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New Oscillation Criteria of Second-Order Nonlinear Delay Dynamic Equations on Time Scales

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Abstract

By using the generalized Riccati transformation and the integral averaging technique, the paper establishes some new oscillation criteria for the second-order nonlinear delay dynamic equations on time scales. The results in this paper unify the oscillation of the second-order nonlinear delay differential equation and the second-order nonlinear delay difference equation on time scales. The Theorems in this paper are new even in the continuous and the discrete cases.

Keywords

Oscillation Criterion, Dynamic Equations, Time Scale

1. Introduction

According to the important academic value and application background in Quantum Physics (especially in Nuclear Physics), engineering mechanics and control theory, the oscillation theory of dynamic equations on time scales has become one of the research hotspots. The paper will deal with the oscillatory behavior of all solutions of second-order nonlinear delay dynamic equation

$$\left(a(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+q(t)f\left(x\left(\tau(t)\right)\right)=0,\quad t\in\mathbb{T},\quad t\geq t_0 \tag{1}$$

In order to obtain the main results, we give the following hypotheses:

(H_1) $\mathbb{T}$ is a time scale (*i.e.*, a nonempty closed subset of the real numbers $\mathbb{R}$) which is unbounded above, and $t_0\in\mathbb{T}$ with $t_0>0$. We define the time scale interval of the form $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}}=[t_0,\infty)\cap\mathbb{T}$.

(H_2) $\gamma\geq 1$ is the ratio of two positive odd integers.

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(H_3) a, q are positive real-valued right-dense continuous functions on an arbitrary time scale $\mathbb{T}$.

(H_4) $\tau \in C_{rd}^1\left(\left[t_0,\infty\right)_{\mathbb{T}},\mathbb{T}\right)$ is a strictly increasing function such that $\tau(t)\le t$ and $\tau(t)\to\infty$ as $t\to\infty$ and $\tilde{\mathbb{T}} := \tau(\mathbb{T}) \subset \mathbb{T}$.

(H_5) $f \in C(\mathbb{R},\mathbb{R})$ is a continuous function, for some positive constant L which satisfies

$$\frac{f(x)}{x} \ge L \quad \text{for all } x \ne 0$$

According to the solution of (1), we mean a nontrivial real-valued function x satisfying (1) for $t \in \mathbb{T}$. We recall that a solution x of Equation (1) is said to be oscillatory on $\left[t_0,\infty\right)_{\mathbb{T}}$ in case it is neither eventually positive nor eventually negative; otherwise, the solution is said to be nonoscillatory. Equation (1) is said to be oscillatory in case all of its solutions are oscillatory. Our attention is restricted on those solutions of (1) which are not eventually identically zero. Since $a(t) > 0$, we shall consider both the cases

$$\int_{t_0}^{\infty}\left(\frac{1}{a(t)}\right)^{\frac{1}{\gamma}}\Delta t = \infty \tag{2}$$

and

$$\int_{t_0}^{\infty}\left(\frac{1}{a(t)}\right)^{\frac{1}{\gamma}}\Delta t < \infty \tag{3}$$

It is easy to see that (1) can be transformed into a second-order nonlinear delay dynamic equation

$$\left(a(t)x^{\Delta}(t)\right)^{\Delta} + q(t)f\left(x\left(\tau(t)\right)\right) = 0,\quad t\in\mathbb{T},\quad t\ge t_0 \tag{4}$$

where $\gamma = 1$. In (1), if $f(x) = x^{\gamma}$, $\tau(t) = t$, then (1) is simplified to an equation

$$\left(a(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + q(t)x^{\gamma}(t) = 0,\quad t\in\mathbb{T},\quad t\ge t_0 \tag{5}$$

In (4), if $a(t) = 1$, then (4) is simplified to an equation

$$x^{\Delta\Delta}(t) + q(t)f\left(x\left(\tau(t)\right)\right) = 0,\quad t\in\mathbb{T},\quad t\ge t_0 \tag{6}$$

In (6), if $f(x) = x$, then (6) is simplified to an equation

$$x^{\Delta\Delta}(t) + q(t)x\left(\tau(t)\right) = 0,\quad t\in\mathbb{T},\quad t\ge t_0 \tag{7}$$

After the careful consideration of the linear delay dynamic equations by Agarwal, Bohner and Saker in 2005 [1] (7) and the nonlinear delay dynamic equations by Sahiner [2] (6), some sufficient conditions for oscillation of (7) and (6) have been established. In 2007, Erbe, Peterson and Saker [3] considered the general nonlinear delay dynamic equations (4) and obtained some new oscillation criteria, which improved the results given by Sahiner [2]. Saker [4] in 2005 and Grace, Bohner and Agarwal [5] in 2009 considered the half-linear dynamic equations (5), and established some sufficient conditions for oscillation of (5). For other related results, we recommend the references [6]-[10]. On the basis of these, by using the generalized Riccati transformation and integral averaging technique, we continue to discuss the oscillation of solutions of (1) and obtain some new oscillatory criteria of Philos-type for (1).

