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Countably Many Positive Solutions for Nonlinear Singular *n*-Point Boundary Value Problems

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Abstract

In this paper, a fixed-point theorem has been used to investigate the existence of countable positive solutions of *n*-point boundary value problem. As an application, we also give an example to demonstrate our results.

Keywords

Boundary Value Problem, Existence of Positive Solutions, Fixed-Point Theorem

1. Introduction

The multi-point boundary value problems arising from applied mathematics and physics have received a great deal of attention in the literature (for instance, [1]-[4] and references therein). But, by so far, few results are about the existence of more than five solutions. To the author's knowledge, there are very few papers concerned with the existence of countable positive solutions for multiple point BVPS (for instance, [5] and references therein). In [5], the authors discussed the existence of countable positive solutions of *n*-point boundary value problems for a *p*-Laplace operator on the half-line. Directly inspired by [5], in this paper, by using a fixed-point theorem, we study the existence of countable positive solutions of the following *n*-point boundary value problems.

$$u''(t)+a(t)f(u(t))=0, \qquad t\in(0,1), \tag{1.1}$$

$$u'(0)=0, \qquad u(1)=\sum_{i=1}^{n-2}a_i u(\xi_i), \tag{1.2}$$

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where $a_i \in [0,+\infty)$, $0 < \sum_{i=1}^{n-2} a_i < 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{n-2} < 1$, $f \in C([0,+\infty),[0,+\infty))$. $a(t)$: $[0,1] \to [0,+\infty)$ and $a(t) \not\equiv 0$ has countable many singularities in $\left[0,\frac{1}{2}\right)$.

This kind of problem arises in the study of a number of chemotherapy, population dynamics, ecology, industrial robotics and physics phenomena. Moreover, many problems in optimal control system, neural network (for example in BAM neural network) and information systems for computational science and engineering (especially in Internet-based computing) can be established as differential equation models with boundary condition (see, for instance, [6] and references therein).

At the end of this section, we state some definitions and lemmas which will be used in Section 2 and Section 3.

Definition 1.1 A map α is said to be a nonnegative, continuous, concave function on a cone P of a real Banach space E, if $\alpha: P \to [0,+\infty)$ is continuous, and

$$\alpha(tx+(1-t)y) \ge t\alpha(x)+(1-t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0,1]$.

Definition 1.2 Given a nonnegative continuous function γ on a cone $P \subset E$, for each $d > 0$, we define the set $P(\gamma,d) = \{x \in P: \ \gamma(x) < d\}$

Lemma 1.1 [7] Let E be a Banach space and $P \subset E$ be a cone in E. Let α, β, γ, be three increasing, nonnegative and continuous functions on P, satisfying for some $c > 0$ and $M > 0$ such that

$$\gamma(x) \le \beta(x) \le \alpha(x), \qquad \|x\| \le M\gamma(x)$$

for all $x \in \overline{P(\gamma,c)}$. Suppose that there exists a completely continuous operator $T: \overline{P(\gamma,c)} \to P$ and $0 < a < b < c$ such that

1) $\gamma(Tx) < c$, for $x \in \partial P(\gamma,c)$.
2) $\beta(Tx) > b$, for $x \in \partial P(\beta,b)$.
3) $P(\alpha,a) \ne \varnothing$, and $\alpha(Tx) < a$, for $x \in \partial P(\alpha,a)$.

Then T has at least three fixed points $x_1, x_2, x_3 \in P(\gamma,c)$ such that

$$0 \le \alpha(x_1) < a < \alpha(x_2), \qquad \beta(x_2) < b < \beta(x_3), \qquad \gamma(x_3) < c.$$

This paper is organized as follows: The preliminary lemmas are in Section 2. The main results are given in Section 3. Finally, in Section 4, we give an example to demonstrate our results.

2. The Preliminary Lemmas

In this paper, we will use the following space $E = C[0,1]$ and E is a Banach space with the norm-$\|u\| = \sup_{t\in[0,1]} |u(t)|$. Let $J = [0,1]$, we define a cone $K \subset E$ by

$$K = \{u \in E: \ u(t) \text{ is a non-increasing and nonnegative concave function on } J\}.$$

For convenience, let us list some conditions.

(H_1) $f \in C([0,+\infty),[0,+\infty))$ and on any subinterval of J and when u is bounded, $f(u(t))$ is bounded on J.

