Applied Mathematics, 2014, 5, 2547-2557
Published Online September 2014 in SciRes. http://www.scirp.org/journal/am
http://dx.doi.org/10.4236/am.2014.516244

Scientific Research

Asymptotic Harmonic Behavior in the Prime Number Distribution

Maurice H. P. M. van Putten

Sejong University, Seoul, South Korea
Email: mvp@sejong.ac.kr

Received 16 June 2014; revised 19 July 2014; accepted 29 July 2014

Abstract

We consider $\Phi(x) = x^{-\frac{1}{4}}\left[1 - 2\sqrt{x}\sum e^{-p^2\pi x}\ln p\right]$ on $x > 0$, where the sum is over all primes p. If Φ is bounded on $x > 0$, then the Riemann hypothesis is true or there are infinitely many zeros $\mathrm{Re}z_k > \frac{1}{2}$. The first 21 zeros give rise to asymptotic harmonic behavior in $\Phi(x)$ defined by the prime numbers up to one trillion.

Keywords

Prime Number Distribution, Summation, Regularization

1. Introduction

The Riemann-zeta function is the analytic extension of

$$\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \cdots = \prod\left(1 - p^{-z}\right)^{-1} \quad (\mathrm{Re}\, z > 1), \tag{1}$$

where Euler's identity on the right hand side expresses the relation of the integers to the primes. The zeros z_k of Riemann's analytic continuation of (1) comprise the negative even integers, $-2, -4, \cdots$, and an infinite number of nontrivial zeros $z_k = a_k + iy_k$ in the strip $0 < a_k < 1$.

A general approach to find zeros is by continuation [1]. If $z(0) = z_0$ is a starting point of a path $z(\lambda)$ with tangent $\tau = z'(\lambda)$,

$$\tau \frac{\zeta'(z)}{\zeta(z)} = -1, \tag{2}$$

How to cite this paper: van Putten, M.H.P.M. (2014) Asymptotic Harmonic Behavior in the Prime Number Distribution. *Applied Mathematics*, **5**, 2547-2557. http://dx.doi.org/10.4236/am.2014.516244

then the endpoint $z_* = \lim_{\lambda\to\infty} z(\lambda)$ is a zero of $\zeta(z)$, all of which are isolated. All known nontrivial zeros satisfy $\operatorname{Re} z_k = \frac{1}{2}$ to within numerical precision, the first three of which are $z_1 = \frac{1}{2} \pm 14.1347i$, $z_2 = \frac{1}{2} \pm 21.0220i$, $z_3 = \frac{1}{2} \pm 25.0109i$. By the symmetry

$$\zeta(s) = \zeta(1-s)\frac{\chi(1-s)}{\chi(s)}, \quad \chi(s) = \pi^{\frac{s}{2}}\Gamma\left(\frac{s}{2}\right), \quad \Gamma(z) = \int_0^\infty t^{z-1}\mathrm{e}^{-t}\mathrm{d}t, \tag{3}$$

it suffices to study zeros in the half plane $\operatorname{Re}(z) \geq \frac{1}{2}$. **Figure 1** illustrates root finding by (2) for the first few zeros.

Continuation (2) is determined by the prime numbers, since

$$-\frac{\zeta'(z)}{\zeta(z)} = -\sum \frac{\ln p}{p^z - 1} = \sum \xi(mz), \quad \xi(z) = \sum p^{-z}\ln p \quad (\operatorname{Re} z > 1), \tag{4}$$

whereby

$$\xi(z) = -\frac{\zeta'(z)}{\zeta(z)} - \sum_{m\geq 2}\xi(mz). \tag{5}$$

The poles of $\xi(z)$ at the zeros are therefore expressed by the prime number distribution.

In this paper, we study the distribution of zeros z_k by Fourier analysis of the function

$$\Phi(x) = x^{-\frac{1}{4}}\left[1 - 2\sqrt{x}\varphi(x)\right] \tag{6}$$

on $x > 0$, where

$$\varphi(x) = \sum \mathrm{e}^{-p^2\pi x}\log p \tag{7}$$

with summation over all primes. In what follows, we put

$$Z(\lambda) = \sum \alpha_k \mathrm{e}^{-\lambda\left(z_k - \frac{1}{2}\right)}, \quad \alpha_k = \gamma(z_k), \quad \gamma(z) = \frac{\Gamma\left(\frac{z}{2}\right)}{\pi^{\frac{z}{2}}}. \tag{8}$$

The α_k are absolutely summable by Stirling's formula and the asymptotic distribution of z_k.

Theorem 1.1. *In the limit as* $x > 0$ *becomes small, we have the asymptotic behavior*

$$\Phi(x) = \frac{1}{2}\gamma\left(\frac{1}{2}\right) + Z\left(\ln\sqrt{x}\right) + \frac{1}{3}\gamma\left(\frac{1}{3}\right)x^{\frac{1}{12}} + o\left(x^{\frac{1}{12}}\right). \tag{9}$$

In (9), Z is evidently unbounded in the limit as x approaches zero whenever a finite number of zeros z_k exists off the critical line $\operatorname{Re} z = \frac{1}{2}$.

