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A Block Procedure with Linear Multi-Step Methods Using Legendre Polynomials for Solving ODEs

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Abstract

In this article, we derive a block procedure for some *K*-step linear multi-step methods (for *K* = 1, 2 and 3), using Legendre polynomials as the basis functions. We give discrete methods used in block and implement it for solving the non-stiff initial value problems, being the continuous interpolant derived and collocated at grid and off-grid points. Numerical examples of ordinary differential equations (ODEs) are solved using the proposed methods to show the validity and the accuracy of the introduced algorithms. A comparison with fourth-order Runge-Kutta method is given. The obtained numerical results reveal that the proposed method is efficient.

Keywords

Collocation Methods with Legendre Polynomials, Initial Value Problems, Perturbation Function, Fourth-Order Runge-Kutta Method

1. Introduction

Numerous problems in Physics, Chemistry, Biology and Engineering science are modeled mathematically by ordinary differential equations (ODEs), e.g., series circuits, mechanical systems with several springs attached in series lead to a system of differential equations [1]. Since most realistic differential equations do not have exact analytic solutions, therefore, approximate and numerical techniques [2]-[14] are used extensively. Recently introduced power series method [15] has been used for solving a wide range of problems. This method has proven rather successful in dealing with both linear as well as nonlinear problems, as it yields analytical solutions and offers certain advantages over standard numerical methods. It is free from rounding off errors since it does not involve discretization, and is computationally inexpensive. Ercan and Mustafa [16] have applied this method to

a system of differential-algebraic equations.

In this paper, we present an efficient numerical method to solve numerically the ODEs. The proposed method is a block procedure for some K-step linear multi-step methods (for $K = 1$, 2 and 3) [17] using Legendre polynomials as the basis functions. In addition, we give discrete methods used in block and implement it for solving the non-stiff IVPs that were the continuous interpolant derived and collocated at grid and off-grid points. In this article, we consider the general form of the first order initial value problem

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0. \tag{1}$$

The plan of the paper is as follows: In Section 2, the derivation of the proposed methods is presented. In Section 3, stability and convergence analysis of the block schemes is given. In Section 4, two numerical examples are considered. The paper ends with summary and conclusions in Section 5.

2. The Derivation of the Proposed Methods

In this section, we derive discrete methods to solve (1) at a sequence of nodal points $x_n = x_0 + nh$ where $h > 0$ is the step-length or grid-size defined by $h = x_{n+1} - x_n$ and $y(x)$ denotes the true solution to (1) while the approximate solution is denoted by $\overline{y}(x) = \{y_n, y_{n+1}, \cdots, y_N\}$, for some positive number N. The proposed method depends on the perturbed collocation method with respect to the power series method is used as the basis for collocation approximation with the Legendre polynomials as the perturbation term.

In the first we consider the approximate solution of the perturbed form of (1) in the following power series [18]-[21]

$$y_K(x) = \sum_{i=0}^{K} c_i \psi_i(x), \quad x_n \le x \le x_{n+K}. \tag{2}$$

where

$$\psi_i(x) = x^i, \quad i = 0, 1, \cdots, K. \tag{3}$$

Substituting from Equation (2) in Equation (1) we have

$$\sum_{i=1}^{K} c_i \psi_i'(x) = f(x, y) + \lambda L_K(x), \tag{4}$$

where $L_K(x)$ is the Legendre polynomial of degree K, valid in $x_n \le x \le x_{n+K}$ and λ is a perturbed parameter. In particular, we shall be dealing with cases $K = 1$, 2 and 3 in (2) and (3).

The well-known Legendre polynomials are defined on the interval $[-1, 1]$ and can be determined with the aid of the following recurrence formula [22] [23]

$$L_{i+1}(x) = \frac{2i+1}{i+1} x L_i(x) - \frac{i}{i+1} L_{i-1}(x), \quad i = 1, 2, \cdots,$$

where the first four polynomials are

$$L_0(x) = 1, \quad L_1(x) = x, \quad L_2(x) = \frac{1}{2}(3x^2 - 1), \quad L_3(x) = \frac{1}{2}(5x^3 - 3x). \tag{5}$$

In order to use these polynomials on the interval $[x_n, x_{n+K}]$ we define the so called shifted Legendre polynomials by introducing the change of variable

$$x = \frac{2\overline{x} - (x_{n+K} + x_n)}{(x_{n+K} - x_n)}, \quad K = 1, 2, 3. \tag{6}$$