(H_2) There exists a sequence $\{t_i\}_{i=1}^{\infty}$ such that $t_{i+1} < t_i$, $t_1 < \frac{1}{2}$, $\lim_{i\to\infty} t_i = t_0$, $\lim_{t\to t_i} a(t) = \infty$, $i = 1,2,\cdots$, and $\int_0^{+\infty} a(t)\mathrm{d}t < +\infty$.

Lemma 2.1. Let $0 < \left(1-\sum_{i=1}^{n-2} a_i\right) < 0$, $h(t) \in C[0,1]$ and $h(t) \ge 0$ on $(0,1)$, then the boundary value problem

$$u''(t) + h(u(t)) = 0, \qquad t \in (0,1), \tag{2.1}$$

$$u'(0)=0, \qquad u(1)=\sum_{i=1}^{n-2} a_i u(\xi_i), \tag{2.2}$$

has a unique solution

$$u(t)=-\int_0^t (t-s)h(u(s))\mathrm{d}s+\frac{1}{1-\sum_{i=1}^{n-2} a_i}\int_0^1 (t-s)h(u(s))\mathrm{d}s-\sum_{i=1}^{n-2} a_i\int_0^{\xi_i}(\xi_i-s)h(u(s))\mathrm{d}s$$

Proof. The proof is easy, so we omit it.

By $u'(t)=-\int_0^t h(u(s))\mathrm{d}s\le 0,\ u''(t)=-h(u(s))\le 0$, we know $u(t)$ is decreasing and concave on $[0,1]$. Then we have

$$\max_{t\in[0,1]} u(t)=u(0)=\frac{\int_0^1(1-s)h(u(s))\mathrm{d}s-\sum_{i=1}^{n-2} a_i\int_0^{\xi_i}(\xi_i-s)h(u(s))\mathrm{d}s}{1-\sum_{i=1}^{n-2} a_i}\le\frac{\int_0^1 h(u(s))\mathrm{d}s}{1-\sum_{i=1}^{n-2} a_i} \tag{2.3}$$

$$\max_{t\in[0,1]} u(t)=u(1)=\frac{1}{1-\sum_{i=1}^{n-2} a_i}\left(\sum_{i=1}^{n-2} a_i\int_{\xi_i}^1(1-s)h(u(s))\mathrm{d}s+\sum_{i=1}^{n-2} a_i\int_0^{\xi_i}(1-\xi_i)h(u(s))\mathrm{d}s\right)\ge 0 \tag{2.4}$$

From (2.3), (2.4) and the concavity of $u(t)$, we can easily get the following lemma.

Lemma 2.2. Let $0<\left(1-\sum_{i=1}^{n-2} a_i\right)<0$, if $h(t)\in C[0,1]$ and $h(t)\ge 0$ on $C[0,1]$, then the unique solution $u(t)$ of (2.1)-(2.2) satisfies $u(t)\ge 0,\ t\in[0,1]$ and $\inf_{0\le t\le 1} u(t)\ge\lambda\|u\|$, where $\lambda=\frac{\sum_{i=1}^{n-2} k_i(1-\xi_i)}{1-\sum_{i=1}^{n-2} k_i\xi_i}<1$.

For $u\in K$, we define an operator $A: K\to E$ by

$$(Au)(t)=-\int_0^t (t-s)a(s)f(u(s))\mathrm{d}s+\frac{1}{1-\sum_{i=1}^{n-2} a_i}\left(\int_0^1(1-s)a(s)f(u(s))\mathrm{d}s-\sum_{i=1}^{n-2} a_i\int_0^{\xi_i}(\xi_i-s)a(s)f(u(s))\mathrm{d}s\right) \tag{2.5}$$

For $u\in K$, then $u\in E,\ \sup_{t\in J}|u(t)|<+\infty$, by (H_1), we know $f(u(t))$ is bounded on J.

So there exists $M_0\ge 0$, such that

$$\max_{s\in J} f(u(s))\le M_0. \tag{2.6}$$

It is easy to see that $(Au)(t)$ is decreasing and concave on $[0,1]$. Then for $u\in K$, we have $Au\in K$, that is

$$AK\subset K. \tag{2.7}$$

From (H_2), (2.3) and (2.6), we have

$$\max_{t\in[0,1]}(Au)(t)=(Au)(0)\le\frac{M_0\int_0^1 a(s)\mathrm{d}s}{1-\sum_{i=1}^{n-2} a_i}<+\infty. \tag{2.8}$$

From (2.7), (2.8), we can get the following lemma.

