Applied Mathematics, 2014, 5, 2643-2649
Published Online October 2014 in SciRes. http://www.scirp.org/journal/am
http://dx.doi.org/10.4236/am.2014.517252

A New Scheme for Discrete HJB Equations

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Received 2 August 2014; revised 28 August 2014; accepted 10 September 2014

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Abstract

In this paper we propose a relaxation scheme for solving discrete HJB equations based on scheme II [1] of Lions and Mercier. The convergence of the new scheme has been established. Numerical example shows that the scheme is efficient.

Keywords

Iterative Algorithm, Relaxation Scheme, HJB Equation, Convergence, Existence

1. Introduction

Consider the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned} &\max_{1\le i\le k}\left\{L^i u - f^i\right\} = 0 \quad \text{in } \Omega, \\ &u = 0 \qquad\qquad\qquad\quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in R^d, $L^i, i = 1,\cdots,k,$ are elliptic operators of second order. Equation (1.1) is arising in stochastic control problems. See [2] and the references therein.

Equation (1.1) can be discretized by finite difference method or finite element method. See [1] [3] and the references therein. Then we obtain the following discrete HJB equation:

$$\max_{1\le i\le k}\left\{A^j U - F^j\right\} = 0, \tag{1.2}$$

where $A^j \in R^{n\times n}, F^j \in R^n, j = 1,\cdots,k$. Equation (1.2) is a system of nonsmooth nonlinear equations. Many numerical algorithms for solving (1.2) have been proposed. See [4]-[12] and the references therein.

[1] has given two iterative algorithms for solving (1.2). At each iteration, a linear complementarity subproblem or a linear equation system subproblem is solved. See also [4].

Scheme I.

Step 1: Given $\varepsilon > 0, m := 1,$ for some j we find $U^{0,k}$ such that

How to cite this paper: Zou, Z.Y. (2014) A New Scheme for Discrete HJB Equations. *Applied Mathematics*, **5**, 2643-2649.
http://dx.doi.org/10.4236/am.2014.517252

$$A^jU^{0,k}=F^j.$$

Step 2: Let $N=(m-1)k, U^{N,0}=U^{N,k}$. For $j=1,\cdots,k,$ we find $U^{N+j,j}$ such that

$$\max\left\{A^jU^{N+j,j}-F^j, U^{N+j,j}-U^{N+j-1,j-1}\right\}=0.$$

Step 3: If $\left\|U^{mk,k}-U^{N,0}\right\|<\varepsilon,$ then the output is $U^{mk,k}$, otherwise $m=:m+1$ and it goes to *Step* 2.
Assume $A^j=\left(a^j_{ls}\right), F^j=\left(F^j_l\right)$. Let

$$A\left(p_1,\cdots,p_n\right)=\left(a^{p_l}_{ls}\right), F\left(p_1,\cdots,p_n\right)=\left(F^{p_l}_l\right). \tag{1.3}$$

That is: the lth row of matrix $A\left(p_1,\cdots,p_n\right)$ is the lth row of matrix A^{p_l}; the lth component of vector $F\left(p_1,\cdots,p_n\right)$ is the lth component of vector F^{p_l}. Now we formulate Scheme II of Lions and Mercier in the notation above.

Scheme II.

Step 1: $m:=0,$ for some j we find U^0 such that

$$A^jU^0=F^j. \tag{1.4}$$

Step 2: For $l=1,\cdots,n,$ we find p^m_l such that

$$p^m_l=\min\left\{j\in\{1,\cdots,k\}:\left(A^jU^m-F^j\right)_l\right\}=\max_{1\le j\le k}\left\{\left(A^jU^m-F^j\right)_l\right\}. \tag{1.5}$$

Step 3: Compute U^{m+1} as the solution of

$$A\left(p^m_1,\cdots,p^m_n\right)U^{m+1}=F\left(p^m_1,\cdots,p^m_n\right). \tag{1.6}$$

Step 4: If $U^{m+1}=U^m$ then the output is U^m, otherwise $m=:m+1$ and it goes to *Step* 2.

In the last decade many numerical schemes have been given for solving (1.2). But the above schemes are still playing a very important role. See [4]-[6] and the references therein.

In this paper we propose, based on Scheme II above, a relaxation scheme with a parameter ω, which for $\omega=1$ is just Scheme II. In our numerical example, the new scheme with $\omega=0.8, 0.9$ is faster than Scheme II $(\omega=1)$. The monotone convergence of the new scheme has been proved.