Corollary 1.2. *If* Φ *is bounded, then the Riemann hypothesis is true or there are infinitely many zeros* $\operatorname{Re} z_k > \frac{1}{2}$.

A similar relation between the distribution of z_k and the primes is [2] [3]

$$\frac{u - \psi_C(u)}{\sqrt{u}} = \sum \frac{u^{z_k - \frac{1}{2}}}{z_k} + \frac{\ln(2\pi) + \ln\sqrt{1-u^{-2}}}{\sqrt{u}} \tag{10}$$

based on the Chebyshev functions

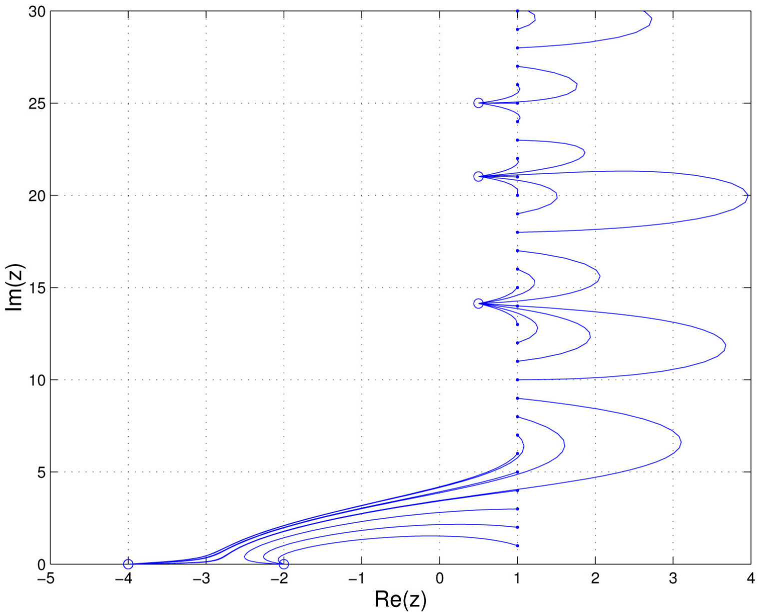

Figure 1. Shown are the trajectories of continuation $z(\lambda)$ in the complex plane z by numerical integration of (2) with initial data $z_0 = 1 + ni$ $(n = 1, 2, 3, \cdots)$ indicated by small dots on $\mathrm{Re}(z) = 1$. Continuation produces roots indicated by open circles, defined by finite endpoints of $z(\lambda)$ in the limit as λ approaches infinity. The roots produced by the choice of initial data are the first three on $\mathrm{Re}\, z = \frac{1}{2}$ and −2 and −4 of the trivial roots.

$$\psi_C(u) = \sum_{p^k \le u} \ln(p), \quad \vartheta_C(u) = \sum_{p \le u} \ln p, \tag{11}$$

where the sum is over all primes p and integers k. In (9), $\Phi(x)$ has a normalization by $x^{\frac{1}{4}}$ according to and Z is absolutely convergent for all $x > 0$, whereas in (10) $\psi_C(u)$ is normalized by $\sqrt{u}$ and the sum $\sum \frac{u^{z_k - \frac{1}{2}}}{z_k}$ is not absolutely convergent. Similar to Corollary 1.2, the left hand side of (10) will be bounded in the limit of large u if the Riemann hypothesis is true.

Section 2 presents some preliminaries on $\zeta(z)$. Section 3 gives an integral representation of $\zeta(z)$ and a discussion on its singularity at $z = 1$. In Section 4, Cauchy's integral formula is applied to derive a sum of residues associated with the z_k. The proof Theorem 1.1 follows from a Fourier transform and asymptotic analysis (Section 5). In Section 6, we illustrate a direct evaluation of $\Phi(x)$ using the primes up to one trillion, showing harmonic behavior arising from Z by the first few zeros z_k. We summarize our findings in Section 7.

2. Background

Our analysis begins with some known properties of $\zeta(z)$ in, e.g., [4]-[9].

Riemann obtained an analytic extension of $\zeta(z)$ by expressing n^{-z} in terms of $\Gamma\left(\frac{z}{2}\right)$,

$$\gamma(z)\zeta(z) = \int_0^\infty x^{\frac{z}{2}-1}\theta_1(x)\,\mathrm{d}x, \tag{12}$$

where

$$\theta_1(x) = \frac{\theta(x) - 1}{2}, \quad \theta(x) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{-n^2 \pi x}. \tag{13}$$

Here, $\theta_1(x)$ satisfies $\theta_1(x)\sim\frac{1}{2\sqrt{x}}$ as x approaches zero by the identity $\theta(x^{-1})=\sqrt{x}\theta(x)$ for the Jacobi function $\theta(x)$ [1]. On $\mathrm{Re}\, z>1$, it obtains the meromorphic expression (e.g. Borwein *et al.*, 2006)

$$\gamma(z)\zeta(z)=\frac{1}{z(z-1)}+f(z),\quad f(z)=\int_1^\infty\left(x^{\frac{z}{2}-1}+x^{-\frac{z}{2}-\frac{1}{2}}\right)\theta_1(x)\mathrm{d}x, \tag{14}$$

which gives a maximal analytic continuation of $\zeta(z)$ and shows a simple pole at $z=1$ with residue 1.