Lemma 2.3. Suppose (H_1) and (H_2) are satisfied. Then $A: K\to K$ is bounded.

Lemma 2.4. Assume $(H_1),(H_2)$ are satisfied, then $A: K\to K$ is completely continuous.

Proof. From Lemma 2.2, we know A is bounded. If $W\in K$ is a bounded subset of K, then AW is uniformly bounded on $I=[0,1]$.

For any $u \in W$, $t_1, t_2 \in [0,1]$, without loss generality, we may assume $t_2 > t_1$, by (2.5), (2.6), (H_2), we have

$$\begin{aligned}\left|(Au)(t_1)-(Au)(t_2)\right| &= \left|\int_0^{t_1}(t_1-s)a(s)f(u(s))\mathrm{d}s-\int_0^{t_2}(t_2-s)a(s)f(u(s))\mathrm{d}s\right| \\ &\le \left|\int_{t_1}^{t_2} t_2 a(s)f(u(s))\mathrm{d}s\right|+\left|(t_2-t_1)\int_0^{t_1}a(s)f(u(s))\mathrm{d}s\right|+\left|\int_{t_1}^{t_2}sa(s)f(u(s))\mathrm{d}s\right|\to 0,\end{aligned}$$

uniformly as $t_1 \to t_2$.

So AW is equi-continuous on $I=[0,1]$.

At last, by (2.5), (H_2), the Lebesgue dominated convergence theorem and continuity of f, we know A is continuous. Then by the Arzela-Ascoli theorem, we can get that $A: K \to K$ is completely continuous.

3. Main Results

Let $\theta_k \in (t_k, t_{k+1})$, $\theta_k < r_k < 1-\theta_k$ and $\gamma_k, \beta_k, \alpha_k$ be three nonnegative, decreasing and continuous functions with

$$\gamma_k(u)=\max_{r_k\le t\le 1-\theta_k} u(t),\quad \beta_k(u)=\min_{\theta_k\le t\le r_k} u(t),\quad \alpha_k(u)=\max_{\theta_k\le t\le 1-\theta_k} u(t).$$

Obviously, for $\forall u \in K$ we have $\gamma_k(u)\le \beta_k(u)\le \alpha_k(u)$.

In the following, we let

$$\rho=\left(1-\sum_{i=1}^{n-2}a_i\right)^{-1}\int_0^1 a(s)\mathrm{d}s,\quad \eta=\left(1-\sum_{i=1}^{n-2}a_i\right)^{-1}\left(\sum_{i=1}^{n-2}a_i\int_{\xi_i}^1(1-s)a(s)\mathrm{d}s\right)$$

Then it is easy to see $\rho > \eta$.

The main result of this paper is as follows.

Theorem 3.1. Assume that (H_1)-(H_2) hold. Let $\{\theta_k\}_{k=1}^{\infty}$ be such that $\theta_k \in (t_k, t_{k+1})$ $(k=1,2,\cdots)$, $\{a_k\}_{k=1}^{\infty}$, $\{b_k\}_{k=1}^{\infty}$ and $\{c_k\}_{k=1}^{\infty}$ be such that $c_{k+1}<a_k<\lambda b_k<c_k$ and $\rho b_k < \eta c_k$. $(k=1,2,\cdots)$.

Furthermore for each natural number k we assume that f satisfies:

(H_3) $f(u)<\dfrac{c_k}{\rho}$ for all $0\le u(t)\le c_k/\lambda$;

(H_4) $f(u)<\dfrac{b_k}{\eta}$ for all $b_k\le u(t)\le b_k/\lambda$

(H_5) $f(u)<\dfrac{a_k}{\rho}$ for all $0\le u(t)\le a_k/\lambda$.

Then the BVP (1.1)-(1.2) has at least three infinite families of positive solutions $\{u_{1k}\}_{k=1}^{\infty}$, $\{u_{2k}\}_{k=1}^{\infty}$ and $\{u_{3k}\}_{k=1}^{\infty}$ with

$$0\le \alpha_k(u_{1k})<a_k<\alpha_k(u_{2k}),\quad \beta_k(u_{2k})<b_k<\beta_k(u_{3k}),\quad \gamma_k(u_{3k})<c_k,\ \text{for } k\in N.$$

Proof. From the definition of A, (2.7) and Lemma 2.4, it is easy to see that $A: \overline{K(\gamma_k, c_k)} \to K$, for $k \in N$ is completely continuous.

