\le\frac{\gamma}{16}\|A^{2\alpha}u\|^2+\frac{4C_8^2}{\gamma}\|A^{2\beta}v\|^{\frac{2+n}{4\beta}}\\&\le\frac{\gamma}{16}\|A^{2\alpha}u\|^2+\frac{\eta}{12}\|A^{2\beta}v\|^2+C_9,\end{aligned}\tag{2.12}$$

Here

$$C_9 \ge \frac{8\beta-4-n}{8\beta}\left(\frac{3(4+n)}{2\beta\eta}\right)^{-\frac{(8\beta)^2}{(4+n)(8\beta-4-n)}}\left(\frac{4C_8^2}{\gamma}\right)^{\frac{8\beta}{8\beta-4-n}},$$

$$\begin{aligned}\left|\left(u\nabla v, A^{2\beta}v\right)\right| &\le \|u\|_{L^4}\|\nabla v\|_{L^4}\left\|A^{2\beta}v\right\| \le \|\Delta u\|^{\frac{n}{16\alpha}}\|u\|^{1-\frac{n}{16\alpha}}\|\Delta v\|^{\frac{4+n}{16\beta}}\|v\|^{1-\frac{4+n}{16\beta}}\left\|A^{2\beta}v\right\| \\ &\le C_{10}\|\Delta u\|^{\frac{n}{16\alpha}}\|\Delta v\|^{\frac{4+n}{16\beta}}\left\|A^{2\beta}v\right\| \le C_{11}\left\|A^{2\alpha}u\right\|^{\frac{n}{16\alpha}}\left\|A^{2\beta}v\right\|^{1+\frac{4+n}{16\beta}} \\ &\le \frac{\gamma}{16}\left\|A^{2\alpha}u\right\|^2 + C_{12}\left\|A^{2\beta}v\right\|^{\frac{16\beta+4+n}{16\beta}\cdot\frac{32\alpha}{32\alpha-n}} \le \frac{\gamma}{16}\left\|A^{2\alpha}u\right\|^2+\frac{\eta}{12}\left\|A^{2\beta}v\right\|^2+C_{13},\end{aligned} \tag{2.13}$$

Here

$$C_{12} \ge \frac{32\alpha-n}{32\alpha}C_{11}^{\frac{32\alpha}{32\alpha-n}}\left(\frac{n}{2\gamma\alpha}\right)^{-\frac{(32\alpha)^2}{n(32\alpha-n)}},$$

$$C_{13} \ge \frac{(32\alpha-n)\beta}{16\beta\alpha-\beta n-\alpha(4+n)}C_{12}^{\frac{16\beta\alpha-\beta n-\alpha(4+n)}{(32\alpha-n)\beta}}\left(\frac{12\alpha(16\beta+4+n)}{\eta\beta(32\alpha-n)}\right)^{-\frac{((32\alpha-n)\beta)^2}{\alpha(16\beta+4+n)(16\beta\alpha-\beta n-\alpha(4+n))}},$$

$$\begin{aligned}\left|\left(v\nabla u, A^{2\beta}v\right)\right| &\le \|v\|_{L^4}\|\nabla u\|_{L^4}\left\|A^{2\beta}v\right\| \le \|\Delta v\|^{\frac{n}{16\beta}}\|v\|^{1-\frac{n}{16\beta}}\|\Delta u\|^{\frac{4+n}{16\alpha}}\|u\|^{1-\frac{4+n}{16\alpha}}\left\|A^{2\beta}v\right\| \\ &\le C_{14}\|\Delta v\|^{\frac{n}{16\beta}}\|\Delta u\|^{\frac{4+n}{16\alpha}}\left\|A^{2\beta}v\right\| \le C_{15}\left\|A^{2\alpha}u\right\|^{\frac{4+n}{16\alpha}}\left\|A^{2\beta}v\right\|^{1+\frac{n}{16\beta}} \\ &\le \frac{\gamma}{16}\left\|A^{2\alpha}u\right\|^2 + C_{16}\left\|A^{2\beta}v\right\|^{\frac{16\beta+n}{16\beta}\cdot\frac{32\alpha}{32\alpha-4-n}} \le \frac{\gamma}{16}\left\|A^{2\alpha}u\right\|^2+\frac{\eta}{12}\left\|A^{2\beta}v\right\|^2+C_{17},\end{aligned} \tag{2.14}$$

Here

$$C_{16} \ge \frac{32\alpha-4-n}{32\alpha}C_{15}^{\frac{32\alpha}{32\alpha-4-n}}\left(\frac{4+n}{2\gamma\alpha}\right)^{-\frac{(32\alpha)^2}{(4+n)(32\alpha-4-n)}},$$

$$C_{17} \ge \frac{(32\alpha-4-n)\beta}{16\beta\alpha-\alpha n-\beta(4+n)}C_{16}^{\frac{16\beta\alpha-\alpha n-\beta(4+n)}{(32\alpha-4-n)\beta}}\left(\frac{12\alpha(16\beta+n)}{\eta\beta(32\alpha-4-n)}\right)^{-\frac{((32\alpha-4-n)\beta)^2}{\alpha(16\beta+n)(16\beta\alpha-\alpha n-\beta(4+n))}},$$

According to the Poincare's inequalities

$$\left\|A^{2\alpha}u\right\|^2 \ge \lambda_1^{2\alpha}\left\|A^{\alpha}u\right\|^2,\quad \left\|A^{2\beta}v\right\|^2 \ge \lambda_1^{2\beta}\left\|A^{\beta}v\right\|^2 \tag{2.15}$$

$$\left|\left(f(x), A^{2\alpha}u\right)\right| \le \left\|A^{\alpha}u\right\|\left\|A^{\alpha}f\right\| \le \frac{\gamma\lambda_1^{2\alpha}}{4}\left\|A^{\alpha}u\right\|^2+\frac{1}{\gamma\lambda_1^{2\alpha}}\left\|A^{\alpha}f\right\|^2, \tag{2.16}$$

$$\left|\left(g(x), A^{2\beta}v\right)\right| \le \left\|A^{\beta}v\right\|\left\|A^{\beta}g\right\| \le \frac{\eta\lambda_1^{2\beta}}{4}\left\|A^{\beta}v\right\|^2+\frac{1}{\eta\lambda_1^{2\beta}}\left\|A^{\beta}g\right\|^2, \tag{2.17}$$