2. New Scheme and Convergence

We propose a new scheme which is an extension of Scheme II.

New Scheme II.

Step 1: Given $\varepsilon>0, \omega\in(0,1]$ $m:=0,$ for some j find U^0 such that

$$A^jU^0=F^j. \tag{2.1}$$

Step 2: For $l=1,\cdots,n,$ find p^m_l such that

$$p^m_l=\min\left\{j\in\{1,\cdots,k\}:\left(A^jU^m-F^j\right)_l\right\}=\max_{1\le j\le k}\left\{\left(A^jU^m-F^j\right)_l\right\}. \tag{2.2}$$

Step 3: Compute V^{m+1} as the solution of

$$A\left(p^m_1,\cdots,p^m_n\right)V^{m+1}=F\left(p^m_1,\cdots,p^m_n\right). \tag{2.3}$$

Step 4: Compute

$$U^{m+1}=(1-\omega)U^m+\omega V^{m+1}. \tag{2.4}$$

Step 5: If $\left\|U^{m+1}-U^m\right\|<\varepsilon$ then output U^m otherwise $m=:m+1$ and go to *Step* 2.

In [13] we proposed the following conditions for (1.2).

Condition A^* All the matrices $A\left(p_1,\cdots,p_n\right), p_l=1,\cdots,m, l=1,\cdots,n,$ are *M*-matrices.

In [13] we have proved the following theorem.

Theorem 2.1 If *Condition* A^* holds then (1.2) has a unique solution.

We have the following convergence theorem.

Theorem 2.2 Assume that *Condition* A^* holds, and that $U^m, m = 0,1,2,\cdots$ are produced by New Scheme II. Then U^m is monotonely decreasing and convergent to the solution of (1.2).

Proof Since all $A(p_1,\cdots,p_n), p_l = 1,\cdots,k, l = 1,\cdots,n,$ are M-matrices, $U^m, m = 0,1,\cdots$ in New Scheme II are well defined.

First, we prove U^m is decreasing monotonically, *i.e.*,

$$\cdots \le U^{m+1} \le U^m \le \cdots \le U^1 \le U^0. \tag{2.5}$$

By (2.3) we have

$$A\left(p_1^0,\cdots,p_n^0\right)V^1 = F\left(p_1^0,\cdots,p_n^0\right), \tag{2.6}$$

which combining with (2.1) and (2.2) yields

$$\begin{aligned} A\left(p_1^0,\cdots,p_n^0\right)U^0 - F\left(p_1^0,\cdots,p_n^0\right) &\ge A^j U^0 - F^j = 0 \\ &= A\left(p_1^0,\cdots,p_n^0\right)V^1 - F\left(p_1^0,\cdots,p_n^0\right). \end{aligned} \tag{2.7}$$

Since $A\left(p_1^0,\cdots,p_n^0\right)$ are M-matrices, (2.7) means

$$V^1 \le U^0. \tag{2.8}$$

By (2.4) we obtain

$$U^1 = (1-\omega)U^0 + \omega V^1. \tag{2.9}$$

By $\omega \in (0,1]$, (2.8) and (2.9) we know

$$U^1 \le U^0, \tag{2.10}$$

and

$$V^1 \le U^1, \tag{2.11}$$

which and (2.10) implies

$$V^1 \le U^1 \le U^0.$$

Similarly, by (2.3) we derive

$$A\left(p_1^1,\cdots,p_n^1\right)V^2 = F\left(p_1^1,\cdots,p_n^1\right),$$

which combining with (2.2) and (2.6) implies

$$\begin{aligned} A\left(p_1^1,\cdots,p_n^1\right)V^1 - F\left(p_1^1,\cdots,p_n^1\right) &\ge A\left(p_1^0,\cdots,p_n^0\right)V^1 - F\left(p_1^0,\cdots,p_n^0\right) \\ &= A\left(p_1^1,\cdots,p_n^1\right)V^2 - F\left(p_1^1,\cdots,p_n^1\right). \end{aligned}$$

Hence we have

$$V^2 \le V^1. \tag{2.12}$$

By (2.4), we have

$$U^2 = (1-\omega)U^1 + \omega V^2. \tag{2.13}$$

By (2.12), (2.13) and $\omega \in (0,1]$, we know

$$U^2 \le (1-\omega)U^1 + \omega V^1, \tag{2.14}$$

which combining with $\omega \in (0,1]$ and (2.11) we derive

$$U^2 \le U^1. \tag{2.15}$$

By (2.11), (2.12) and (2.13) ,we get

$$V^2 \le U^2,$$

which combining with (2.15) implies

$$V^2 \le U^2 \le U^1.$$

It is easy to derive by induction that

$$V^{m+1} \le U^{m+1} \le U^m, m = 0,1,\cdots, \tag{2.16}$$

and

$$V^{m+1} \le V^m, m = 0,1,\cdots. \tag{2.17}$$

It follows that (2.5) holds.