Next we show all the conditions of Lemma 1.2 hold.

For any $u \in K$, it is easy to see $\gamma_k(u)\le\beta_k(u)\le\alpha_k(u)$. From Lemma 2.2, we have

$$\gamma_k(u)=\max_{r_k\le t\le 1-\theta_k}u(t)\ge \inf_{0\le t\le 1}u(t)\ge \lambda(t)\|u\|,\ \text{so}\ \|u\|\le \lambda^{-1}\gamma_k(u) \tag{3.1}$$

First, we choose $u\in\partial K(\gamma_k,c_k)$, then we have $\gamma_k(u)=\max_{r_k\le t\le 1-\theta_k}u(t)=u(r_k)=c_k$. From $u(t)\le\|u\|$ and (3.1), we can get $0\le u(t)\le\|u\|\le\lambda^{-1}u(r_k)=\lambda^{-1}c_k$, for $t\in J$. Then with (H_3), it implies that $f(u)<\dfrac{c_k}{\rho}$, for

$t \in J$.

$$\text{So} \quad \gamma_k(Au) = \max_{r_k \le t \le 1-\theta_k} (Au)(t) = Au(r_k) \le (Au)(0) < \frac{c_k}{\rho} \frac{\int_0^1 a(s)\,\mathrm{d}s}{1-\sum_{i=1}^{n-2} a_i} = c_k$$

Therefore, the first condition of Lemma 1.2 satisfies.

Next, we select $u \in \partial(\beta_k, b_k)$. Then $\beta_k(u) = \min_{\theta_k \le t \le r_k} u(t) = u(r_k) = b_k$, we have $u(t) \ge b_k$, for $\theta_k \le t \le r_k$. Again from $u(t) \le \|u\|$, and Lemma (2.2) we can get that

$$u(t) \le \|u\| \le \lambda^{-1} u(r_k) = b_k/\lambda.$$

Then $b_k \le u(t) \le b_k/\lambda$, for $\theta_k \le t \le r_k$. By (H_4), we have $f(u) > \frac{b_k}{\eta}$, for $\theta_k \le t \le r_k$.

So, there has

$$\begin{aligned}\beta_k(Au) &= \min_{r_k \le t \le \theta_k} (Au)(t) \ge Au(1) \ge \left(1-\sum_{i=1}^{n-2} a_i\right)^{-1} \left(\sum_{i=1}^{n-2} a_i \int_{\xi_i}^1 (1-s) a(s) f(u(s))\,\mathrm{d}s\right) \\ &> \frac{b_k}{\eta}\left(1-\sum_{i=1}^{n-2} a_i\right)^{-1} \left(\sum_{i=1}^{n-2} a_i \int_{\xi_i}^1 (1-s) a(s)\,\mathrm{d}s\right) = b_k.\end{aligned}$$

This implies the second condition of Lemma 1.2 is satisfied.

Finally, we only need to show the third condition of Lemma 1.2 is also satisfied.

We select $u(t) \equiv a_k/2$, for $t \in J$. Obviously, $u(t) \in K(\alpha_k, a_k)$, hence $K(\alpha_k, a_k)$ is nonempty. $\forall u(t) \in \partial(\alpha_k, a_k)$, we have $\alpha_k(u) = \max_{\theta_k \le t \le 1-\theta_k} u(t) = u(\theta_k) = a_k$. Also from $u(t) \le \|u\|$ and Lemma (2.4), we can get $0 \le u(t) \le \|u\| \le \lambda^{-1} u(\theta_k) = \lambda^{-1} a_k$, for $t \in J$. Then from (H_5), we have $f(u) < \frac{a_k}{\rho}$.

$$\text{So} \quad \alpha_k(Au) = \max_{\theta_k \le t \le 1-\theta_k} (Au)(t) = (Au)(\theta_k) \le (Au)(0) < \frac{a_k}{\rho}\left(1-\sum_{i=1}^{n-2} a_i\right)^{-1} \int_0^1 a(s)\,\mathrm{d}s = a_k.$$

Then all the conditions of Lemma 1.2 are satisfied. From Lemma 1.2, we get the conclusion in Theorem 3.1.

4. Example

Now we consider an example to illustrate our results.