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Since we are interested in oscillatory behavior, we suppose that the time scale under consideration is not bounded above, *i.e.*, sup $\mathbb{T} = \infty$. On any time scale we define the forward and the backward jump operators by

$$\sigma(t) = \inf\{s\in\mathbb{T}: s>t\} \text{ and } \rho(t) = \sup\{s\in\mathbb{T}: s<t\}$$

A point $t\in\mathbb{T}$ is said to be left-dense if $\rho(t) = t$, right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. The graininess μ of the time scale is defined by $\mu(t) = \sigma(t) - t$. A function

$f:\mathbb{T}\to\mathbb{R}$ is said to be rd-continuous if it is continuous at each right-dense point and if there exists a finite left limit at all left-dense points.

Throughout this paper, we will make use of the following product and quotient rules for the derivative of the product *fg* and the quotient f/g of two differentiable functions *f* and *g*

$$(fg)^{\Delta}(t)=f^{\Delta}(t)g(t)+f(\sigma(t))g^{\Delta}(t)=f(t)g^{\Delta}(t)+f^{\Delta}(t)g(\sigma(t)) \tag{8}$$

$$\left(\frac{f}{g}\right)^{\Delta}(t)=\frac{f^{\Delta}(t)g(t)-f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))} \tag{9}$$

For $b,\ c\in\mathbb{T}$ and a differentiable function *f*, the Cauchy integral of f^{Δ} is defined by

$$\int_b^c f^{\Delta}(t)\Delta t=f(c)-(b)$$

The integration by parts formula reads

$$\int_b^c f^{\Delta}(t)g(t)\Delta t=f(c)g(c)-f(b)g(b)-\int_b^c f^{\sigma}(t)g^{\Delta}(t)\Delta t \tag{10}$$

and infinite integrals are defined by

$$\int_b^{\infty} f(s)\Delta s=\lim_{t\to\infty}\int_b^t f(s)\Delta s$$

For more details, see [11] [12].

2. Main Results

In order to obtain the main results, the following lemmas are first introduced.

Lemma 1 (*Han et al.* [[10], *Lemma* 2.2]) *Assume that* $\tau:\mathbb{T}\to\mathbb{R}$ *is strictly increasing and* $\tilde{\mathbb{T}}:=\tau(\mathbb{T})\subset\mathbb{T}$ *is a time scale,* $\tau(\sigma(t))=\sigma(\tau(t))$. *Let* $x:\tilde{\mathbb{T}}\to\mathbb{R}$. *If* $\tau^{\Delta}(t)$, *and let* $x^{\Delta}(\tau(t))$ *exist for* $t\in\mathbb{T}^k$, *then* $(x(\tau(t)))^{\Delta}$ *exist, and*

$$(x(\tau(t)))^{\Delta}=x^{\Delta}(\tau(t))\tau^{\Delta}(t) \tag{11}$$

Lemma 2 (*Bohner et al.* [[11], *Theorem* 1.90]) *Assume that* $x(t)$ *is* Δ*-differentiable and eventually positive or eventually negative, then*

$$\left((x(t))^{\gamma}\right)^{\Delta}=\gamma\int_0^1\left[hx(\sigma(t))+(1-h)x(t)\right]^{\gamma-1}x^{\Delta}(t)\mathrm{d}h \tag{12}$$

Lemma 3 (*Sun et al.* [[13], *Lemma* 2.1]) *Assume that the conditions* (H_1)-(H_5) *and* (2) *hold, and let* $x(t)$ *be an eventually position solution of* (1), *then there exists* $t_1\in[t_0,\infty)_{\mathbb{T}}$ *such that*

$$x^{\Delta}(t)>0,\quad \left(a(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}<0,\quad t\in[t_1,\infty)_{\mathbb{T}} \tag{13}$$

Next, we will provide a new sufficient condition for oscillation of all solutions of (1), which can be considered as the extension of the result of Philos [14] for oscillation of second-order differential equations.

Theorem 1 *Assume that the conditions* (H_1) - (H_5), (2) *hold and* $\tau(\sigma(t))=\sigma(\tau(t))$. *Let* $H:D_{\mathbb{T}}\equiv\{(t,s):t\ge s\ge t_0,t,s\in[t_0,\infty)_{\mathbb{T}}\}\to\mathbb{R}$ *be a rd-continuous function such that*

$$H(t,t)=0\ \text{for}\ t\ge t_0,\ H(t,s)>0\ \text{for}\ t>s\ge t_0,\ t,s\in[t_0,\infty)_{\mathbb{T}}$$

and H has a non-positive continuous Δ*-partial derivative* $H^{\Delta_s}(t,s)$ *with respect to the second variable. Furthermore, let* $h:D_{\mathbb{T}}\to\mathbb{R}$ *be a rd-continuous function, and satisfies*

$$-H^{\Delta_s}(t,s)=h(t,s)\sqrt{H(t,s)}\ \text{for all}\ (t,s)\in D_{\mathbb{T}}$$