Cases Study

Case 1: If $K = 1$

In this case, we take the polynomial $L_1(x) = x$ and use (6), then collocate this equation at x_n and x_{n+1}, we obtain

$$L_1(x_n) = \frac{2x_n - x_{n+1} - x_n}{x_{n+1} - x_n} = -1,$$

$$L_1(x_{n+1}) = \frac{2x_{n+1} - x_{n+1} - x_n}{x_{n+1} - x_n} = 1.$$

In addition, from Equation (3) we can deduce that $\psi_0'(x) = 0$, $\psi_1'(x) = 1$. Then, the Equation (4) will reduce to the following form

$$c_1 = f(x, y) + \lambda L_1(x). \tag{7}$$

We now collocate Equation (7) at x_{n+i}, $i = 0,1$ and interpolate (2) at $x = x_n$, we get a system of three Equations with $c_i (i = 0,1)$ and parameter λ

$$\begin{aligned} c_0 + c_1 x_n &= y_n, \\ c_1 + \lambda &= f_n, \\ c_1 - \lambda &= f_{n+1}. \end{aligned} \tag{8}$$

Using a suitable method to solve the above system to obtain

$$\lambda = \frac{1}{2}(f_n - f_{n+1}), \quad c_1 = f_n - \lambda, \quad c_0 = y_n - x_n(f_n - \lambda)$$

From (2), we have

$$\overline{y}(x) = c_0 + c_1 x. \tag{9}$$

Now, the required numerical scheme of the proposed method will be obtained if we collocate the above Equation (9) at $x = x_{n+1}$ and substitute c_0, c_1, λ as follows

$$y_{n+1} = y_n + \frac{h}{2}(f_{n+1} + f_n). \tag{10}$$

Which is the well-known trapezoidal rule.

Case 2: If $K = 2$

In this case, we take the polynomial $L_2(x) = \frac{1}{2}(3x^2 - 1)$ and use (6), then collocating this equation at x_n, x_{n+1} and x_{n+2}, we obtain

$$L_2(x_n) = 1, \quad L_2(x_{n+1}) = -\frac{1}{2}, \quad L_2(x_{n+2}) = 1.$$

In addition, from Equation (3) we can deduce that $\psi_0'(x) = 0$, $\psi_1'(x) = 1$, $\psi_2'(x) = 2x$. Then, the Equation (4) will reduce to the following form

$$c_1 + 2xc_2 = f(x, y) + \lambda L_2(x). \tag{11}$$

We now collocate Equation (11) at x_{n+i}, $i = 0,1,2$ and interpolate (2) at $x = x_n$, we get a system of four equations with $c_i (i = 0,1,2)$ and parameter λ

$$\begin{aligned} c_0 + c_1 x_n + c_2 x_n^2 &= y_n, \\ c_1 + 2x_n c_2 &= f_n + \lambda, \\ c_1 + 2x_{n+1} c_2 &= f_{n+1} - \frac{1}{2}\lambda, \\ c_1 + 2x_{n+2} c_2 &= f_{n+2} + \lambda. \end{aligned} \tag{12}$$

Using a suitable method to solve the above system to obtain

$$\lambda = \frac{1}{3}\left(2f_{n+1} - f_n - f_{n+2}\right),$$

$$c_2 = \frac{1}{4h}\left(f_{n+2} - f_n\right),$$

$$c_1 = \frac{1}{3}\left(2f_n + 2f_{n+1} - f_{n+2}\right) + \frac{1}{2h}x_n\left(f_n - f_{n+2}\right),$$

$$c_0 = y_n - \frac{1}{3}x_n\left(2f_n + 2f_{n+1} - f_{n+2}\right) + \frac{1}{4h}x_n^2\left(f_{n+2} - f_n\right).$$

From (2), we have

$$\overline{y}(x) = c_0 + c_1 x + c_2 x^2. \tag{13}$$

Now, the required numerical scheme of the proposed method will be obtained if we collocate the above Equation (13) at $x = x_{n+1}$ and substitute c_0, c_1, c_2, λ as follows

$$y_{n+1} = y_n + \frac{h}{12}\left(5f_n + 8f_{n+1} + 11f_{n+2}\right). \tag{14}$$

Case 3: If $K = 3$

For case $K = 3$, we collocate the continuous scheme

$$\overline{y}(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3, \tag{15}$$

at grid and off-grid points $x = x_n$, $x = x_{n+1}$, $x = x_{n+2}$ and $x = x_{n+3}$ and this gives the required block schemes

$$y_{n+1} = y_n + \frac{h}{98}\left(39f_n + 107f_{n+1} + 54f_{n+2} - 7f_{n+3}\right). \tag{16}$$

3. Stability and Convergence Analysis of the Block Schemes

In this section, we present a summary on the order, the error constant and the convergence of the proposed block schemes. This summary in given in the following table.

Step	$K = 1$	$K = 2$	$K = 3$
Order	2	2	4
Error Constant	$-\dfrac{1}{12}$	$-\dfrac{1}{12}$	$\dfrac{1}{96}$
Convergence	convergent	convergent	convergent

4. Numerical Examples

In this section, we implement the proposed method with $K = 2$ and $K = 3$ to solve two first order initial value problems, and compare the obtained numerical results with those obtained from using the fourth-order Runge-Kutta method (RK4).

Example 1.