Riemann further introduced the symmetric form $Q(z)\zeta(z)$, $Q(z)=\frac{1}{2}z(z-1)\gamma(z)$ satisfying $Q(z)\zeta(z)=Q(1-z)\zeta(1-z)$, whereby

$$\zeta(z)=\pi^{z-1}\zeta(1-z)\frac{\Gamma\left(\frac{1}{2}-\frac{z}{2}\right)}{\Gamma\left(\frac{z}{2}\right)}=\frac{\pi^z\zeta(1-z)}{\cos\left(\frac{1}{2}\pi z\right)\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{1}{2}+\frac{z}{2}\right)}=\frac{\pi^{z-\frac{1}{2}}2^{z-1}\zeta(1-z)}{\cos\left(\frac{1}{2}\pi z\right)\Gamma(z)} \tag{15}$$

using $\Gamma\left(\frac{1}{2}-\frac{z}{2}\right)\Gamma\left(\frac{1}{2}+\frac{z}{2}\right)=\frac{\pi}{\cos(\pi z)}$ and $\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)=2^{1-2z}\sqrt{\pi}\Gamma(2z)$. Along $z=1-iy$, $\zeta(z)$ is non-vanishing [10]-[13], allowing

$$\frac{\zeta'(z)}{\zeta(z)}=-\frac{\zeta'(1-z)}{\zeta(1-z)}+\ln(2\pi)+\frac{\pi}{2}\tan\left(\frac{\pi z}{2}\right)-\psi(z) \tag{16}$$

in terms of the digamma function

$$\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}\sim\ln(z)+O\left(z^{-1}\right) \tag{17}$$

in the limit of large $|z|$.

Lemma 2.1. *In the limit of large* y, *the logarithmic derivative of* $\zeta(z)$ *satisfies*

$$\frac{\zeta'(iy)}{\zeta(iy)}=-\frac{\zeta'(1-iy)}{\zeta(1-iy)}+O(\ln y). \tag{18}$$

Proof. The result follows from (17) and (16). □

Lemma 2.2. *Along the line* $z=iy$, *we have the asymptotic expansion* $|\gamma(iy)|\sim\sqrt{\frac{2\pi}{y}}\mathrm{e}^{-\frac{\pi}{2}y}$ *in the limit of large* y, *whereby the* α_k *are absolutely summable.*

Proof. Recall (8) and the asymptotic expansion $\Gamma(z)=\sqrt{2\pi}z^{z-\frac{1}{2}}\mathrm{e}^{-z}\left[1+O\left(z^{-1}\right)\right]$ with a branch cut along the negative real axis. In the limit of large y_k, $y_k\sim\frac{2\pi k}{\ln k}$, and hence $|Z_k|\sim\mathrm{e}^{-\frac{\pi^2k}{\ln k}}$, since $|\arg z_k|\sim\frac{\pi}{2}$ as k becomes large. Hence, the Z_k are absolutely summable. Numerically, their sum is small, $\sum|\alpha_k|=3.5\times10^{-5}$ based on a large number of known zeros z_k. □

Lemma 2.3. *In the limit of large* y, *we have*

$$\left|\gamma(iy)\frac{\zeta'(iy)}{\zeta(iy)}\right|=O\left(y^{-\frac{1}{2}}\mathrm{e}^{-\frac{\pi}{2}y}\ln y\right). \tag{19}$$

---

[1]When $z=n$ is an integer, $\frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$ is one-half the surface area of S^n.

Proof. By Lemma 2.1-2.2, we have

$$\left|\gamma(iy)\frac{\zeta'(iy)}{\zeta(iy)}\right| \sim \sqrt{\frac{2\pi}{y}}\left(\left|\frac{\zeta'(1-iy)}{\zeta(1-iy)}\right| + O(\ln y)\right)\mathrm{e}^{-\frac{\pi}{2}y} \tag{20}$$

for large y. Also [4] [14] [15]

$$\left|\frac{\zeta'(1-iy)}{\zeta(1-iy)}\right| \le c(\ln y)^{\frac{2}{3}}(\ln\ln y)^{\frac{1}{3}} \tag{21}$$

on $y > \delta$ for some positive constants c, δ. □

3. An Integral Representation of $\xi(z)$

Following the same steps leading to the Riemann integral for $\zeta(z)$, we have

$$\gamma(z)\xi(z) = \int_0^\infty x^{\frac{z}{2}-1}\varphi(x)\mathrm{d}x = \frac{1}{z-1} + g(z), \tag{22}$$

where $1/(z-1)$ absorbs the simple pole in $\xi(z)$ at $z=1$ due to the simple pole in $\zeta(z)$ at $z=1$, leaving $g(z)$ analytic at $z=1$. Following a decomposition $g(z) = g_2(z) - g_1(z)$,

$$g_1(z) = \frac{1}{2}\int_0^1 x^{\frac{2z-1}{4}}\Phi(x)\frac{\mathrm{d}x}{x}, \quad g_2(z) = \int_1^\infty x^{\frac{z}{2}-1}\varphi(x)\mathrm{d}x, \tag{23}$$

and substitution $x = \mathrm{e}^{2\lambda}$, $g(z)$ appears as the Laplace transforms

$$g_1(z) = \int_{-\infty}^0 \Phi\left(\mathrm{e}^{2\lambda}\right)\mathrm{e}^{\lambda\left(z-\frac{1}{2}\right)}\mathrm{d}\lambda, \quad g_2(z) = 2\int_0^\infty \varphi\left(\mathrm{e}^{2\lambda}\right)\mathrm{e}^{\lambda z}\mathrm{d}\lambda. \tag{24}$$

These integral expressions allow continuations to $\operatorname{Re} z > 1$, respectively, the entire complex plane.