Assume that there exists a positive nondecreasing Δ*-differentiable function* $\delta\in C_{rd}^1([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that for* *every positive constant M,*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[LH(t,s)\delta(s)q(s)-\frac{(a(\tau(s)))^{\frac{1}{\gamma}}}{4(M)^{\frac{\gamma-1}{\gamma}}\delta(s)\tau^{\Delta}(s)}\left(\delta^{\Delta}(s)\sqrt{H(t,s)}-\delta(s)h(t,s)\right)^{2}\right]\Delta s=\infty \quad (14)$$

for $t>s\geq t_0$, $t,\ s\in[t_0,\infty)_{\mathbb{T}}$. *Then* (1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1) on $[t_0,\infty)_{\mathbb{T}}$. Without loss of generality, we assume that $x(t)>0$ and $x(\tau(t))>0$ for all $t\in[t_1,\infty)_{\mathbb{T}}$, $t_1\in[t_0,\infty)_{\mathbb{T}}$, and we shall only consider this case. When $x(t)$ is eventually negative, the proof is similar. By Lemma 3, we have (23). Define the function $W(t)$ by

$$W(t)=\delta(t)\frac{a(t)\left(x^{\Delta}(t)\right)^{\gamma}}{x(\tau(t))},\quad t\in[t_1,\infty)_{\mathbb{T}} \quad (15)$$

Then on $[t_1,\infty)_{\mathbb{T}}$, we have $W(t)>0$, and by (8)-(9), we obtain

$$W^{\Delta}(t)=\frac{\delta(t)}{x(\tau(t))}\left(a(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+a(\sigma(t))\left(x^{\Delta}(\sigma(t))\right)^{\gamma}\frac{x(\tau(t))\delta^{\Delta}(t)-\delta(t)\left(x(\tau(t))\right)^{\Delta}}{x(\tau(t))x(\tau(\sigma(t)))}$$

$t\in[t_1,\infty)_{\mathbb{T}}$. Based on (1) and (15), we can obtain

$$W^{\Delta}(t)\leq -Lq(t)\delta(t)+\frac{\delta^{\Delta}(t)}{\delta(\sigma(t))}W(\sigma(t))-\frac{\delta(t)a(\sigma(t))\left(x^{\Delta}(\sigma(t))\right)^{\gamma}\left(x(\tau(t))\right)^{\Delta}}{x(\tau(t))x(\tau(\sigma(t)))}$$

by using (11), we have $\left(x(\tau(t))\right)^{\Delta}=x^{\Delta}(\tau(t))\tau^{\Delta}(t)$ thus

$$W^{\Delta}(t)\leq -Lq(t)\delta(t)+\frac{\delta^{\Delta}(t)}{\delta(\sigma(t))}W(\sigma(t))-\frac{\delta(t)a(\sigma(t))\left(x^{\Delta}(\sigma(t))\right)^{\gamma}x^{\Delta}(\tau(t))\tau^{\Delta}(t)}{x(\tau(t))x(\tau(\sigma(t)))} \quad (16)$$

By $\left(a(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}<0$ and $x^{\Delta}(t)>0$, we have

$$a(\tau(t))\left(x^{\Delta}(\tau(t))\right)^{\gamma}\geq a(\sigma(t))\left(x^{\Delta}(\sigma(t))\right)^{\gamma}\ \text{and}\ x(\tau(t))\leq x(\tau(\sigma(t))) \quad (17)$$

Substituting (17) in (16), we obtain

$$\begin{aligned}W^{\Delta}(t)&\leq -Lq(t)\delta(t)+\frac{\delta^{\Delta}(t)}{\delta(\sigma(t))}W(\sigma(t))-\left(\frac{a(\sigma(t))}{a(\tau(t))}\right)^{\frac{1}{\gamma}}\frac{\delta(t)\tau^{\Delta}(t)a(\sigma(t))\left(x^{\Delta}(\sigma(t))\right)^{\gamma+1}}{\left(x(\tau(\sigma(t)))\right)^{2}}\\&=-Lq(t)\delta(t)+\frac{\delta^{\Delta}(t)}{\delta(\sigma(t))}W(\sigma(t))-\left(\frac{a(\sigma(t))}{a(\tau(t))}\right)^{\frac{1}{\gamma}}\frac{\delta(t)\tau^{\Delta}(t)}{a(\sigma(t))\left(\delta(\sigma(t))\right)^{2}}\left(W(\sigma(t))\right)^{2}\frac{1}{\left(x^{\Delta}(\sigma(t))\right)^{\gamma-1}}.\end{aligned} \quad (18)$$

$t\in[t_1,\infty)_{\mathbb{T}}$. Now, due to the fact that $a(t)\left(x^{\Delta}(t)\right)^{\gamma}$ is positive and nonincreasing, there exists an $T\in[t_1,\infty)_{\mathbb{T}}$ sufficiently large such that $a(t)\left(x^{\Delta}(t)\right)^{\gamma}\leq\frac{1}{M}$ for some positive constant M and $t\in[T,\infty)_{\mathbb{T}}$, and we have $a(\sigma(t))\left(x^{\Delta}(\sigma(t))\right)^{\gamma}\leq\frac{1}{M}$, so that

$$\frac{1}{\left(x^{\Delta}\left(\sigma(t)\right)\right)^{\gamma-1}} \geq \left(Ma\left(\sigma(t)\right)\right)^{\frac{\gamma-1}{\gamma}} \tag{19}$$

