

# A Monotonicity Condition for Strong Convergence of the Mann Iterative Sequence for Demicontractive Maps in Hilbert Spaces

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## Abstract

Let  $H$  be a real Hilbert space and  $C$  be a nonempty closed convex subset of  $H$ . Let  $T : C \rightarrow C$  be a demicontractive map satisfying  $\langle Tx, x \rangle \geq \|x\|^2$  for all  $x \in D(T)$ . Then the Mann iterative sequence given by  $x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n$ , where  $\alpha_n \in (0, 1) \forall n \geq 0$ , converges strongly to an element of  $F(T) := \{x \in C : Tx = x\}$ . This strong convergence is obtained without the compactness-type assumptions on  $C$ , which many previous results (see e.g. [1]) employed.

## Keywords

Demicontractive Maps, Mann Iterative Sequence, Strong Convergence, Monotonicity, Hilbert Spaces

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## 1. Introduction

Let  $H$  be a real Hilbert space. A mapping  $T : H \rightarrow H$  is said to be demicontractive if there exists a constant  $k > 0$  such that

$$\|Tx - p\|^2 \leq \|x - p\|^2 + k\|x - Tx\|^2 \quad (1.1)$$

for all  $(x, p) \in H \times F(T)$ , where  $F(T) := \{x \in H : Tx = x\} \neq \emptyset$ . More often than not,  $k$  is assumed to be in

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the interval  $(0,1)$ . However, this is a restriction of convenience. If  $k = 1$ , then  $T$  is called a hemicontractive map.

On the otherhand,  $T$  is said to satisfy condition (A) if there exists  $\lambda > 0$  such that

$$\langle x - Tx, x - p \rangle \geq \lambda \|x - Tx\|^2, \tag{1.2}$$

for all  $(x, p) \in H \times F(T)$ . Inequality (1.2) is equivalent to

$$\langle Tx - p, x - p \rangle \leq \|x - p\|^2 - \lambda \|x - Tx\|^2. \tag{1.3}$$

The above classes of maps were studied independently by Hicks and Kubicek [2] and Maruster [3]. It is however shown in [4] that the two classes of maps coincide if  $k \in (0,1)$  and  $\lambda \in \left(0, \frac{1}{2}\right)$ .

The class of demicontractive maps includes the class of quasi-nonexpansive and the class of strictly pseudocontractive maps. Any strictly pseudocontractive mapping with a nonempty fixed point set is demicontractive.

If  $C$  is a closed convex subset of any Banach space  $E$  and  $T : C \rightarrow C$  is any map, then the Mann iteration sequence [5] is given by  $x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n$ , where  $\alpha_n \in (0,1) \forall n \geq 0$ , satisfying certain conditions. Several authors (see e.g. [1]-[3] [5] [6]) have studied the convergence of the Mann iteration sequence to fixed points of certain mappings in certain Banach spaces. However, the Mann iteration sequence is very suitable for the study of convergence to fixed points of demicontractive mappings. It is well known (see e.g. [4]) that demicontractivity alone is not sufficient for the convergence of the Mann iteration sequence. Some additional smoothness properties of  $T$  such as continuity and demiclosedness are necessary.

A map  $T$  is said to be demiclosed at a point  $x_0$  if whenever  $\{x_n\}$  is a sequence in the domain of  $T$  such that  $\{x_n\}$  converges weakly to  $x_0 \in D(T)$  and  $\{Tx_n\}$  converges strongly to  $y_0$ , then  $Tx_0 = y_0$ .

In [7], Maruster studied the convergence of the Mann iteration sequence for demicontractive maps, in finite dimensional spaces, with an application to the study of the so-called relaxation algorithm for the solution of a particular convex feasibility problem. More precisely, he proved the following:

**Theorem 1 [7]:** Let  $T : \mathfrak{R}^m \rightarrow \mathfrak{R}^m$  be a nonlinear mapping, where  $\mathfrak{R}^m$  is the  $m$ -Euclidean space. Suppose the following are satisfied:

- 1)  $I - T$  is demiclosed at 0.
- 2)  $T$  is demicontractive with constant  $k$ , or equivalently  $T$  satisfies condition A with  $\lambda = \frac{1-k}{2}$ .

Then the Mann iteration sequence converges to a point of  $F(T)$  for any starting  $x_0$ .