From (2.12)-(2.17), we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|A^{\alpha}u\right\|^2+\left\|A^{\beta}v\right\|^2\right)+\frac{\gamma}{4}\left\|A^{2\alpha}u\right\|^2+\frac{\gamma\lambda_1^{2\alpha}}{2}\left\|A^{\alpha}u\right\|^2+\frac{\eta}{4}\left\|A^{2\beta}v\right\|^2+\frac{\eta\lambda_1^{2\beta}}{2}\left\|A^{\beta}v\right\|^2 \\ &\le \frac{\gamma\lambda_1^{2\alpha}}{4}\left\|A^{\alpha}u\right\|^2+\frac{1}{\gamma\lambda_1^{2\alpha}}\left\|A^{\alpha}f\right\|^2+\frac{\eta\lambda_1^{2\beta}}{4}\left\|A^{\beta}v\right\|^2+\frac{1}{\eta\lambda_1^{2\beta}}\left\|A^{\beta}g\right\|^2+C_{18},\end{aligned}$$

Here

$$C_{18} \geq C_6 + C_9 + C_{13} + C_{17},$$

So

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|A^{\alpha}u\right\|^2 + \left\|A^{\beta}v\right\|^2\right) + \frac{\gamma\lambda_1^{2\alpha}}{4}\left\|A^{\alpha}u\right\|^2 + \frac{\eta\lambda_1^{2\beta}}{4}\left\|A^{\beta}v\right\|^2 \leq \frac{1}{\gamma\lambda_1^{2\alpha}}\left\|A^{\alpha}f\right\|^2 + \frac{1}{\eta\lambda_1^{2\beta}}\left\|A^{\beta}g\right\|^2 + C_{18}.$$

We obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|A^{\alpha}u\right\|^2 + \left\|A^{\beta}v\right\|^2\right) + a\left(\left\|A^{\alpha}u\right\|^2 + \left\|A^{\beta}v\right\|^2\right) \leq \frac{1}{a}\left(\left\|A^{\alpha}f\right\|^2 + \left\|A^{\beta}g\right\|^2\right) + 2C_{18}.$$

Using the Gronwall's inequality, the Lemma 2 is proved. □

3. Global Attractor and Dimension Estimation

Theorem 1. Assume that $(f(x), g(x)) \in L^2(\Omega) \times L^2(\Omega)$ and $(u_0, v_0) \in H^{2\alpha}(\Omega) \times H^{2\beta}(\Omega)$, so problem (1.1) exist a unique solution $w(u(x,t), v(x,t)) \in L^2(0,\infty; H^{2\alpha}(\Omega) \times H^{2\beta}(\Omega))$.

Proof. By the method of Galerkin and Lemma 1-Lemma 2,we can easily obtain the existence of solutions. Next, we prove the uniqueness of solutions in detail.

Assume $w_1(u_1, v_1), w_2(u_2, v_2)$ are two solutions of problem (1.1), let $w(u,v) = w_1(u_1,v_1) - w_2(u_2,v_2)$, Here $u = u_1 - u_2, v = v_1 - v_2$, so the difference of the two solution satisfies

$$\begin{cases} \dfrac{\partial u_1}{\partial t} + (u_1 \cdot \nabla)u_1 - (v_1 \cdot \nabla)v_1 + \gamma A^{2\alpha}u_1 = f(x), \\ \dfrac{\partial v_1}{\partial t} + (u_1 \cdot \nabla)v_1 - (v_1 \cdot \nabla)u_1 + \eta A^{2\beta}v_1 = g(x), \\ \nabla u_1 = \nabla v_1 = 0, \\ (u_1, v_1)(x,0) = (u_{10}, v_{10})(x), \\ u_1(x,t)|_{\partial\Omega} = v_1(x,t)|_{\partial\Omega} = 0. \end{cases} \tag{3.1}$$

$$\begin{cases} \dfrac{\partial u_2}{\partial t} + (u_2 \cdot \nabla)u_2 - (v_2 \cdot \nabla)v_2 + \gamma A^{2\alpha}u_2 = f(x), \\ \dfrac{\partial v_2}{\partial t} + (u_2 \cdot \nabla)v_2 - (v_2 \cdot \nabla)u_2 + \eta A^{2\beta}v_2 = g(x), \\ \nabla u_2 = \nabla v_2 = 0, \\ (u_2, v_2)(x,0) = (u_{20}, v_{20})(x), \\ u_2(x,t)|_{\partial\Omega} = v_2(x,t)|_{\partial\Omega} = 0. \end{cases} \tag{3.2}$$

The two above formulae subtract and obtain

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\nabla u_1 + u_2\nabla u - v\nabla v_1 - v_2\nabla v + \gamma A^{2\alpha}u = 0, \\ \dfrac{\partial v}{\partial t} + u\nabla v_1 + u_2\nabla v - v\nabla u_1 - v_2\nabla u + \eta A^{2\beta}v = 0, \\ \nabla u = \nabla v = 0, \\ (u, v)(x,0) = (u_0, v_0)(x), \\ u(x,t)|_{\partial\Omega} = v(x,t)|_{\partial\Omega} = 0. \end{cases} \tag{3.3}$$

For the problem (3.3) multiply the first equation by u with both sides and obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + (u\nabla u_1 + u_2\nabla u - v\nabla v_1 - v_2\nabla v, u) + \gamma\left\|A^{\alpha}u\right\|^2 = 0, \tag{3.4}$$

For the problem (3.3) multiply the second equation by v with both sides and obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2+\left(u\nabla v_1+u_2\nabla v-v\nabla u_1-v_2\nabla u,v\right)+\eta\left\|A^\beta v\right\|^2=0, \tag{3.5}$$

According to

$$b\left(u_2,u,u\right)=b\left(u_2,v,v\right)=0,\ \ b\left(v_2,v,u\right)=-b\left(v_2,u,v\right). \tag{3.6}$$

According to (3.1) + (3.2), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2+\|v\|^2\right)+\left(u\nabla u_1-v\nabla v_1,u\right)+\left(u\nabla v_1-v\nabla u_1,v\right)+\gamma\left\|A^\alpha u\right\|^2+\eta\left\|A^\beta v\right\|^2=0, \tag{3.7}$$

According to Sobolev inequality, when $n<4$

$$\|u\|_\infty\le C_{19}\|\Delta u\|^{\frac{n}{4}}\|u\|^{\frac{4-n}{4}}\le C_{20}\|\Delta u\|\le C_{20}\lambda_1^{\alpha-1}\left\|A^\alpha u\right\|, \tag{3.8}$$

$$\|v\|_\infty\le C_{21}\|\Delta v\|^{\frac{n}{4}}\|v\|^{\frac{4-n}{4}}\le C_{22}\|\Delta v\|\le C_{22}\lambda_1^{\beta-1}\left\|A^\beta v\right\|, \tag{3.9}$$