Substituting (19) into (18), we obtain

$$\begin{aligned} W^{\Delta}(t) &\leq -Lq(t)\delta(t)+\frac{\delta^{\Delta}(t)}{\delta(\sigma(t))}W(\sigma(t))-(M)^{\frac{\gamma-1}{\gamma}}\frac{\delta(t)\tau^{\Delta}(t)}{\left(a(\tau(t))\right)^{\frac{1}{\gamma}}\left(\delta(\sigma(t))\right)^{2}}\left(W(\sigma(t))\right)^{2} \\ &= -Lq(t)\delta(t)+\frac{\delta^{\Delta}(t)}{\delta(\sigma(t))}W(\sigma(t))-\bar{\delta}(t)\frac{\left(W(\sigma(t))\right)^{2}}{\left(\delta(\sigma(t))\right)^{2}}. \end{aligned} \tag{20}$$

where $\bar{\delta}(t)=(M)^{\frac{\gamma-1}{\gamma}}\frac{\delta(t)\tau^{\Delta}(t)}{\left(a(\tau(t))\right)^{\frac{1}{\gamma}}}$. Thus, for every $t,\ T\in[t_1,\infty)_{\mathbb{T}}$ with $t\geq T\geq t_1$, by (10), we obtain

$$\begin{aligned} &\int_T^t LH(t,s)\delta(s)q(s)\Delta s \\ &\leq H(t,T)W(T)-\int_T^t\left(-H^{\Delta_s}(t,s)\right)W(\sigma(s))\Delta s+\int_T^t H(t,s)\frac{\delta^{\Delta}(s)}{\delta(\sigma(s))}W(\sigma(s))\Delta s \\ &\quad -\int_T^t H(t,s)\bar{\delta}(s)\frac{\left(W(\sigma(s))\right)^{2}}{\left(\delta(\sigma(s))\right)^{2}}\Delta s \\ &= H(t,T)W(T)+\int_T^t\frac{\delta^{\Delta}(s)\sqrt{H(t,s)}-\delta(s)h(t,s)\sqrt{H(t,s)}}{\delta(\sigma(s))}W(\sigma(s))\Delta s-\int_T^t H(t,s)\bar{\delta}(s)\frac{\left(W(\sigma(s))\right)^{2}}{\left(\delta(\sigma(s))\right)^{2}}\Delta s \\ &\leq H(t,T)W(T)+\int_T^t\frac{\delta^{\Delta}(s)\sqrt{H(t,s)}-\delta(s)h(t,s)}{\delta(\sigma(s))}\sqrt{H(t,s)}W(\sigma(s))\Delta s-\int_T^t H(t,s)\bar{\delta}(s)\frac{\left(W(\sigma(s))\right)^{2}}{\left(\delta(\sigma(s))\right)^{2}}\Delta s \\ &= H(t,T)W(T)-\int_T^t\left[\sqrt{\bar{\delta}(s)}\frac{\sqrt{H(t,s)}W(\sigma(s))}{\delta(\sigma(s))}-\frac{\delta^{\Delta}(s)\sqrt{H(t,s)}-\delta(s)h(t,s)}{2\sqrt{\bar{\delta}(s)}}\right]^{2}\Delta s \\ &\quad +\int_T^t\frac{\left(a(\tau(s))\right)^{\frac{1}{\gamma}}}{4(M)^{\frac{\gamma-1}{\gamma}}\delta(s)\tau^{\Delta}(s)}\left[\delta^{\Delta}(s)\sqrt{H(t,s)}-\delta(s)h(t,s)\right]^{2}\Delta s \\ &\leq H(t,T)W(T)+\int_T^t\frac{\left(a(\tau(s))\right)^{\frac{1}{\gamma}}}{4(M)^{\frac{\gamma-1}{\gamma}}\delta(s)\tau^{\Delta}(s)}\left[\delta^{\Delta}(s)\sqrt{H(t,s)}-\delta(s)h(t,s)\right]^{2}\Delta s. \end{aligned} \tag{21}$$

By (21), we obtain

$$\begin{aligned} &\int_T^t\left[LH(t,s)\delta(s)q(s)-\frac{\left(a(\tau(s))\right)^{\frac{1}{\gamma}}}{4(M)^{\frac{\gamma-1}{\gamma}}\delta(s)\tau^{\Delta}(s)}\left(\delta^{\Delta}(s)\sqrt{H(t,s)}-\delta(s)h(t,s)\right)^{2}\right]\Delta s \\ &\leq H(t,T)W(T)\leq H(t,t_0)W(T). \end{aligned}$$