Lemma 3.1. *Analytic extension of* $g_1(z)$ *extends to* $z > \frac{1}{2}$.

Proof. With $z = a + ib$, the second term on the right hand side in (5) satisfies

$$\sum_{m\ge 3}\left|\xi(mz)\right| \le \sum_{n\ge 3}\frac{n^{-3a}\log n}{1-n^{-a}} < -\frac{\sqrt{2}\zeta'(3a)}{\sqrt{2}-1}, \tag{25}$$

which is bounded in $\operatorname{Re} z = a > \frac{1}{2}$. Since the second term $\xi(2z)$ in (5) is analytic in $\operatorname{Re} z = a > \frac{1}{2}$, it follows that $g(a)$ in is analytic on $a > \frac{1}{2}$. Following (5) as a approaches $\frac{1}{2}$ from the right, we have

$$\xi(a) = -\frac{1}{2a-1} + u_1(a), \tag{26}$$

where $u_1(a)$ is analytic at $a = \frac{1}{2}$. By (22), as a approaches $\frac{1}{2}$ from the right, we have

$$g_1(a) = -\frac{1}{2a-1} + u_2(a), \tag{27}$$

where $u_2(a)$ is analytic about $a = \frac{1}{2}$. □

Figure 2 shows a numerical evaluation of $\Phi(x)$ for small x evaluated for the 37.6 billion primes up to one trillion, allowing x down to 2.6×10^{-23} $(\lambda = -26)$ in view of the requirement for an accurate truncation in $\varphi(x)$ as defined by (7). The result shows asymptotic harmonic behavior in the limit as x becomes small.

If the integral

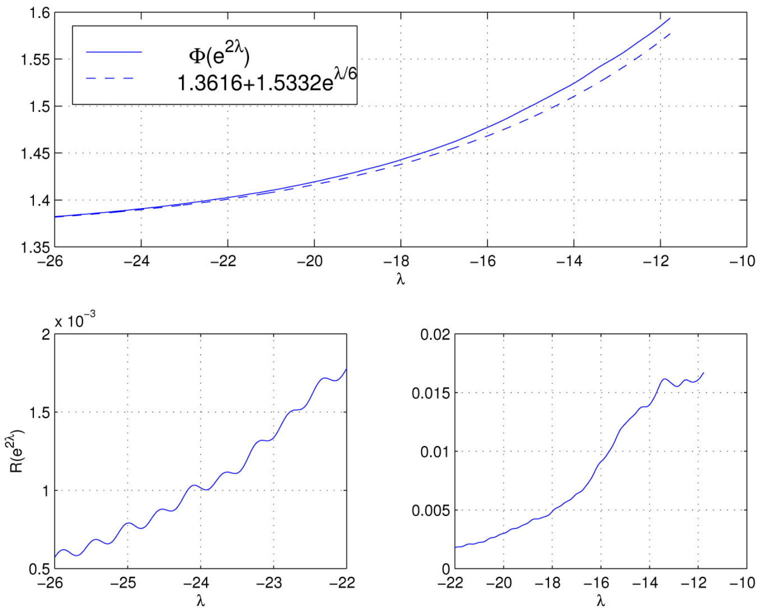

Figure 2. The top window shows $\Phi\left(e^{2\lambda}\right)$ on $\lambda \in [-26,-11.7756]$ and its leading order approximation $1.3616+1.5332e^{\frac{\lambda}{6}}$. The asymptotic harmonic behavior is apparent in the residual difference (52) between the two, shown in the bottom two windows, including the period of 2.2496 in λ associated with the first zero $z_* = \frac{1}{2} \pm 14.1347i$.

$$\int_\epsilon^1 x^{\frac{2z-1}{4}} \Phi(x) \frac{dx}{x} \tag{28}$$

is absolutely convergent as $\epsilon > 0$ approaches zero, e.g., when $\Phi(x)$ is of one sign in some neighborhood of $z = 0$, as in the numerical evaluation shown in **Figure 2**, then $g_1(z)$ has an analytic extension into $\operatorname{Re} z > \frac{1}{2}$ with no singularities, implying the absence of z_k in this region. However, this requires information on the point wise behavior of $\Phi(x)$, which goes beyond the relatively weaker integrability property (23).

To make a step in this direction, we next apply a linear transform to (5) to derive the asymptotic behavior of $\Phi(x)$ in terms of the distribution z_k.