According to (3.8)-(3.9),we can get

$$\left|\left(u\nabla u_1,u\right)\right|\le\|u\|\|\nabla u\|\|u_1\|_\infty\le C_{23}\|u\|\left\|A^\alpha u\right\|\le\frac{\gamma}{12}\left\|A^\alpha u\right\|^2+\frac{3C_{23}^2}{\gamma}\|u\|^2, \tag{3.10}$$

$$\left|\left(v\nabla v_1,u\right)\right|\le\|v\|\|\nabla u\|\|v_1\|_\infty\le C_{24}\|v\|\left\|A^\alpha u\right\|\le\frac{\gamma}{12}\left\|A^\alpha u\right\|^2+\frac{3C_{24}^2}{\gamma}\|v\|^2, \tag{3.11}$$

$$\left|\left(v\nabla u_1,v\right)\right|\le\|v\|\|\nabla v\|\|u_1\|_\infty\le C_{25}\|v\|\left\|A^\beta v\right\|\le\frac{\eta}{12}\left\|A^\beta v\right\|^2+\frac{3C_{25}^2}{\eta}\|v\|^2, \tag{3.12}$$

$$\left|\left(u\nabla v_1,v\right)\right|\le\|u\|\|\nabla v\|\|v_1\|_\infty\le C_{26}\|u\|\left\|A^\beta v\right\|\le\frac{\eta}{12}\left\|A^\beta v\right\|^2+\frac{3C_{26}^2}{\eta}\|u\|^2, \tag{3.13}$$

From (3.10)-(3.13),

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2+\|v\|^2\right)+\frac{\gamma}{2}\left\|A^\alpha u\right\|^2+\frac{\gamma\lambda_1^{2\alpha}}{2}\|u\|^2+\frac{\eta}{2}\left\|A^\beta v\right\|^2+\frac{\eta\lambda_1^{2\beta}}{2}\|v\|^2\\&\le\frac{\gamma}{4}\left\|A^\alpha u\right\|^2+\frac{\eta}{4}\left\|A^\beta v\right\|^2+\gamma\lambda_1^{2\alpha}\|u\|^2+\eta\lambda_1^{2\beta}\|v\|^2.\end{aligned}$$

Here $\gamma\lambda_1^{2\alpha}\ge\frac{3C_{23}^2}{\gamma}+\frac{3C_{26}^2}{\eta},\ \eta\lambda_1^{2\beta}\ge\frac{3C_{24}^2}{\gamma}+\frac{3C_{25}^2}{\eta}.$

So, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2+\|v\|^2\right)+\frac{\gamma}{4}\left\|A^\alpha u\right\|^2+\frac{\eta}{4}\left\|A^\beta v\right\|^2\le\frac{\gamma\lambda_1^{2\alpha}}{2}\|u\|^2+\frac{\eta\lambda_1^{2\beta}}{2}\|v\|^2.$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2+\|v\|^2\right)\le\gamma\lambda_1^{2\alpha}\|u\|^2+\eta\lambda_1^{2\beta}\|v\|^2.$$

Let $b=\max\left\{\gamma\lambda_1^{2\alpha},\eta\lambda_1^{2\beta}\right\}$, so we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2+\|v\|^2\right)\le b\left(\|u\|^2+\|v\|^2\right).$$

According to the consistent Gronwall inequality,

$$\left(\|u\|^2+\|v\|^2\right)^2\le\left(\|u_0\|^2+\|v_0\|^2\right)\mathrm{e}^{bt}=0.$$

So we can get $u=v=0$, the uniqueness is proved. □

Theorem 2. [9] Let E be a Banach space, and $\{S(t)\}\ (t\ge0)$ are the semigroup operators on E.

$S(t): E \to E, S(t)\cdot S(\tau) = S(t+\tau), S(0) = I$, here *I* is a unit operator. Set $S(t)$ satisfy the follow conditions

1) $S(t)$ is bounded. Namely $\forall R > 0$, $\|u\|_\infty \le R$, it exists a constant $C(R)$, so that $\|S(t)u\|_E \le C(R)(t \in [0,+\infty))$;

2) It exists a bounded absorbing set $B_0 \subset E$, namely $\forall B \subset E$, it exists a constant t_0, so that $S(t)B \subset B_0\ (t > t_0)$;

3) When $t > 0$, $S(t)$ is a completely continuous operator *A*.

Therefore, the semigroup operators $S(t)$ exist a compact global attractor.

Theorem 3. Assume $(u_0, v_0) \in E = H^{2\alpha}(\Omega)\times H^{2\beta}(\Omega)$, $(f(x), g(x)) \in L^2(\Omega)\times L^2(\Omega)$, $\alpha > \frac{n}{2}$, $\beta > \frac{n}{2}$. Problem (1.1) have global attractor $A = w(B_0) = \bigcap_{s\ge 0}\overline{\bigcup_{t\ge s} S(t)B_0}$.

Proof.

1) When $\|u_0\|_{H^{2\alpha}(\Omega)} + \|v_0\|_{H^{2\beta}(\Omega)} \le R_1 + R_2$. From Lemma 1,

$$\|S(t)u_0\|_{H^{2\alpha}(\Omega)} + \|S(t)v_0\|_{H^{2\beta}(\Omega)} = \|u\|_{H^{2\alpha}(\Omega)} + \|v\|_{H^{2\beta}(\Omega)} \le C_{27}\left(\|u_0\|_{H^{2\alpha}(\Omega)} + \|v_0\|_{H^{2\beta}(\Omega)}\right) \le C_{27}(R_1+R_2).$$

So $S(t)$ in *E* is uniformly bounded.

2) $(u(t), v(t)) = S(t)(u_0, v_0)$ has *E* in a bounded absorbing set

$$B_0 = \left\{(u,v)\in E : \|u_0\|_{H^{2\alpha}(\Omega)} + \|v_0\|_{H^{2\beta}(\Omega)} \le R_1 + R_2\right\}.$$

From Lemma 2, when $\|u_0\|_{H^{2\alpha}(\Omega)} \le R_1, \|v_0\|_{H^{2\beta}(\Omega)} \le R_2$, there is

$$\|A^{2\alpha}u\| + \|A^{2\beta}v\| = \|u\|_{H^{2\alpha}(\Omega)} + \|v\|_{H^{2\beta}(\Omega)} \le C_{28}\left(\|u_0\|_{H^{2\alpha}(\Omega)} + \|v_0\|_{H^{2\beta}(\Omega)}\right) \le C_{28}(R_1+R_2).$$

Since $E \to E$ is tightly embedded, so B_0 is $S(t)$ in the tight absorbing set in *E*.