From the above inequality, denoting $T=T_0$, we obtain

$$\int_{t_0}^{t}\left[LH(t,s)\delta(s)q(s)-\frac{\left(a(\tau(s))\right)^{\frac{1}{\gamma}}}{4(M)^{\frac{\gamma-1}{\gamma}}\delta(s)\tau^{\Delta}(s)}\left(\delta^{\Delta}(s)\sqrt{H(t,s)}-\delta(s)h(t,s)\right)^{2}\right]\Delta s$$

$$=\left\{\int_{t_0}^{T_0}+\int_{t_0}^{t}\right\}\left[LH(t,s)\delta(s)q(s)-\frac{\left(a(\tau(s))\right)^{\frac{1}{\gamma}}}{4(M)^{\frac{\gamma-1}{\gamma}}\delta(s)\tau^{\Delta}(s)}\left(\delta^{\Delta}(s)\sqrt{H(t,s)}-\delta(s)h(t,s)\right)^{2}\right]\Delta s$$

$$\leq H(t,t_0)\left\{\int_{t_0}^{T_0}L\delta(s)q(s)\Delta s+W(T_0)\right\}.$$

The above inequality implies that

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[LH(t,s)\delta(s)q(s)-\frac{\left(a(\tau(s))\right)^{\frac{1}{\gamma}}}{4(M)^{\frac{\gamma-1}{\gamma}}\delta(s)\tau^{\Delta}(s)}\left(\delta^{\Delta}(s)\sqrt{H(t,s)}-\delta(s)h(t,s)\right)^{2}\right]\Delta s$$

$$\leq\int_{t_0}^{T_0}L\delta(s)q(s)\Delta s+W(T_0).$$

So we have a contradiction to the condition (14). This completes the proof.

Remark 1 *From Theorem* 1, *we can obtain different conditions for oscillation of all solutions of* (1) *with different choices of* $\delta(t)$ *and* $H(t,s)$. *For example,* $H(t,s)=(t-s)^m$ *or* $H(t,s)=\left(\ln\frac{t+1}{s+1}\right)^m$.

Now, let us consider the function $H(t,s)$ defined by

$$H(t,s)=(t-s)^m,\ m\geq 1,\ t\geq s\geq t_0,\ t,s\in[t_0,\infty)_{\mathbb{T}}.$$

Then $H(t,t)=0$ for $t\geq t_0$, and $H(t,s)>0$, $H^{\Delta_s}(t,s)\leq 0$ for $t>s\geq t_0$, $t,s\in[t_0,\infty)_{\mathbb{T}}$. Furthermore, the function *h* with $h(t,s)=m(t-s)^{\frac{m-2}{2}}$ for $t>s\geq t_0$, $t,s\in[t_0,\infty)_{\mathbb{T}}$. Hence we have the following results.

Corollary 1 *Assume that the conditions* (H_1) - (H_5), (2) *hold and* $\tau(\sigma(t))=\sigma(\tau(t))$. *Furthermore, assume that there exists a positive nondecreasing* Δ*-differentiable function* $\delta\in C_{rd}^{1}\left([t_0,\infty)_{\mathbb{T}},\mathbb{R}\right)$ *such that for every positive constant M and* $m\geq 1$,

$$\limsup_{t\to\infty}\frac{1}{t^m}\int_{t_0}^{t}(t-s)^m\left[L\delta(s)q(s)-\frac{\left(a(\tau(s))\right)^{\frac{1}{\gamma}}}{4(M)^{\frac{\gamma-1}{\gamma}}\delta(s)\tau^{\Delta}(s)}\left(\delta^{\Delta}(s)-m\frac{\delta(s)}{t-s}\right)^{2}\right]\Delta s=\infty \tag{22}$$

for $t>s\geq t_0$, $t,s\in[t_0,\infty)_{\mathbb{T}}$. *Then* (1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Now, when (3) holds, we give the oscillatory criteria of Philos-type for (1).

Theorem 2 *Assume that the conditions* (H_1) - (H_5), (3) *hold and* $\tau(\sigma(t))=\sigma(\tau(t))$, *and let H, h and* δ *be defined as in Theorem* 1 *and the condition* (14) *holds. Furthermore, assume that for every* $t_1\in[t_0,\infty)_{\mathbb{T}}$,

$$\int_{t_1}^{\infty}\left[\frac{1}{a(s)}\int_{t_1}^{s}\theta(u)q(u)\Delta u\right]^{\frac{1}{\gamma}}\Delta s=\infty \tag{23}$$

where

$$\theta(t)=\int_{t}^{\infty}\left(\frac{1}{a(s)}\right)^{\frac{1}{\gamma}}\Delta s$$

Then (1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1) on $[t_0,\infty)_{\mathbb{T}}$. Without loss of generality, we assume that $x(t)>0$ and $x(\tau(t))>0$ for all $t\in[t_1,\infty)_{\mathbb{T}}$, $t_1\in[t_0,\infty)_{\mathbb{T}}$, and we shall only consider this case. When $x(t)$ is eventually negative, the proof is similar. Since $a(t)\left(x^{\Delta}(t)\right)^{\gamma}$ is decreasing, it is eventually of one sign and hence $x^{\Delta}(t)$ is eventually of one sign. Thus, we shall distinguish the following two cases:

(1) $x^{\Delta}(t)>0$ for $t\geq t_1$; and

(2) $x^{\Delta}(t)<0$ for $t\geq t_1$.