It follows from (2.2) and (2.3) that

$$\begin{aligned}\max_{1\le j\le k}\left\{A^j V^m - F^j\right\} &= A\left(p_1^m,\cdots,p_n^m\right)V^m - F\left(p_1^m,\cdots,p_n^m\right)\\ &= A\left(p_1^m,\cdots,p_n^m\right)\left(V^m - V^{m+1}\right), \quad m = 0,1,\cdots.\end{aligned} \tag{2.18}$$

Since the set $\left\{\left(p_1,\cdots,p_n\right): p_l = 1,\cdots,k, l = 1,\cdots,n\right\}$ is a finite set there exist positive integers q and m with $q > k$ such that

$$\left(p_1^q,\cdots,p_n^q\right) = \left(p_1^m,\cdots,p_n^m\right).$$

Therefore, we have

$$A\left(p_1^q,\cdots,p_n^q\right) = A\left(p_1^m,\cdots,p_n^m\right),$$

$$F\left(p_1^q,\cdots,p_n^q\right) = F\left(p_1^m,\cdots,p_n^m\right).$$

Then by (2.2) we obtain

$$V^{q+1} = V^{m+1},$$

which and (2.17) results in

$$V^{q+1} = V^q = \cdots = V^{m+2} = V^{m+1}. \tag{2.19}$$

From (2.4), (2.16) and (2.19) we have

$$U^{q+1} = U^q = \cdots = U^{m+2} = U^{m+1}. \tag{2.20}$$

It follows from (2.18), (2.19) and (2.20) that

$$\max_{1\le j\le k}\left\{A^j U^{m+1} - F^j\right\} = 0,$$

which means U^{m+1} is a solution of (1.2). The existence of solution has been proved.

Finally, we prove the uniqueness of solution. Assume U and U^* are solutions of (1.2), *i.e.*,

$$\max_{1\le j\le k}\left\{A^j U - F^j\right\} = 0, \tag{2.21}$$

$$\max_{1\le j\le k}\left\{A^j U^* - F^j\right\} = 0. \tag{2.22}$$

It is easy to see from (2.21) and (2.22) that there exist $(p_1,\cdots,p_n)$ and $(p_1^*,\cdots,p_n^*)$ such that

$$A(p_1,\cdots,p_n)U - F(p_1,\cdots,p_n) = 0, \tag{2.23}$$

$$A(p_1^*,\cdots,p_n^*)U^* - F(p_1^*,\cdots,p_n^*) = 0, \tag{2.24}$$

$$A(p_1^*,\cdots,p_n^*)U - F(p_1^*,\cdots,p_n^*) \le 0, \tag{2.25}$$

$$A(p_1,\cdots,p_n)U^* - F(p_1,\cdots,p_n) \le 0. \tag{2.26}$$

(2.23) and (2.26) implie $U^* \le U$. But (2.24) and (2.25) implies $U^* \ge U$. Hence $U^* = U$. The proof is complete. ☐

3. Numerical Example

We use example 2 in [4], *i.e.*, $k = n = 2, \Omega = (0,1)\times(0,1)$.

$$\begin{aligned} &\max_{1\le i\le 2}\left\{L^i u - f^i\right\} = 0 && \text{in } \Omega, \\ &u = 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

where $\Omega = \{(x,y): 0 < x, y < 1\}$,

$$L^1 = -(x+6)^2\frac{\partial^2}{\partial x^2} - (x+6)(y+2)\frac{\partial^2}{\partial x\partial y} - (y+2)^2\frac{\partial^2}{\partial y^2}$$

$$+\left[0.5(x+6)-4\right]\frac{\partial}{\partial x} + 0.5(y+2)\frac{\partial}{\partial y} + 1,$$

$$L^2 = -(x+6)^2\frac{\partial^2}{\partial x^2} - 0.8(x+6)(y+2)\frac{\partial^2}{\partial x\partial y} - 0.75(y+2)^2\frac{\partial^2}{\partial y^2}$$