4. A Sum of Residues *Z* Associated with the Non-Trivial Zeros

Consider

$$h(z) = \gamma(z)\frac{\zeta'(z)}{\zeta(z)} + \frac{1}{z-1} \tag{29}$$

and its Fourier transform

$$H(\lambda) = \int_{a-i\infty}^{a+i\infty} h(z) e^{-\lambda z} \frac{dz}{2\pi i}. \tag{30}$$

Lemma 4.1. *$h(z)$ has a simple pole at $z = 1$ with residue 1 and simple poles at each of the nontrivial zeros z_k of $\zeta(z)$ with residue Z_k.*

Proof. We have (e.g. Borwein *et al.* 2006)

$$\zeta_1(z) = \frac{1}{2} z(z-1)\gamma(z)\zeta(z), \quad \frac{\zeta_1'(z)}{\zeta_1(z)} = B + \sum_k \left(\frac{1}{z-z_k} + \frac{1}{z_k} \right), \tag{31}$$

where B is a constant, so that

$$\gamma(z)\frac{\zeta'(z)}{\zeta(z)}+\frac{1}{z-1}=\gamma(z)\left[B+\Sigma_k\left(\frac{1}{z-z_k}+\frac{1}{z_k}\right)\right]+A(z). \tag{32}$$

Here

$$A(z)=\frac{1-\gamma(z)}{z-1}-\frac{\gamma(z)}{z}-2\frac{\gamma(z)\psi(z)}{\ln\pi}, \tag{33}$$

where $\psi(z)$ denotes the digamma function as before, includes contributions from the logarithmic derivative of the factor to $\zeta(z)$ in (31), whose singularities are restricted to the trivial zeros of $\zeta(z)$. □

We now consider the Fourier integral over $\operatorname{Re} z=a$ as part of contour integration closed over $z=x\pm iY$ and $\operatorname{Re} z=0$.

Proposition 4.2. *The Fourier transform of* $h(z)$ *over* $\operatorname{Re} z>\sup a_k$ *satisfies*

$$H(\lambda)=e^{-\frac{\lambda}{2}}Z(\lambda)+O(1) \tag{34}$$

in the limit of large $\lambda<0$.

Proof. Integration over $z=x+iY$ $(0<x<a)$ gives

$$e^{-iY}\int_{iY}^{iY+a}h(z)e^{-\lambda x}\frac{dx}{2\pi i}=e^{-iY}\int_{iY}^{iY+a}\frac{\Gamma\left(\frac{z}{2}\right)}{\pi^{\frac{z}{2}}}\frac{\zeta'(z)}{\zeta(z)}\frac{dx}{2\pi i}+O\left(Y^{-1}\right), \tag{35}$$

where we choose Y to be between two consecutive values of y_k. We have

$$\int_{iY}^{iY+a}\frac{\Gamma\left(\frac{z}{2}\right)}{\pi^{\frac{z}{2}}}\frac{\zeta'(z)}{\zeta(z)}\frac{dx}{2\pi i}\sim\gamma_k\int_{iY}^{iY+a}\frac{\zeta'(z)}{\zeta(z)}\frac{dx}{2\pi i}\sim\frac{\gamma_k}{2\pi i}\ln(2a-1)\left[1+\frac{4ai(y_k-Y)}{1-2a}+\pi i\right]. \tag{36}$$

In the limit as k approaches infinity, y_k-Y approaches zero and $|\gamma_k|$ becomes small by Lemma 2.2., whence

$$\left(\int_{iY}^{iY+a}-\int_{-iY}^{-iY+a}\right)\frac{\Gamma\left(\frac{z}{2}\right)}{\pi^{\frac{z}{2}}}\frac{\zeta'(z)}{\zeta(z)}\frac{dx}{2\pi i}\sim\ln(2a-1)\operatorname{Im}\gamma_k=O\left(\ln(2a-1)\sqrt{\frac{2\pi}{y_k}}e^{-\frac{\pi}{2}y_k}\right). \tag{37}$$

Next, integration over $z=iy$ with a small semicircle around $z=0$ obtains an $O(1)$ result in the limit of large λ by application of Lemma 2.1-2.3 and the Riemann-Lebesgue Lemma. The result now follows in the limit as k approaches infinity, taking into account the residue sum $e^{-\frac{\lambda}{2}}Z(\lambda)$ associated with the z_k and absolute summability of the α_k. □

5. Proof of Theorem 1.1

Multiplying (5) by $\gamma(z)$, we have

$$\gamma(z)\xi(z)=-\gamma(z)\frac{\zeta'(z)}{\zeta(z)}-\gamma(z)\sum_{m\ge2}\xi(mz), \tag{38}$$

that is, by (22) and (29),

$$\frac{1}{z-1}+g(z)=-h(z)+\frac{1}{z-1}-\gamma(z)\sum_{m\ge2}\xi(mz). \tag{39}$$

We thus consider

$$g_1(z)=g_2(z)+h(z)+\gamma(z)\sum_{m\ge2}\xi(mz), \tag{40}$$

which *ab initio* is defined on $\operatorname{Re} z>1$ by Euler's identity with Fourier transform

$$G_1(\lambda)=\int_{a-i\infty}^{a+i\infty} g_1(z)\mathrm{e}^{-\lambda z}\frac{\mathrm{d}z}{2\pi i}=\mathrm{e}^{-\frac{\lambda}{2}}\Phi\left(\mathrm{e}^{2\lambda}\right). \tag{41}$$

Turning to the right hand side of (40), we consider the coefficients

$$c_m(z)=\frac{\gamma(z)}{\gamma(mz)},\ C_m=\frac{1}{m}\gamma\left(\frac{1}{m}\right)\ \ (m\geq 1). \tag{42}$$

Here, $C_m=m^{-1}\gamma(1/m)$ since $\gamma(1)=1$. In particular, $C_2=\frac{1}{2}\gamma\left(\frac{1}{2}\right)$ and $m^{-1}c_m\left(m^{-1}z\right)=1+\left(\frac{1}{2}\ln\pi+\frac{1}{2}\gamma\right)z+O\left(z^2\right)$ has a well defined limit and $C_m\to 2$ in the limit as m becomes arbitrarily large.