Case (1). The proof when $x^{\Delta}(t)$ is an eventually positive is similar to that of the proof of Theorem 1 and it hence is omitted.

Case (2). For $s\geq t\geq t_1$, we have

$$a(s)\left(-x^{\Delta}(s)\right)^{\gamma}\geq a(t)\left(-x^{\Delta}(t)\right)^{\gamma}$$

and hence

$$-x^{\Delta}(s)\geq\left(\frac{a(t)}{a(s)}\right)^{\frac{1}{\gamma}}\left(-x^{\Delta}(t)\right) \tag{24}$$

Integrating (24) from $t\geq t_1$ to $u\geq t$ and letting $u\to\infty$ yields

$$x(t)\geq\left[\int_t^{\infty}\left(\frac{1}{a(s)}\right)^{\frac{1}{\gamma}}\Delta s\right]\left(a(t)\right)^{\frac{1}{\gamma}}\left(-x^{\Delta}(t)\right)=-\theta(t)a^{\frac{1}{\gamma}}(t)x^{\Delta}(t)\ \text{ for } t\in[t_1,\infty)_{\mathbb{T}}$$

and thus

$$\left(x(t)\right)^{\gamma}\geq-\left(\theta(t)\right)^{\gamma}a(t)\left(x^{\Delta}(t)\right)^{\gamma}\geq-\left(\theta(t)\right)^{\gamma}a(t_1)\left(x^{\Delta}(t_1)\right)^{\gamma}=b^{\gamma}\left(\theta(t)\right)^{\gamma}\ \text{ for } t\in[t_1,\infty)_{\mathbb{T}} \tag{25}$$

where $b=-a^{\frac{1}{\gamma}}(t_1)x^{\Delta}(t_1)>0$. Using (25) in Equation (1), we find

$$-\left(a(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}\geq Lq(t)x(\tau(t))\geq Lq(t)x(t)\geq bL\theta(t)q(t)\ \text{ for } t\in[t_1,\infty)_{\mathbb{T}} \tag{26}$$

Integrating (26) from t_1 to t, we have

$$-a(t)\left(x^{\Delta}(t)\right)^{\gamma}\geq-a(t_1)\left(x^{\Delta}(t_1)\right)^{\gamma}+bL\int_{t_1}^{t}\theta(s)q(s)\Delta s\geq bL\int_{t_1}^{t}\theta(s)q(s)\Delta s$$

so that

$$-x^{\Delta}(t)\geq\left[\frac{bL}{a(t)}\int_{t_1}^{t}\theta(s)q(s)\Delta s\right]^{\frac{1}{\gamma}} \tag{27}$$

Integrating (27) from t_1 to t, we obtain

$$\infty>x(t_1)\geq-x(t)+x(t_1)\geq\int_{t_1}^{t}\left[\frac{bL}{a(s)}\int_{t_1}^{s}\theta(u)q(u)\Delta u\right]^{\frac{1}{\gamma}}\Delta s\to\infty\ \text{ as } t\to\infty$$

by (23), which is a contradiction. This completes the proof. □

Remark 2 *In the past, the usual result is that the condition* (3) *was established, then every solution of the Equation* (1) *is either oscillatory or converges to zero. But now Theorem* 2 *in our paper prove that if the condition* (3) *is satisfied, every solution of the Equation* (1) *is oscillatory.*

Similar to the Corollary 1, by applying Theorem 2 with

$$H(t-s)=(t-s)^{m},\ m\geq1,\ t\geq s\geq t_0,\ t,s\in[t_0,\infty)_{\mathbb{T}}$$

we have the following results.

Corollary 2 *Assume that the conditions* (H_1) - (H_5), (3), (22), (23) *hold and* $\tau(\sigma(t))=\sigma(\tau(t))$, *then* (1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Next, we give a result of a succinctness and convenient to application.

Theorem 3 *Assume that the conditions* (H_1) - (H_5), (2) *hold and* $\tau(\sigma(t)) = \sigma(\tau(t))$, *and let* $H : D_{\mathbb{T}} \equiv \{(t,s) : t \geq s \geq t_0, t, s \in [t_0,\infty)_{\mathbb{T}}\} \to \mathbb{R}$ *be a rd-continuous function such that*

$$H(t,t) = 0 \ \ \text{for} \ \ t \geq t_0, \ H(t,s) > 0 \ \ \text{for} \ \ t > s \geq t_0, \ \ t, s \in [t_0,\infty)_{\mathbb{T}}$$