$$+\left[(x+6)-2\right]\frac{\partial}{\partial x} + (y+2)\frac{\partial}{\partial y} + 4,$$

$$u = x(1-x)y(1-y),$$

$$f^1 = f^2 = \max\left(L^1u, L^2u\right).$$

The discretization of the above second order derivatives are:

$$\frac{\partial^2}{\partial x^2} \approx h^{-2}D_{h,x}^+D_{h,x}^-, \quad \frac{\partial^2}{\partial y^2} \approx h^{-2}D_{h,y}^+D_{h,y}^-,$$

$$\frac{\partial^2}{\partial x\partial y} \approx \frac{1}{2}h^{-2}\left[D_{h,x}^+D_{h,y}^+ + D_{h,x}^-D_{h,y}^-\right],$$

where $D_{h,x}^\pm, D_{h,y}^\pm$ denote the forward and backward difference respectively in x and y, $h = 1/10$, $h = 1/20$. We use New Scheme II to solve the discrete problem. Take $\varepsilon = 10^{-5}$, $\omega = 0.1, 0.5, 0.8, 0.9, 1.0$ and 1.1, 1.3, 1.5, 1.8, 1.9 respectively.

Table 1 and **Table 2** show the ∞-norm of the residual $R = \max_{1\le j\le k}\left\{A^jU^m - F^j\right\}$ when iteration terminates.

We see that $R \approx 0$ for $\omega \le 1$ and R is big for $\omega \in (1,2)$.

Table 3 shows the relation between iteration number m and relaxation number $\omega\left(\omega \in (0,1]\right)$. **Table 4** and **Table 5** show the value of U^m at $(x,y)^{\mathrm{T}} = (0.5,0.5)^{\mathrm{T}}$ for $h = 1/10$ and $h = 1/20$ respectively.

We can see from **Table 3** that the algorithm for $\omega = 0.8, 0.9$ is faster than that for $\omega = 1$. **Table 4** and **Table 5** display the monotonicity of the algorithm.

Table 1. ∞-norm of the residual *R*.

ω	0.1	0.5	0.8	0.9	1.0
$\|R\|_\infty$					
$h=1/10$	3.419e–004	2.099e–011	9.464e–012	6.861e–012	6.651e–012
$h=1/20$	6.630e–003	1.784e–008	6.653e–011	6.062e–011	8.169e–006

Table 2. ∞-norm of the residual *R*.

ω	1.1	1.3	1.5	1.8	1.9
$\|R\|_\infty$					
$h=1/10$	3.440e–000	2.314e+001	4.670e+001	8.421e+001	9.730e–000
$h=1/20$	1.667e–003	4.323e+001	1.754e+002	4.323e+001	2.089e+002

Table 3. Iteration number *m*.

ω	0.1	0.5	0.8	0.9	1.0
m					
$h=1/10$	200	198	107	90	124
$h=1/20$	600	495	282	258	400

Table 4. The value of U^m at $(x,y)^{\mathrm{T}}=(0.5,0.5)^{\mathrm{T}}$.

ω	0.1	0.5	0.8	0.9	1.0
$h=1/10$					
$m=1$	1.091409800	1.086033962	1.082002083	1.080658123	1.079314164
$m=2$	1.089751377	1.080022728	1.074891194	1.073533844	1.076283661
$m=3$	1.088256293	1.075449958	1.072050161	1.071072814	1.073086733
$m=4$	1.086758364	1.073060086	1.069451302	1.068586924	1.072407806
Last m	1.065963994	1.065887109	1.065887109	1.065887109	1.065887109

Table 5. The value of U^m at $(x,y)^{\mathrm{T}}=(0.5,0.5)^{\mathrm{T}}$.

ω	0.1	0.5	0.8	0.9	1.0
$h=1/20$					
$m=1$	1.077654026	1.073664734	1.070672766	1.069675443	1.068678121
$m=2$	1.076493553	1.069008305	1.065427282	1.065027950	1.068036835
$m=3$	1.075236529	1.065915940	1.063091196	1.062134520	1.066011200
$m=4$	1.073996351	1.063479656	1.060857772	1.060476760	1.065563176
Last m	1.054467308	1.054409847	1.054409847	1.054409847	1.054409847

Funding

This work was supported by Educational Commission of Guangdong Province, China (No. 2012LYM-0066) and the National Social Science Foundation of China (No. 14CJL016).

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