Lemma 5.1. *The sum* $\sum_{m\geq n}\xi(mz)$ *is well-defined on* $\mathrm{Re}\,z>\frac{1}{n}$.

Proof. The result follows from the case $n=2$. By the Prime Number Theorem, $p_k\sim k\ln k$, whereby summation over the tails $k\geq n$ satisfy

$$\sum_{k\geq n}^{\infty}\frac{\ln(p_k)}{p_k^{2a}}\sim\sum_{k\geq n}^{\infty}\left[\frac{1}{k^{2a}\ln(k)^{2a-1}}+\frac{\ln\ln(k)}{(k\ln(k))^{2a}}\right]<\infty \tag{43}$$

whenever $a>\frac{1}{2}$. Hence, for $z=a+iy$, $\left|\sum p^{-2z}\ln p\right|\leq\sum p^{-2a}\ln p<\infty$ whenever $a>\frac{1}{2}$. It follows that

$$\left|\sum\nolimits_{m\geq 2}\xi(mz)\right|\leq\sum\nolimits_{m\geq 2}\sum\nolimits_{p}p^{-(m-2)a}p^{-2a}\ln p\leq\sum\nolimits_{m\geq 0}2^{-m}\sum\nolimits_{p}p^{-2a}\ln p=\sum\nolimits_{p}p^{-2a}\ln p<\infty \tag{44}$$

on $\mathrm{Re}\,z>\frac{1}{2}$. □

Lemma 5.2. *For any* $m\geq 2$, *the Fourier transform of* $\frac{c_m(z)}{mz-1}$ *over* $\mathrm{Re}\,z=a>\frac{1}{2}$ *satisfies*

$$D_m(\lambda)=C_m\mathrm{e}^{-\frac{\lambda}{m}}+o(1) \tag{45}$$

Proof. The Fourier integral can be obtained in a contour integration with closure over $z=iy$ and the edges $z=x+iY$ $(0<x<a)$ for large $\pm Y$. In the notation (42), it obtains a residue $C_m=m^{-1}c_m(1/m)=m^{-1}\gamma(1/m)$ at $z=1/m$, since $\gamma(1)=1$, whence

$$D_m(\lambda)=C_m\mathrm{e}^{-\frac{\lambda}{m}}+\mathrm{e}^{\frac{\lambda}{2}}\int_{-\infty}^{\infty}\frac{c_m(iy)}{imy-1}\mathrm{e}^{-i\lambda y}\frac{\mathrm{d}y}{2\pi}. \tag{46}$$

The integral (46) exists by virtue of a removable singularity of $c_m(z)$ at $z=0$. It asymptotically decays to zero for large λ when $m\geq 2$ by the Riemann-Lebesgue Lemma. □

We now consider (40) with (22),

$$g_1(z)=g_2(z)+h(z)+\sum_{m\geq 2}\left(\frac{c_m(z)}{mz-1}+c_m(z)g(mz)\right)=h(z)+\sum_{m=2}^{N}\frac{c_m(z)}{mz-1}+r_N(z) \tag{47}$$

with a remainder

$$r_N(z)=g_2(z)+\sum_{m\geq 2}c_m(z)g(mz)+\gamma(z)\sum_{m\geq N+1}\xi(mz). \tag{48}$$

Lemma 5.3. *For* $N\geq 3$, *the Fourier transform*

$$\mathrm{e}^{\frac{\lambda}{2}}R_N(\lambda)=\int_{a-i\infty}^{a+i\infty}r_N(z)\mathrm{e}^{-\lambda\left(z-\frac{1}{2}\right)}\frac{\mathrm{d}z}{2\pi i}=o(1) \tag{49}$$

in the limit of large $\lambda < 0$.

Proof. Since $r_N(z)$ is analytic in $\mathrm{Re}\,z > \frac{1}{3}$, we are at liberty to consider the transform $\mathrm{e}^{\frac{\lambda}{2}} R_N(\lambda)$ on $a = 1/2$. The result follows from the Riemann-Lebesgue Lemma. □

Proof of Theorem 1.1. The Fourier transform of (47) is

$$G_1(\lambda) = H(\lambda) + D_2(\lambda) + D_3(\lambda) + R_3(\lambda). \tag{50}$$