and H has a non-positive continuous Δ*-partial derivative* $H^{\Delta_s}(t,s)$ *with respect to the second variable. Furthermore, assume that there exists a positive* Δ*-differentiable function* $\delta \in C^1_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ *such that for every positive constant M,*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s) \left[Lq(s)\delta(s) - \frac{(a(\tau(s)))^{\frac{1}{\gamma}} (\delta^{\Delta}(s))^2}{4(M)^{\frac{\gamma-1}{\gamma}} \delta(s)\tau^{\Delta}(s)} \right] \Delta s = \infty \tag{28}$$

for $t > s \geq t_0$, $t, s \in [t_0,\infty)_{\mathbb{T}}$. *Then* (1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1) on $[t_0,\infty)_{\mathbb{T}}$. Without loss of generality, assume that $x(t) > 0$ and $x(\tau(t)) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$, $t_1 \in [t_0,\infty)_{\mathbb{T}}$, which we shall only consider this case. When $x(t)$ is eventually negative, the proof is similar. Proceeding as in the proof of Theorem 1, we obtain (20), thus

$$\begin{aligned} W^{\Delta}(t) &\leq -Lq(t)\delta(t) + \frac{\delta^{\Delta}(t)}{\delta(\sigma(t))} W(\sigma(t)) - (M)^{\frac{\gamma-1}{\gamma}} \frac{\delta(t)\tau^{\Delta}(t)}{(a(\tau(t)))^{\frac{1}{\gamma}} (\delta(\sigma(t)))^2} (W(\sigma(t)))^2 \\ &= -Lq(t)\delta(t) + \frac{(a(\tau(t)))^{\frac{1}{\gamma}} (\delta^{\Delta}(t))^2}{4(M)^{\frac{\gamma-1}{\gamma}} \delta(t)\tau^{\Delta}(t)} - \left[\frac{\sqrt{(M)^{\frac{\gamma-1}{\gamma}} \delta(t)\tau^{\Delta}(t)}}{\delta(\sigma(t))\sqrt{(a(\tau(t)))^{\frac{1}{\gamma}}}} W(\sigma(t)) - \frac{\sqrt{(a(\tau(t)))^{\frac{1}{\gamma}}}\, \delta^{\Delta}(t)}{2\sqrt{(M)^{\frac{\gamma-1}{\gamma}} \delta(t)\tau^{\Delta}(t)}} \right]^2 \\ &\leq -\left[Lq(t)\delta(t) - \frac{(a(\tau(t)))^{\frac{1}{\gamma}} (\delta^{\Delta}(t))^2}{4(M)^{\frac{\gamma-1}{\gamma}} \delta(t)\tau^{\Delta}(t)} \right], \end{aligned} \tag{29}$$

for all $t \in [t_1,\infty)_{\mathbb{T}}$. Then from (29), we have

$$Lq(s)\delta(s) - \frac{(a(\tau(s)))^{\frac{1}{\gamma}} (\delta^{\Delta}(s))^2}{4(M)^{\frac{\gamma-1}{\gamma}} \delta(s)\tau^{\Delta}(s)} \leq -W^{\Delta}(s)$$

for all $s \in [t_1,\infty)_{\mathbb{T}}$, and therefore, for all $t > s \geq t_1$,

$$\begin{aligned} &\int_{t_1}^{t} H(t,s) \left[Lq(s)\delta(s) - \frac{(a(\tau(s)))^{\frac{1}{\gamma}} (\delta^{\Delta}(s))^2}{4(M)^{\frac{\gamma-1}{\gamma}} \delta(s)\tau^{\Delta}(s)} \right] \Delta s \leq -\int_{t_1}^{t} H(t,s) W^{\Delta}(s) \Delta s \\ &\overset{(20)}{=} -H(t,s)W(s)\Big|_{t_1}^{t} + \int_{t_1}^{t} H^{\Delta_s}(t,s) W(\delta(s)) \Delta s \leq H(t,t_1)W(t_1) \leq H(t,t_0)W(t_1), \end{aligned}$$

and hence, for all $t > s \geq t_1$,

$$\begin{aligned} \int_{t_0}^{t} H(t,s) \left[Lq(s)\delta(s) - \frac{(a(\tau(s)))^{\frac{1}{\gamma}} (\delta^{\Delta}(s))^2}{4(M)^{\frac{\gamma-1}{\gamma}} \delta(s)\tau^{\Delta}(s)} \right] \Delta s &= \left\{ \int_{t_0}^{t_1} + \int_{t_1}^{t} \right\} \left[H(t,s) \left(Lq(s)\delta(s) - \frac{(a(\tau(s)))^{\frac{1}{\gamma}} (\delta^{\Delta}(s))^2}{4(M)^{\frac{\gamma-1}{\gamma}} \delta(s)\tau^{\Delta}(s)} \right) \right] \Delta s \\ &\leq H(t,t_0) \left\{ \int_{t_0}^{t_1} Lq(s)\delta(s)\Delta s + W(t_1) \right\}. \end{aligned}$$

Thus

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\left[Lq(s)\delta(s)-\frac{\left(a(\tau(s))\right)^{\frac{1}{\gamma}}\left(\delta^{\Delta}(s)\right)^2}{4(M)^{\frac{\gamma-1}{\gamma}}\delta(s)\tau^{\Delta}(s)}\right]\Delta s$$
$$\le\int_{t_0}^{t_1}Lq(s)\delta(s)\Delta s+W(t_1)<\infty$$

which is contradicted with (28). This completes the proof. □

Now, applying Theorem 3 with

$$H(t,s)=(t-s)^m,\ m\ge1,\ t\ge s\ge t_0,\ t,s\in[t_0,\infty)_{\mathbb{T}}$$

we have the following results.