3) So the semigroup operator $S(t): E \to E$ is completely continuous. □

In order to estimate the Hausdorff and fractal dimension of the global attractor A of problem (1.1), let problem (1.1) linearize and obtain

$$\begin{cases} \dfrac{\partial U}{\partial t} + (U\cdot\nabla)u + (u\cdot\nabla)U - (V\cdot\nabla)v - (v\cdot\nabla)V + \gamma A^{2\alpha}U = 0, \\ \dfrac{\partial V}{\partial t} + (U\cdot\nabla)v + (u\cdot\nabla)V - (V\cdot\nabla)u - (v\cdot\nabla)U + \eta A^{2\beta}V = 0, \\ U(0) = U_0, V(0) = V_0. \end{cases} \tag{3.14}$$

Assume $(U_0, V_0) \in H^{2\alpha}(\Omega)\times H^{2\beta}(\Omega)$, $(U(t), V(t))$ is the solutions of the problem (3.14). We know $(u,v) \in L^\infty\left(0,\infty; H^{2\alpha}(\Omega)\times H^{2\beta}(\Omega)\right)$. It is easy to prove the problem (3.14) has the uniqueness of solutions $(U(t), V(t)) \in L^\infty\left(0,\infty; H^{2\alpha}\left(\Omega\times H^{2\beta}(\Omega)\right)\right)$.

To prove $S(t)$ in (u_0, v_0) has differential, let $(u(t), v(t)) = S(t)(u_0, v_0)$, so there has

$$\left(DS(t)(u_0, v_0)\right)(U_0, V_0) = (U(t), V(t)).$$

Theorem 4. Assume R_3, R_4, R_5, R_6 and *T* are constants, so it exists a constant $C_{23} = C_{29}(R_3, R_4, R_5, R_6, T)$, and $\forall u_0, u_0', v_0, v_0', t$ has $\|u_0\|_{H^{2\alpha}(\Omega)} \le R_3$, $\|u_0'\|_{H^{2\alpha}(\Omega)} \le R_4$, $\|v_0\|_{H^{2\beta}(\Omega)} \le R_5$, $\|v_0'\|_{H^{2\beta}(\Omega)} \le R_6$, $\|t\| \le T$, so there is

$$\begin{aligned} &\left\|S(t)(u_0+u_0', v_0+v_0') - S(t)(u_0,v_0) - \left(DS(t)(u_0,v_0)\right)(u_0', v_0')\right\|^2_{H^{2\alpha}(\Omega)\times H^{2\beta}(\Omega)} \\ &\le C_{29}\left(\|u_0'\|^2_{H^{2\alpha}(\Omega)} + \|v_0'\|^2_{H^{2\beta}(\Omega)}\right). \end{aligned} \tag{3.15}$$

Proof. Meet the initial value problem (3.14) of respectively for (u_0, v_0), (u_0+u_0', v_0+v_0') solutions for (u,v), (u_1, v_1), let $\theta_1 = u_1 - u$, $\theta_2 = v_1 - v$. So θ_1, θ_2 satisfies

$$\begin{cases} \dfrac{\partial\theta_1}{\partial t}+u_1\nabla u_1-u\nabla u-v_1\nabla v_1+v\nabla v+\gamma A^{2\alpha}\theta_1=0, \\ \dfrac{\partial\theta_2}{\partial t}+u_1\nabla v_1-u\nabla v-v_1\nabla u_1+v\nabla u+\eta A^{2\beta}\theta_2=0, \\ \theta_1(x,0)=u_0',\theta_2(x,0)=v_0'. \end{cases} \tag{3.16}$$

Here

$$u_1\nabla u_1-u\nabla u-v_1\nabla v_1+v\nabla v=\theta_1\nabla u_1+u\nabla\theta_1-\theta_2\nabla v_1-v\nabla\theta_2, \tag{3.17}$$

$$u_1\nabla v_1-u\nabla v-v_1\nabla u_1+v\nabla u=\theta_1\nabla v_1+u\nabla\theta_2-\theta_2\nabla u_1-v\nabla\theta_1, \tag{3.18}$$

For the problem (3.16) multiply the first equation by θ_1 with both sides and for the problem (3.16) multiply the second equation by θ_2 with both sides and obtain

$$\begin{cases} \dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}\|\theta_1\|^2+\left(\theta_1\nabla u_1+u\nabla\theta_1-\theta_2\nabla v_1-v\nabla\theta_2,\theta_1\right)+\gamma\left\|A^{\alpha}\theta_1\right\|^2=0, \\ \dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}\|\theta_2\|^2+\left(\theta_1\nabla v_1+u\nabla\theta_2-\theta_2\nabla u_1-v\nabla\theta_1,\theta_2\right)+\eta\left\|A^{\beta}\theta_2\right\|^2=0, \end{cases} \tag{3.19}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\theta_1\|^2+\|\theta_2\|^2\right)\le 2a\left(\|\theta_1\|^2+\|\theta_2\|^2\right), \tag{3.20}$$

Here $a=\min\left\{\dfrac{\gamma\lambda_1^{2\alpha}}{2},\dfrac{\eta\lambda_1^{2\beta}}{2}\right\}$.

For the problem (3.16) multiply the first equation by $A^{2\alpha}\theta_1$ with both sides and for the problem (3.16) multiply the second equation by $A^{2\beta}\theta_2$ with both sides and obtain

$$\begin{cases} \dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}\left\|A^{\alpha}\theta_1\right\|^2+\left(\theta_1\nabla u_1+u\nabla\theta_1-\theta_2\nabla v_1-v\nabla\theta_2,A^{2\alpha}\theta_1\right)+\gamma\left\|A^{2\alpha}\theta_1\right\|^2=0, \\ \dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}\left\|A^{\beta}\theta_2\right\|^2+\left(\theta_1\nabla v_1+u\nabla\theta_2-\theta_2\nabla u_1-v\nabla\theta_1,A^{2\beta}\theta_2\right)+\eta\left\|A^{2\beta}\theta_2\right\|^2=0, \end{cases} \tag{3.21}$$

According to the Sobolev's interpolation inequalities

$$\|\nabla u\|_\infty\le C_{30}\left\|A^{\alpha}u\right\|^{\frac{2+n}{4\alpha}}\|u\|^{\frac{4\alpha-2-n}{4\alpha}}, \tag{3.22}$$

$$\|\nabla v\|_\infty\le C_{31}\left\|A^{\beta}v\right\|^{\frac{2+n}{4\beta}}\|v\|^{\frac{4\beta-2-n}{4\beta}}, \tag{3.23}$$