By Proposition 4.2 and Lemmas 5.1-5.2, we have

$$\mathrm{e}^{-\frac{\lambda}{2}}\Phi\left(\mathrm{e}^{2\lambda}\right) = \mathrm{e}^{-\frac{\lambda}{2}} Z(\lambda) + C_2 \mathrm{e}^{-\frac{\lambda}{2}} + C_3 \mathrm{e}^{-\frac{\lambda}{3}} + o\left(\mathrm{e}^{-\frac{\lambda}{2}}\right). \tag{51}$$

With $x = \mathrm{e}^{2\lambda}$, Theorem 1.1 now follows. □

6. Numerical Illustration of Asymptotic Harmonic Behavior

The harmonic behavior emerges in

$$R(x) = -\Phi(x) - C_2 - C_3 x^{\frac{1}{12}}. \tag{52}$$

To search for higher harmonics $Z_i(\lambda)$ associated with the zeros z_i in $\lambda \in [-26, -11.7759] = [\lambda_1 - \lambda_2, \lambda_1 + \lambda_2]$, we compare the spectrum of $\Phi\left(\mathrm{e}^{2\lambda}\right)$ by taking a Fast Fourier Transform with respect to α,

$$\lambda(\alpha) = \lambda_1 + \lambda_2 \cos\alpha \quad (\alpha \in [0, 2\pi]), \tag{53}$$

and compare the results with an analytic expression for the Fourier coefficients of the $Z_i(\lambda)$ $(i = 1, 2, \cdots)$,

$$c_{ni}[\lambda_1, \lambda_2] = 2\,\mathrm{Re}\left\{(-i)^n \gamma_i \mathrm{e}^{-i\lambda_1 z_i} J_n(-\lambda_2 z_i)\right\}, \tag{54}$$

where $J_n(z)$ denotes the Bessel function of the first of order n. **Figure 3** shows the first 21 harmonics in our evaluation of $\Phi(x)$, which is about the maximum that can be calculated by direct summation in quad precision.

7. Conclusions

The zeros $z_k = a_k + iy_k$ of the Riemann-zeta function are endpoints of continuation, defined by an expressed by a regularized sum $\Phi(x)$ over the prime numbers defined by (6).

The zeros z_k of $\zeta(z)$ introduce asymptotic harmonic behavior in $\Phi\left(\mathrm{e}^{2\lambda}\right)$ as a function of $\lambda < 0$, defined by the sum $Z(\lambda)$ of residues of the z_k, shown in **Figure 2**, **Figure 3**. Primes up to 4 billion are needed to identify the first 4 harmonics, up to 70 billion for the 10 and up to 1 trillion for the first 21. It appears that, effectively, the prime number range scales exponentially with the number of harmonics it contains.

Theorem 1.1 describes a correlation between the distribution of the primes and the distribution of the non-trivial zeros z_k. Suppose there are a finite number of zeros z_k in $\mathrm{Re}\,z > \frac{1}{2}$. We may then consider k^* for which $a_{k^*} = \max a_k$ gives rise to dominant exponential growth in $Z(\lambda)$ in the limit as $\lambda < 0$ becomes large. This observation leads to Corollary 1.2. Z can remain bounded in $x > 0$ only if the Riemann hypothesis is true, or if $Z(\lambda)$ remains fortuitously bounded as an infinite sum over $a_k > \frac{1}{2}$ with no maximum in $a < 1$. Conversely, Riemann hypothesis implies

$$\lim_{x\to 0^+} \Phi(x) = \frac{1}{2}\gamma\left(\frac{1}{2}\right) \simeq 1.3616. \tag{55}$$

According to (9) and our numerical calculation shown in **Figure 3**, the zeros z_k explored to large k by