Example 4.1. Consider the boundary value problem

$$x'' + a(t) f(u(t)) = 0, \quad t \in (0,1), \tag{4.1}$$

$$u'(0) = 0, \quad u(1) = \frac{1}{2} u\left(\frac{1}{2}\right), \tag{4.2}$$

Then the BVP (4.1)-(4.2) can be regarded as a BVP of the form (1.1)-(1.2) in $E = C[0,1]$. In this situation, $a_1 = \xi_1 = \frac{1}{2}$.

Let $\delta = \frac{\pi^2}{3} - \frac{9}{4}$, $t' = \frac{3}{8}$, $t_i = t' - \sum_{k=1}^{i} \frac{1}{4(k+1)^4}$, $i = 1, 2, \cdots$.

Consider the function $a(t)$: $[0,1] \to [0,+\infty)$, $a(t) = \sum_{i=1}^{\infty} a_i(t)$, $t \in [0,1]$, where

$$a_i(t)=\begin{cases}\dfrac{1}{i^2\left(t_{i+1}+t_i\right)}, & 0\le t<\dfrac{t_{i+1}+t_i}{2}\\ \dfrac{\sqrt{2}}{\delta\left(t_i-t\right)^{\frac{1}{2}}}, & \dfrac{t_{i+1}+t_i}{2}\le t<t_i,\\ \dfrac{\sqrt{2}}{\delta\left(t-t_i\right)^{\frac{1}{2}}}, & t_i<t\le\dfrac{t_{i-1}+t_i}{2},\\ 0, & \dfrac{t_{i-1}+t_i}{2}<t<1.\end{cases}$$

$$\int_0^1 a(t)\mathrm{d}t=\sum_{i=1}^{+\infty}\int_0^1 a_i(t)\mathrm{d}t=\sum_{i=1}^{+\infty}\left(\frac{1}{2i^2}+\frac{1}{\delta(i+2)^2}+\frac{1}{\delta(i+1)^2}\right)=\frac{\pi^2}{12}+1<+\infty.$$

It is easy to know (H_2) satisfies.

Let $\theta_k\in\left(t_k,t_{k+1}\right)$, $\theta_k<r_k<1-\theta_k$, $\{\theta_k\}_{k=1}^{\infty}$ be such that $\theta_k\in\left(t_k,t_{k+1}\right)$ $(k=1,2,\cdots)$, $\{a_k\}_{k=1}^{\infty}$, $\{b_k\}_{k=1}^{\infty}$ and $\{c_k\}_{k=1}^{\infty}$ be such that $c_{k+1}<a_k<\lambda b_k<c_k$, and $\rho b_k<\eta c_k$ $(k=1,2,\cdots)$.

This with $\lambda<1,\ \rho>\eta$ implies that $\dfrac{c_{k+1}}{\lambda}<\dfrac{a_k}{\lambda}<b_k<\dfrac{c_k}{\lambda}$, $\dfrac{a_k}{\rho}<\dfrac{b_k}{\eta}<\dfrac{c_k}{\rho}$, $(k=1,2,\cdots)$.

$$\text{Let}\quad f(u)=\begin{cases}\dfrac{a_k}{2\rho}, & 0\le u(t)\le\dfrac{a_k}{\lambda},\\ \dfrac{1}{2}\left[\dfrac{a_k}{\rho}+\dfrac{(u(t)-\lambda^{-1}a_k)(\frac{b_k}{\eta}+\frac{c_k-a_k}{\rho})}{\left(b_k-\lambda^{-1}a_k\right)}\right], & \dfrac{a_k}{\lambda}<u(t)\le b_k,\\ \dfrac{1}{2}\left(\dfrac{b_k}{\eta}+\dfrac{c_k}{\rho}\right), & u(t)>b_k.\end{cases}$$

Obviously, (H_3), (H_4), (H_5) are satisfied, and it is easy to prove that (H_1) is also satisfied. So all the conditions of Theorem 3.1 are satisfied, thus the BVP (4.1)-(4.2) has at least three infinite families of positive solutions $\{u_{1k}\}_{k=1}^{\infty}$, $\{u_{2k}\}_{k=1}^{\infty}$ and $\{u_{3k}\}_{k=1}^{\infty}$ satisfying

$$0\le\alpha_k\left(u_{1k}\right)<a_k<\alpha_k\left(u_{2k}\right),\quad \beta_k\left(u_{2k}\right)<b_k<\beta_k\left(u_{3k}\right),\quad \gamma_k\left(u_{3k}\right)<c_k,\ \text{for}\ \ k\in N.$$

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