According to (3.22)-(3.23), we have

$$\begin{aligned}\left|\left(\theta_1\nabla u_1,A^{2\alpha}\theta_1\right)\right|&\le\|\theta_1\|\|\nabla u_1\|_\infty\left\|A^{2\alpha}\theta_1\right\|\le C_{30}\|\theta_1\|\left\|A^{\alpha}u_1\right\|^{\frac{2+n}{4\alpha}}\|u_1\|^{\frac{4\alpha-2-n}{4\alpha}}\left\|A^{2\alpha}\theta_1\right\| \\ &\le C_{32}\|\theta_1\|\left\|A^{2\alpha}\theta_1\right\|\le\frac{\gamma}{8}\left\|A^{2\alpha}\theta_1\right\|^2+\frac{2C_{32}^2}{\gamma}\|\theta_1\|^2,\end{aligned} \tag{3.24}$$

In a similar way, we can obtain

$$\begin{aligned}\left|\left(u\nabla\theta_1,A^{2\alpha}\theta_1\right)\right|&\le\|\theta_1\|\|\nabla u\|_\infty\left\|A^{2\alpha}\theta_1\right\|\le C_{30}\|\theta_1\|\left\|A^{\alpha}u\right\|^{\frac{2+n}{4\alpha}}\|u\|^{\frac{4\alpha-2-n}{4\alpha}}\left\|A^{2\alpha}\theta_1\right\| \\ &\le C_{33}\|\theta_1\|\left\|A^{2\alpha}\theta_1\right\|\le\frac{\gamma}{8}\left\|A^{2\alpha}\theta_1\right\|^2+\frac{2C_{33}^2}{\gamma}\|\theta_1\|^2,\end{aligned} \tag{3.25}$$

$$\begin{aligned}\left|\left(\theta_2\nabla v_1, A^{2\alpha}\theta_1\right)\right| &\le \|\theta_2\|\|\nabla v_1\|_\infty \left\|A^{2\alpha}\theta_1\right\| \le C_{31}\|\theta_2\|\left\|A^\beta v_1\right\|^{\frac{2+n}{4\beta}}\|v_1\|^{\frac{4\beta-2-n}{4\beta}}\left\|A^{2\alpha}\theta_1\right\| \\ &\le C_{34}\|\theta_2\|\left\|A^{2\alpha}\theta_1\right\| \le \frac{\gamma}{8}\left\|A^{2\alpha}\theta_1\right\|^2 + \frac{2C_{34}^2}{\gamma}\|\theta_2\|^2,\end{aligned} \tag{3.26}$$

$$\begin{aligned}\left|\left(v\nabla\theta_2, A^{2\alpha}\theta_1\right)\right| &\le \|\theta_2\|\|\nabla v\|_\infty \left\|A^{2\alpha}\theta_1\right\| \le C_{31}\|\theta_2\|\left\|A^\beta v\right\|^{\frac{2+n}{4\beta}}\|v\|^{\frac{4\beta-2-n}{4\beta}}\left\|A^{2\alpha}\theta_1\right\| \\ &\le C_{35}\|\theta_2\|\left\|A^{2\alpha}\theta_1\right\| \le \frac{\gamma}{8}\left\|A^{2\alpha}\theta_1\right\|^2 + \frac{2C_{35}^2}{\gamma}\|\theta_2\|^2,\end{aligned} \tag{3.27}$$

$$\left|\left(\theta_1\nabla v_1, A^{2\beta}\theta_2\right)\right| \le \|\theta_1\|\|\nabla v_1\|_\infty \left\|A^{2\beta}\theta_2\right\| \le C_{34}\|\theta_1\|\left\|A^{2\beta}\theta_2\right\| \le \frac{\eta}{8}\left\|A^{2\beta}\theta_2\right\|^2 + \frac{2C_{34}^2}{\eta}\|\theta_1\|^2, \tag{3.28}$$

$$\left|\left(u\nabla\theta_2, A^{2\beta}\theta_2\right)\right| \le \|\theta_2\|\|\nabla u\|_\infty \left\|A^{2\beta}\theta_2\right\| \le C_{33}\|\theta_2\|\left\|A^{2\beta}\theta_2\right\| \le \frac{\eta}{8}\left\|A^{2\beta}\theta_2\right\|^2 + \frac{2C_{33}^2}{\eta}\|\theta_2\|^2, \tag{3.29}$$

$$\left|\left(\theta_2\nabla u_1, A^{2\beta}\theta_2\right)\right| \le \|\theta_2\|\|\nabla u_1\|_\infty \left\|A^{2\beta}\theta_2\right\| \le C_{32}\|\theta_2\|\left\|A^{2\beta}\theta_2\right\| \le \frac{\eta}{8}\left\|A^{2\beta}\theta_2\right\|^2 + \frac{2C_{32}^2}{\eta}\|\theta_2\|^2, \tag{3.30}$$

$$\left|\left(v\nabla\theta_1, A^{2\beta}\theta_2\right)\right| \le \|\theta_1\|\|\nabla v\|_\infty \left\|A^{2\beta}\theta_2\right\| \le C_{35}\|\theta_1\|\left\|A^{2\beta}\theta_2\right\| \le \frac{\eta}{8}\left\|A^{2\beta}\theta_2\right\|^2 + \frac{2C_{35}^2}{\eta}\|\theta_1\|^2, \tag{3.31}$$

So, we can get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|A^\alpha\theta_1\right\|^2 + \left\|A^\beta\theta_2\right\|^2\right) + \gamma\left\|A^{2\alpha}\theta_1\right\|^2 + \eta\left\|A^{2\beta}\theta_2\right\|^2 \le \frac{\gamma}{2}\left\|A^{2\alpha}\theta_1\right\|^2 + \frac{\eta}{2}\left\|A^{2\beta}\theta_2\right\|^2 + c\left(\|\theta_1\|^2 + \|\theta_2\|^2\right),$$

Here $c = \max\left\{\frac{2C_{32}^2}{\gamma} + \frac{2C_{33}^2}{\gamma} + \frac{2C_{34}^2}{\eta} + \frac{2C_{35}^2}{\eta}, \frac{2C_{32}^2}{\eta} + \frac{2C_{33}^2}{\eta} + \frac{2C_{34}^2}{\gamma} + \frac{2C_{35}^2}{\gamma}\right\}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|A^\alpha\theta_1\right\|^2 + \left\|A^\beta\theta_2\right\|^2\right) \le 2c\left(\|\theta_1\|^2 + \|\theta_2\|^2\right),$$