Corollary 3 *Assume that the conditions* (H_1) - (H_5), (2) *hold and* $\tau(\sigma(t))=\sigma(\tau(t))$. *If there exists a positive* Δ*-differentiable function* $\delta\in C^1_{rd}\left([t_0,\infty)_{\mathbb{T}},\mathbb{R}\right)$ *such that for every positive constant M and* $m\ge1$,

$$\limsup_{t\to\infty}\frac{1}{t^m}\int_{t_0}^{t}(t-s)^m\left[Lq(s)\delta(s)-\frac{\left(a(\tau(s))\right)^{\frac{1}{\gamma}}\left(\delta^{\Delta}(s)\right)^2}{4(M)^{\frac{\gamma-1}{\gamma}}\delta(s)\tau^{\Delta}(s)}\right]\Delta s=\infty, \tag{30}$$

for $t>s\ge t_0$, $t,\ s\in[t_0,\infty)_{\mathbb{T}}$. *Then* (1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Using the same ideas as in the proof of Theorem 2, when (3) holds, we can now obtain the following result.

Theorem 4 *Assume that the conditions* (H_1) - (H_5), (3), (23) *hold and* $\tau(\sigma(t))=\sigma(\tau(t))$. *Furthermore, let H and* δ *define the same as Theorem* 3 *and the condition* (28) *holds. Then* (1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$

Now, let

$$H(t,s)=(t-s)^m,\ m\ge1,\ t\ge s\ge t_0,\ t,s\in[t_0,\infty)_{\mathbb{T}}$$

we have the following results.

Corollary 4 *Assume that the conditions* (H_1) - (H_5), (3), (23), (30) *hold and* $\tau(\sigma(t))=\sigma(\tau(t))$, *then* (1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Remark 3 *Our results in this paper unify the oscillation of the second-order nonlinear delay differential equation and the second-order nonlinear delay difference equation. As an example, when* $\mathbb{T}=\mathbb{Z}$, *the* (1) *becomes*

$$\Delta\left(a_n(\Delta x_n)^{\gamma}\right)+q_nf(x_{n-\sigma})=0,\ n=0,1,2,\cdots$$

and the condition (30) *becomes*

$$\limsup_{n\to\infty}\frac{1}{n^m}\sum_{l=n_0}^{n}(n-l)^m\left[Lq_l\delta_l-\frac{(a_{l-\sigma})^{\frac{1}{\gamma}}(\Delta\delta_l)^2}{4(M)^{\frac{\gamma-1}{\gamma}}\delta_l}\right]=\infty$$

then Corollary 3 *extends Theorem* 2.1 *in* [15] *and Theorem* 1 *generalizes Theorem* 2.1 *in* [15]. *The Theorem* 2 - 4 *in this paper are new even for the cases* $\mathbb{T}=\mathbb{R}$ *and* $\mathbb{T}=\mathbb{Z}$.

Example 1 *Consider the second-order nonlinear delay* 2*-difference equations*

$$\left(\frac{t+1}{t+2}x^{\Delta}(t)\right)^{\Delta}+\frac{1}{t^2}\left(1+x^2\left(\frac{t}{2}\right)\right)x\left(\frac{t}{2}\right)=0,\ t\in\overline{2^{\mathbb{Z}}},\ t\ge t_0:=2 \tag{31}$$

Here

$$a(t)=\frac{t+1}{t+2},\ q(t)=t^{-2},\ f(x)=x\left(1+x^2\right),\ \tau(t)=\frac{t}{2},\ \gamma=1.$$

The conditions (H_1) - (H_4) and (2) are clearly satisfied, (H_5) holds with L = 1. Now let $\delta(t)=t$ for all $t\ge2$, then

$$\frac{1}{t^2}\int_2^t (t-s)^2\left[Lq(s)\delta(s)-\frac{\left(a(\tau(s))\right)^{\frac{1}{\gamma}}\left(\delta^\Delta(s)\right)^2}{4(M)^{\frac{\gamma-1}{\gamma}}\delta(s)\tau^\Delta(s)}\right]\Delta s$$

$$=\frac{1}{t^2}\int_2^t (t-s)^2\left[\frac{1}{s}-\frac{1}{2s}\frac{s+2}{s+4}\right]\Delta s \ge \frac{1}{t^2}\int_2^t \frac{(t-s)^2}{2s}\Delta s \to \infty \quad \text{as} \quad t\to\infty.$$

so that (30) is satisfied as well. Altogether, by Corollary 3, the equation (31) is oscillatory.

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