Consider the following IVP

$$y'(x) = -y(x), \quad h = 0.1, \quad y(0) = 1.$$

With the exact solution $y(x) = e^{-x}$.

The numerical results of this example are presented in **Table 1** and **Table 2** with the cases $K = 2$ and $K = 3$, respectively. In these tables, we presented a comparison the obtained numerical results using the proposed method with the exact solution and those numerical results obtained from using RK4.

Example 2.

Consider the following IVP

$$y'(x) = x(1-y), \quad h = 0.1, \quad y(0) = 0$$

With the exact solution $y(x) = 1 - e^{-x^2/2}$.

The numerical results of this example are presented in **Figure 1** and **Figure 2** with the cases $K = 2$ and $K = 3$, respectively. In these figures, we presented a comparison the obtained numerical results using the proposed method with the exact solution and those numerical results obtained from using RK4.

Table 1. A comparison the proposed method at $K = 2$ with the exact solution and RK4: Example 1.

x	Block Scheme $K = 2$	Exact Solution	RK4
0.0	1.000000	1.000000	1.000000
0.2	0.818712	0.818730	0.818781
0.4	0.670345	0.670320	0.670541
0.6	0.548848	0.548811	0.548848
0.8	0.449215	0.449328	0.449345
1.0	0.367822	0.367879	0.367836

Table 2. A comparison the proposed method at $K = 3$ with the exact solution and RK4: Example 1.

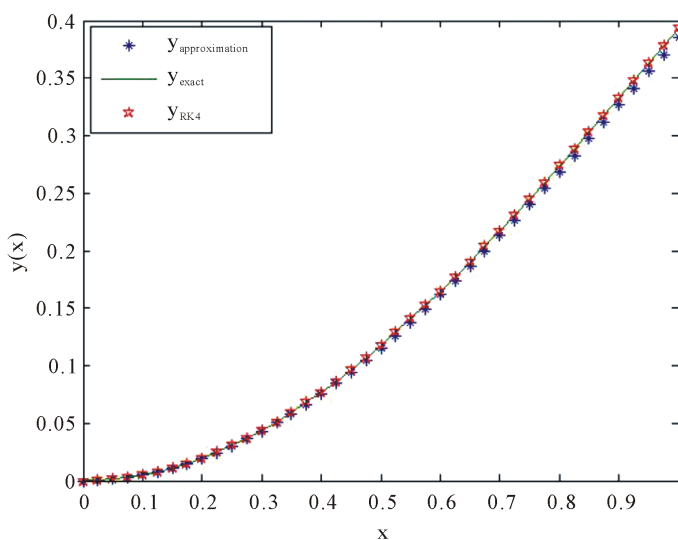| x	Block Scheme $K = 3$	Exact Solution	RK4
0.0	1.000000	1.000000	1.000000
0.2	0.818735	0.818730	0.818738
0.4	0.670328	0.670320	0.670321
0.6	0.548817	0.548811	0.548813
0.8	0.449322	0.449328	0.449325
1.0	0.367871	0.367879	0.367873

Figure 1. A comparison the proposed method at $K = 2$ with the exact solution and RK4: Example 2.
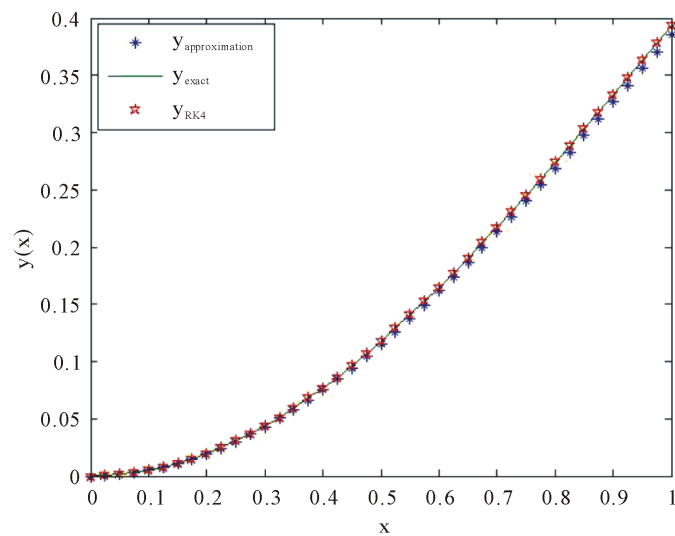
Figure 2. A comparison the proposed method at $K = 3$ with the exact solution and RK4: Example 2.

5. Summary and Conclusions

In this paper, we presented three new block-schemes ($K = 1$, $K = 2$ and $K = 3$) that are convergent and absolutely stable. We used the proposed method to solve numerically a wide-range of linear initial value problems. The results of the presented examples show that our method was capable for solving such problems of IVPs and generates the convergence analysis, and closed to their exact solutions. This method is very simple and effective for a wide-range of ODEs. All computations are made by Matlab.

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