According to the Poincare's inequalities

$$\|\theta_1\|^2 \le \frac{1}{\lambda_1^{2\alpha}}\left\|A^\alpha\theta_1\right\|^2, \quad \|\theta_2\|^2 \le \frac{1}{\lambda_1^{2\beta}}\left\|A^\beta\theta_2\right\|^2. \tag{3.32}$$

Let $d = \max\left\{\frac{c}{\lambda_1^{2\alpha}}, \frac{c}{\lambda_1^{2\beta}}\right\}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|A^\alpha\theta_1\right\|^2 + \left\|A^\beta\theta_2\right\|^2\right) \le 2d\left(\left\|A^\alpha\theta_1\right\|^2 + \left\|A^\beta\theta_2\right\|^2\right),$$

According to Gronwall's inequalities, we obtain

$$\left(\left\|A^\alpha\theta_1\right\|^2 + \left\|A^\beta\theta_2\right\|^2\right) \le \left(\left\|A^\alpha u_0'\right\|^2 + \left\|A^\beta v_0'\right\|^2\right)\mathrm{e}^{2dt}. \tag{3.33}$$

Let (U,V) be the solutions of the linear Equation (3.14), and satisfies $(U(0),V(0)) = (u_0', v_0')$, Assume

$$\begin{aligned}(w_1, w_2) &= (u_1 - u - U, v_1 - v - V) \\ &= S(t)(u_0 + u_0', v_0 + v_0') - S(t)(u_0, v_0) - (DS(t)(u_0, v_0))(u_0', v_0'),\end{aligned} \tag{3.34}$$

So, we can get

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}w_1 + u_1\nabla u_1 - u\nabla u - v_1\nabla v_1 + v\nabla v - U\nabla u - u\nabla U + V\nabla v + v\nabla V + \gamma A^{2\alpha}w_1 = 0. \\ \dfrac{\mathrm{d}}{\mathrm{d}t}w_2 + u_1\nabla v_1 - u\nabla v - v_1\nabla u_1 + v\nabla u - U\nabla v - u\nabla V + V\nabla u + v\nabla U + \eta A^{2\beta}w_2 = 0. \\ w_1(x,0) = 0, w_2(x,0) = 0 \end{cases} \tag{3.35}$$

Here

$$\begin{aligned} &u_1\nabla u_1 - u\nabla u - v_1\nabla v_1 + v\nabla v - U\nabla u - u\nabla U + V\nabla v + v\nabla V \\ &= \theta_1\nabla\theta_1 + w_1\nabla u + u\nabla w_1 - \theta_2\nabla\theta_2 - w_2\nabla v - v\nabla w_2 \end{aligned} \tag{3.36}$$

$$\begin{aligned} &u_1\nabla v_1 - u\nabla v - v_1\nabla u_1 + v\nabla u - U\nabla v - u\nabla V + V\nabla u + v\nabla U \\ &= \theta_1\nabla\theta_2 + w_1\nabla v + u\nabla w_2 - \theta_2\nabla\theta_1 - w_2\nabla u - v\nabla w_1. \end{aligned} \tag{3.37}$$

For the problem (3.33) multiply the first equation by w_1 with both sides and for the problem (3.33) multiply the second equation by w_2 with both sides and obtain

$$\begin{cases} \dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}\|w_1\|^2 + (\theta_1\nabla\theta_1 + w_1\nabla u + u\nabla w_1 - \theta_2\nabla\theta_2 - w_2\nabla v - v\nabla w_2, w_1) + \gamma\|A^\alpha w_1\|^2 = 0 \\ \dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}\|w_2\|^2 + (\theta_1\nabla\theta_2 + w_1\nabla v + u\nabla w_2 - \theta_2\nabla\theta_1 - w_2\nabla u - v\nabla w_1, w_2) + \eta\|A^\beta w_2\|^2 = 0 \end{cases} \tag{3.38}$$

According to (3.8)-(3.9), then

$$|(\theta_1\nabla\theta_1, w_1)| \le \|\theta_1\|\|\nabla\theta_1\|\|w_1\|_\infty \le C_{36}\|A^\alpha\theta_1\|\|A^\alpha w_1\| \le \frac{\gamma}{10}\|A^\alpha w_1\|^2 + \frac{5C_{36}^2}{2\gamma}\|A^\alpha\theta_1\|^2. \tag{3.39}$$

$$|(w_1\nabla u, w_1)| \le \|w_1\|\|\nabla w_1\|\|u\|_\infty \le C_{37}\|w_1\|\|A^\alpha w_1\| \le \frac{\gamma}{10}\|A^\alpha w_1\|^2 + \frac{5C_{37}^2}{2\gamma}\|w_1\|^2. \tag{3.40}$$

$$|(\theta_2\nabla\theta_2, w_1)| \le \|\theta_2\|\|\nabla\theta_2\|\|w_1\|_\infty \le C_{38}\|A^\beta\theta_2\|\|A^\alpha w_1\| \le \frac{\gamma}{10}\|A^\alpha w_1\|^2 + \frac{5C_{38}^2}{2\gamma}\|A^\beta\theta_2\|^2. \tag{3.41}$$

$$|(w_2\nabla v, w_1)| \le \|w_2\|\|\nabla w_1\|\|v\|_\infty \le C_{39}\|w_2\|\|A^\alpha w_1\| \le \frac{\gamma}{10}\|A^\alpha w_1\|^2 + \frac{5C_{39}^2}{2\gamma}\|w_2\|^2. \tag{3.42}$$

$$|(v\nabla w_2, w_1)| \le \|w_1\|\|\nabla w_2\|\|v\|_\infty \le C_{40}\|w_1\|\|A^\beta w_2\| \le \frac{\eta}{10}\|A^\beta w_2\|^2 + \frac{5C_{40}^2}{2\eta}\|w_1\|^2. \tag{3.43}$$

$$|(\theta_1\nabla\theta_2, w_2)| \le \|\theta_1\|\|\nabla\theta_2\|\|w_2\|_\infty \le C_{41}\|A^\beta\theta_2\|\|A^\beta w_2\| \le \frac{\eta}{10}\|A^\beta w_2\|^2 + \frac{5C_{41}^2}{2\eta}\|A^\beta\theta_2\|^2. \tag{3.44}$$

$$|(w_1\nabla v, w_2)| \le \|w_1\|\|\nabla w_2\|\|v\|_\infty \le C_{42}\|w_1\|\|A^\beta w_2\| \le \frac{\eta}{10}\|A^\beta w_2\|^2 + \frac{5C_{42}^2}{2\eta}\|w_1\|^2. \tag{3.45}$$