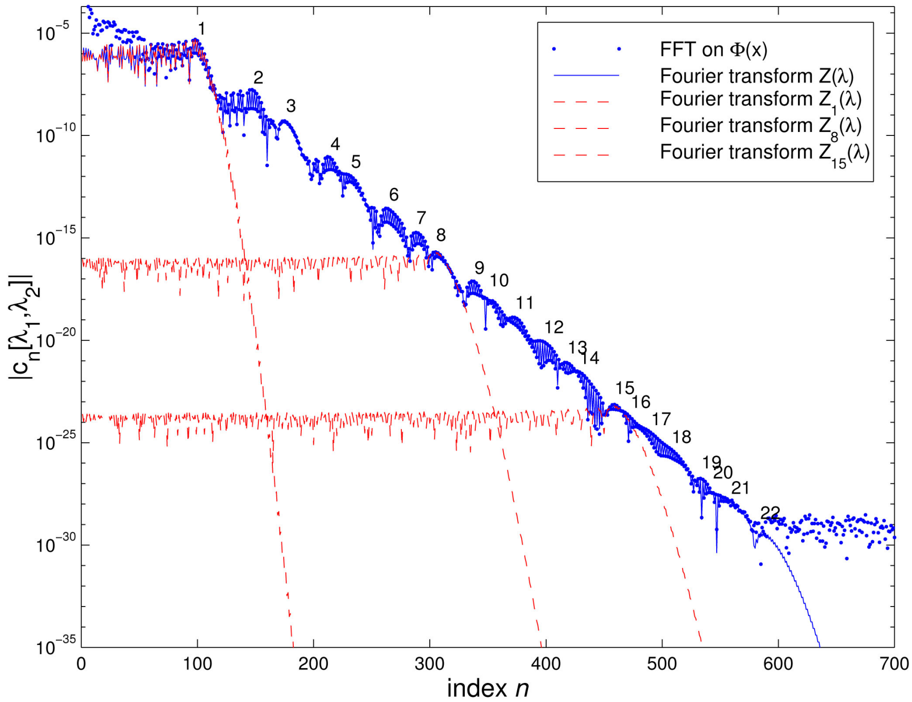

Figure 3. Shown are the absolute values of the Fourier coefficients $c_n[\lambda_1,\lambda_2]$ of $\Phi(e^{2\lambda})$ obtained by a Fast Fourier Transform (FFT) of (52) on the computational domain (53), where $\lambda_1 = -26$, $\lambda_2 = -11.7756$ covers 32 periods of $Z_1(\lambda)$ (*dots*), on the basis of the 37,607,912,2019 primes up to 1,000,000,000,0039. The resulting spectrum is compared with the exact spectra $c_{ni}[\lambda_1,\lambda_2]$ of the $Z_i(\lambda)$ given by the analytic expression (54) for $i = 1,2,3,\cdots$ (*continuous line*). Shown are also the individual spectra of $Z_i(\lambda)$ for $i = 1,8$ and 15 associated with the zeros z_1, z_8 and z_{15}. The match between the computed and exact spectra accurately identifies the first 21 harmonics of $Z(\lambda)$ in Φ out of 22 shown, corresponding to the first 21 nontrivial zeros z_i of $\zeta(z)$.

existing numerical experiments effectively probe (and constrain) a distribution in primes which extends exponentially large in *k*.

Acknowledgements

The author gratefully acknowledges stimulating discussions with Fabian Ziltener and Anton F.P. van Putten. Some of the manuscript was prepared at the Korea Institute for Advanced Study, Dongdaemun-Gu, Seoul. This research was supported in part by the National Science Foundation through TeraGrid resources provided by Purdue University under grant number TG-DMS100033. We specifically acknowledge the assistance of Vicki Halberstadt, Rich Raymond and Kimberly Dillman. The computations have been carried out using Lahey Fortran 95.

References

[1] Keller, H.B. (1987) Numerical Methods in Bifurcation Problems. Springer Verlag/Tata Institute for Fundamental Research, Berlin.

[2] Hadamard, J. (1893) Etude sur les propriétés des fonctions entières et en particulier d'une fonction. *Journal de Mathématiques Pures et Appliquées*, **9**, 171-216.

[3] von Mangoldt, H. (1985) Zu Riemann's Abhandlung "Über die Anzahl der Priemzahlen unter einer gegebenen Grösse". *Journal für die Reine und Angewandte Mathematik*, **114**, 255-305.

[4] Titchmarsh, E.C. (1986) The Theory of the Riemann Zeta-Function. 2nd Edition, Oxford.

[5] Lehmer, D.H. (1988) The Sum of Like Powers of the Zeros of the Riemann Zeta Function. *Mathematics of Computation*, **50**, 265-273. http://dx.doi.org/10.1090/S0025-5718-1988-0917834-X

[6] Dusart, P. (1999) Inégalités explicites pour $\psi(X)$, $\theta(X)$, $\pi(X)$ et les nombres premiers. *Comptes Rendus Mathematiques* (*Mathematical Reports*) *des l'Academie des Sciences*, **21**, 53-59.

[7] Keiper, J.B. (1992) Power Series Expansions of Riemann's ξ Function. *Mathematics of Computation*, **58**, 765-773.

[8] Ford, K. (2002) Zero-Free Regions for the Riemann Zeta Function. *Number Theory for the Millenium*, **2**, 25-26.

[9] Borwein, P., Choi, S., Rooney, B. and Weirathmueller, A. (2006) The Riemann Hypothesis. Springer Verlag, Berlin.

[10] Littlewood, J.E. (1922) Researches in the Theory of the Riemann ξ-Function. *Proceedings of the London Mathematical Society*, *Series* 2, **20**, 22-27.

[11] Littlewood, J.E. (1926) On the Riemann Zeta-Function. *Proceedings of the London Mathematical Society*, *Series* 2, **24**, 175-201. http://dx.doi.org/10.1112/plms/s2-24.1.175

[12] Littlewood, J.E. (1928) Mathematical Notes (5): On the Function $1/\zeta(1+ti)$. *Proceedings of the London Mathematical Society*, *Series* 2, **27**, 349-357. http://dx.doi.org/10.1112/plms/s2-27.1.349

[13] Wintner, A. (1941) On the Asymptotic Behavior of the Riemann Zeta-Function on the Line $\sigma=1$. *American Journal of Mathematics*, **63**, 575-580. http://dx.doi.org/10.2307/2371370

[14] Richert, H.E. (1967) Zur Abschätzung der Riemannschen Zetafunktion in der Nähe der Vertikalen $\sigma=1$. *Mathematische Annalen*, **169**, 97-101. http://dx.doi.org/10.1007/BF01399533

[15] Cheng, Y. (1999) An Explicit Upper Bound for the Riemann Zeta Function near the Line $\sigma=1$. *Rocky Mountain Journal of Mathematics*, **29**, 115-140. http://dx.doi.org/10.1216/rmjm/1181071682