$$|(\theta_2\nabla\theta_1, w_2)| \le \|\theta_2\|\|\nabla\theta_1\|\|w_2\|_\infty \le C_{43}\|A^\alpha\theta_1\|\|A^\beta w_2\| \le \frac{\eta}{10}\|A^\beta w_2\|^2 + \frac{5C_{43}^2}{2\eta}\|A^\alpha\theta_1\|^2. \tag{3.46}$$

$$|(w_2\nabla u, w_2)| \le \|w_2\|\|\nabla w_2\|\|u\|_\infty \le C_{44}\|w_2\|\|A^\beta w_2\| \le \frac{\eta}{10}\|A^\beta w_2\|^2 + \frac{5C_{44}^2}{2\eta}\|w_2\|^2. \tag{3.47}$$

$$|(v\nabla w_1, w_2)| \le \|w_2\|\|\nabla w_1\|\|v\|_\infty \le C_{45}\|A^\alpha w_1\|\|w_2\| \le \frac{\gamma}{10}\|A^\alpha w_1\|^2 + \frac{5C_{45}^2}{2\gamma}\|w_2\|^2. \tag{3.48}$$

According to, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w_1\|^2+\|w_2\|^2\right)+\gamma\left\|A^{\alpha}w_1\right\|^2+\eta\left\|A^{\beta}w_2\right\|^2$$
$$\le\frac{\gamma}{2}\left\|A^{\alpha}w_1\right\|^2+\frac{\eta}{2}\left\|A^{\beta}w_2\right\|^2+e\left(\|w_1\|^2+\|w_2\|^2\right)+k\left(\|\theta_1\|^2+\|\theta_2\|^2\right).$$

Here $e=\max\left\{\frac{5C_{37}^2}{2\gamma}+\frac{5C_{40}^2}{2\eta}+\frac{5C_{42}^2}{2\eta},\frac{5C_{39}^2}{2\gamma}+\frac{5C_{44}^2}{2\gamma}+\frac{5C_{45}^2}{2\eta}\right\},k=\max\left\{\frac{5C_{36}^2}{2\gamma}+\frac{5C_{43}^2}{2\eta},\frac{5C_{38}^2}{2\gamma}+\frac{5C_{41}^2}{2\eta}\right\}$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w_1\|^2+\|w_2\|^2\right)+\frac{\gamma}{2}\left\|A^{\alpha}w_1\right\|^2+\frac{\eta}{2}\left\|A^{\beta}w_2\right\|^2\le e\left(\|w_1\|^2+\|w_2\|^2\right)+k\left(\|\theta_1\|^2+\|\theta_2\|^2\right).$$

We obtain

$$\sup_{t\in[0,T]}\left(\|w_1\|^2+\|w_2\|^2\right)\le\left(\|u_0'\|_{H^{2\alpha}(\Omega)}^2+\|v_0'\|_{H^{2\beta}(\Omega)}^2\right)\mathrm{e}^{2dt}$$

So

$$\left\|S(t)(u_0+u_0',v_0+v_0')-S(t)(u_0,v_0)-\left(DS(t)(u_0,v_0)\right)(u_0',v_0')\right\|_{H^{2\alpha}(\Omega)\times H^{2\beta}(\Omega)}^2$$
$$\le C_{29}\left(\|u_0'\|_{H^{2\alpha}(\Omega)}^2+\|v_0'\|_{H^{2\beta}(\Omega)}^2\right). \quad (3.49)\ \square$$

Let $(U_1(t),V_1(t)),(U_2(t),V_2(t)),\cdots,(U_N(t),V_N(t))$ be the solutions of the linear Equation (3.33) corresponding to the initial value $(U_1(0),V_1(0))=(\zeta_1,\xi_1),U_2(0),V_2(0)=(\zeta_2,\xi_2),\cdots,(U_N(0),V_N(0))=(\zeta_N,\xi_N)$, so there is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|(U_1(t),V_1(t))\wedge(U_2(t),V_2(t))\wedge\cdots\wedge(U_N(t),V_N(t))\right\|^2$$
$$+2tr\left(L(u(t),v(t))\cdot Q_N\right)\left\|(U_1(t),V_1(t))\wedge(U_2(t),V_2(t))\wedge\cdots\wedge(U_N(t),V_N(t))\right\|^2=0. \quad (3.50)$$

$L(u(t),v(t))=L(S(t)(u_0,v_0))$ is linear mapping that is defined in the problem (3.34), $\wedge$ represents the outer product, *tr* represents the trace, Q_N is the orthogonal projection from $L^2(\Omega)$ to the span $\{(U_1(t),V_1(t)),(U_2(t),V_2(t)),\cdots,(U_N(t),V_N(t))\}$.

Theorem 5. Under the assume of Theorem 3, the global attractor *A* of problem (1.1) has finite Hausdorff and fractal dimension, and

$$d_H\le J_0,\ d_F\le 2J_0,$$

Here J_0 is a minimal positive integer of the following inequality

$$J_0=\frac{1}{2C'^{2l}(\gamma+\eta)}\left\{(\gamma+\eta)C'+\left[\left((\gamma+\eta)C'\right)^2+16C'^{2l}(\gamma+\eta)+C_{49}\right]^{\frac{1}{2}}\right\},$$

Proof. By theorem [8], we need to estimate the lower bound of $tr\left(L(u(t),v(t))\cdot Q_N\right)$. Let $(\varphi_1,\psi_1),(\varphi_2,\psi_2),\cdots,(\varphi_N,\psi_N)$ be the orthogonal basis of subspace of $Q_NL^2(\Omega)$,

$$\begin{aligned}tr\left(L(u(t),v(t))\cdot Q_N\right)&=\sum_{j=1}^N\left\{\left(L(u(t))\varphi_j,\varphi_j\right)\right\}+\sum_{j=1}^N\left\{\left(L(v(t))\psi_j,\psi_j\right)\right\}\\&=\sum_{j=1}^N\left\{\left(\varphi_j\nabla u+u\nabla\varphi_j-\psi_j\nabla v-v\nabla\psi_j+\gamma A^{2\alpha}\varphi_j,\varphi_j\right)\right\}\\&\quad+\sum_{j=1}^N\left\{\left(\varphi_j\nabla v+u\nabla\psi_j-\psi_j\nabla u-v\nabla\varphi_j+\eta A^{2\beta}\psi_j,\psi_j\right)\right\}\\&=\sum_{j=1}^N\left\{\left(\varphi_j\nabla u-\psi_j\nabla v-v\nabla\psi_j,\varphi_j\right)+\gamma\left\|A^{\alpha}\varphi_j\right\|^2\right\}\\&\quad+\sum_{j=1}^N\left\{\left(\varphi_j\nabla v-\psi_j\nabla u-v\nabla\varphi_j,\psi_j\right)+\eta\left\|A^{\beta}\psi_j\right\|^2\right\}\end{aligned} \quad (3.51)$$

According to (3.8)-(3.9), we can get

$$\left|\left(\varphi_j\nabla u,\varphi_j\right)\right|\le\|\varphi_j\|\|\nabla u\|\|\varphi_j\|_\infty\le\|\varphi_j\|\|A^\alpha u\|\|A^\alpha\varphi_j\|\le C_{46}\|\varphi_j\|\|A^\alpha\varphi_j\|\le\frac{\gamma}{4}\|A^\alpha\varphi_j\|^2+\frac{C_{46}^2}{\gamma}\|\varphi_j\|^2. \tag{3.52}$$

$$\left|\left(\psi_j\nabla v,\varphi_j\right)\right|\le\|\psi_j\|\|\nabla\varphi_j\|\|v\|_\infty\le C_{47}\|\psi_j\|\|\nabla\varphi_j\|\le\frac{\gamma}{4}\|\nabla\varphi_j\|^2+\frac{C_{47}^2}{\gamma}\|\psi_j\|^2. \tag{3.53}$$

$$\left|\left(v\nabla\psi_j,\varphi_j\right)\right|\le\|\nabla\psi_j\|\|\varphi_j\|\|v\|_\infty\le C_{47}\|\nabla\psi_j\|\|\varphi_j\|\le\frac{\eta}{4}\|\nabla\psi_j\|^2+\frac{C_{47}^2}{\eta}\|\varphi_j\|^2. \tag{3.54}$$

$$\left|\left(\varphi_j\nabla v,\psi_j\right)\right|\le\|\varphi_j\|\|\nabla\psi_j\|\|v\|_\infty\le C_{47}\|\varphi_j\|\|\nabla\psi_j\|\le\frac{\eta}{4}\|\nabla\psi_j\|^2+\frac{C_{47}^2}{\eta}\|\varphi_j\|^2. \tag{3.55}$$

$$\left|\left(\psi_j\nabla u,\psi_j\right)\right|\le\|\psi_j\|\|\nabla u\|\|\psi_j\|_\infty\le\|\psi_j\|\|A^\alpha u\|\|A^\beta\psi_j\|\le C_{48}\|\psi_j\|\|A^\beta\psi_j\|\le\frac{\eta}{4}\|A^\beta\psi_j\|^2+\frac{C_{48}^2}{\eta}\|\psi_j\|^2. \tag{3.56}$$

$$\left|\left(v\nabla\varphi_j,\psi_j\right)\right|\le\|\nabla\varphi_j\|\|\psi_j\|\|v\|_\infty\le C_{47}\|\nabla\varphi_j\|\|\psi_j\|\le\frac{\gamma}{4}\|\nabla\varphi_j\|^2+\frac{C_{47}^2}{\gamma}\|\psi_j\|^2. \tag{3.57}$$

Under the bounded condition, select $\left(\varphi_j(x,y),\psi_j(x,y)\right)=\mathrm{e}^{ik_1x+ik_2y}$ is the standard eigenfunction of $-\Delta u=\lambda u$, $-\Delta v=\lambda v$ and the corresponding eigenvalues are $\lambda_j\,(j=1,2,\cdots)$, and

$$\|\nabla\varphi_j\|^2=\|\nabla\psi_j\|^2=\lambda_j,\ \|\Delta\varphi_j\|^2=\|\Delta\psi_j\|^2=\lambda_j^2,\ \|\varphi_j\|^2=\|\psi_j\|^2=1,$$

$$\|A^\alpha\varphi_j\|^2=\lambda_j^{2\alpha},\ \|A^\beta\varphi_j\|^2=\lambda_j^{2\beta},\ \lambda_j\ge\left[\frac{(j-1)^{\frac{1}{2}}}{2}-1\right]^2\sim C\cdot j^{\frac{2}{n}}.$$

Let $C_{49}\ge\frac{C_{46}^2+2C_{47}^2}{\gamma}+\frac{2C_{47}^2+C_{48}^2}{\eta}$. Therefore, we can get

$$\begin{aligned}tr\left(L\left(u(t),v(t)\right)\cdot Q_N\right)&\ge\gamma\sum_{j=1}^N\lambda_j^{2\alpha}+\eta\sum_{j=1}^N\lambda_j^{2\beta}-\frac{\gamma}{4}\sum_{j=1}^N\lambda_j^{2\alpha}-\frac{\eta}{4}\sum_{j=1}^N\lambda_j^{2\beta}-\left(\frac{\gamma}{2}+\frac{\eta}{2}\right)\sum_{j=1}^N\lambda_j-NC_{49}\\&\ge\frac{3\gamma}{4}\sum_{j=1}^N\lambda_j^{2\alpha}+\frac{3\eta}{4}\sum_{j=1}^N\lambda_j^{2\beta}-\left(\frac{\gamma}{2}+\frac{\eta}{2}\right)\sum_{j=1}^N\lambda_j-NC_{49},\end{aligned}$$

Let $l=\min\{\alpha,\beta\}$.

$$tr\left(L\left(u(t),v(t)\right)\cdot Q_N\right)\ge\left(\frac{3\gamma}{4}+\frac{3\eta}{4}\right)\sum_{j=1}^N\lambda_j^{2l}-\left(\frac{\gamma}{2}+\frac{\eta}{2}\right)\sum_{j=1}^N\lambda_j-NC_{49},$$

By $\lambda_j\ge C'j^{\frac{2}{n}}$ and $n=2,3$

$$\sum_{j=1}^N j^{\frac{4l}{n}}\ge\sum_{j=1}^N j^2=\frac{N(N+1)(2N+1)}{6}>\frac{N^3}{3},$$

$$\sum_{j=1}^N j=\frac{N(N+1)}{2}>\frac{N^2}{2},$$

So, we can obtain

$$N>\frac{1}{2C'^{2l}(\gamma+\eta)}\left\{(\gamma+\eta)C'+\left[\left((\gamma+\eta)C'\right)^2+16C'^{2l}(\gamma+\eta)+C_{49}\right]^{\frac{1}{2}}\right\}=J_0,$$

We have

$$tr\left(L\left(u(t),v(t)\right)\cdot Q_N\right) > 0.$$

Therefore

$$d_H \le J_0,\ d_F \le 2J_0.\quad \square$$

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