Maruster [4] noted that in infinite dimensional spaces, demicontractivity and demiclosedness of  $T$  are not sufficient for strong convergence. However, the two conditions ensure weak convergence. More precisely, he proved the following:

**Theorem 2 [3]:** Let  $T : C \rightarrow C$  be a nonlinear mapping with  $F(T) \neq \emptyset$ , where  $C$  is a closed convex subset of a real Hilbert space  $H$ . Suppose the following conditions are satisfied:

- 1)  $I - T$  is demiclosed at 0.
- 2)  $T$  is demicontractive with constant  $k$ , or equivalently  $T$  satisfies condition A with  $\lambda = \frac{1-k}{2}$ .
- 3)  $0 < a \leq \alpha_n \leq b < 2\lambda = 1 - k$ .

Then the Mann iteration sequence converges weakly to a fixed point of  $F(T)$ , for any starting  $x_0$ .

## 2. Strong Convergence

As noted above, demicontractivity and demiclosedness of  $T$  are not sufficient for strong convergence of the Mann iteration sequence in infinite dimensional spaces. Some additional conditions on  $T$ , or some modifications of the Mann iteration sequence are required for strong convergence to fixed points of demicontractive maps. Such additional conditions or modifications have been studied by several authors (see e.g. [1] [2] [6] [8] [9]).

There is however an interesting connection between the strong convergence of the Mann iteration sequence to a fixed point of a demicontractive map,  $T$ , and the existence of a non-zero solution of a certain variational inequality. This connection was observed by Maruster [3], and has been studied by several authors. More precisely,

Maruster proved the following theorem:

**Theorem 3 [3]:** Suppose  $T$  satisfies the conditions of Theorem 2. If in addition there exists  $0 \neq h \in H$  such that

$$\langle x - Tx, h \rangle \leq 0 \quad (1.4)$$

for all  $x \in D(T)$ , then starting from a suitable  $x_0$ , the Mann iteration sequence converges strongly to an element of  $F(T)$ .

The conditions of/and the variational inequality in Theorem 3 have been used and generalized by several authors (see e.g. [8] [9]). The existence of a non-zero solution to the variational inequality is sometimes gotten under very stringent conditions. In [4] remark 4, Maruster and Maruster made the following observation "It would therefore be interesting to study more closely the existence of a non-zero solution of the variational inequality".

The purpose of this paper is to provide a monotonicity condition under which the Mann iteration sequence converges strongly to a fixed point of a demicontractive map. The convergence does not need to pass through the variational inequality (1.4). The condition is embodied in the following theorem:

Before we state and prove our theorem, we give the following definition which will be useful in the sequel.

**Definition 1:** Let  $H$  be a real Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and norm  $\|\cdot\|$  and let  $C$  be a nonempty closed convex subset of  $H$ . The orthogonal projection  $P_C x$  of  $H$  onto  $C$  is defined by  $P_C x = \arg \min_{y \in C} \|x - y\|$ , and has the following properties:

- 1)  $\langle x - P_C x, z - P_C x \rangle \leq 0$ , for all  $z \in C$ .
- 2)  $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle$ , for all  $x, y \in H$ .

**Theorem 4:** Suppose  $T$  satisfies:

- 1) The conditions of Theorem 2.
- 2)  $\langle Tx, x \rangle \geq \|x\|^2$  for all  $x \in D(T)$ . Then starting from a suitable  $x_0$ , the Mann iteration sequence converges strongly to an element of  $F(T)$ .

**Proof.** Choose  $x_0 \in C$  such that  $\langle x_0 - p, p \rangle \geq 0$ . This implies there exists  $\epsilon_0 > 0$  such that  $\langle x_0 - p, p \rangle \geq \epsilon_0 \|x_0 - p\|^2$ . Suppose  $\langle x_n - p, p \rangle \geq \epsilon_0 \|x_n - p\|^2$ . Then using (1.3) and condition (ii) of Theorem 4, we have

$$\begin{aligned} \langle x_{n+1} - p, p \rangle &= \langle (1 - \alpha_n)[x_n - p] + \alpha_n [Tx_n - p], p \rangle \\ &= (1 - \alpha_n) \langle x_n - p, p \rangle + \alpha_n \langle Tx_n - p, p \rangle \\ &= (1 - \alpha_n) \langle x_n - p, p \rangle + \alpha_n \langle Tx_n - x_n, p \rangle + \alpha_n \langle x_n - p, p \rangle \\ &= \langle x_n - p, p \rangle + \alpha_n \langle Tx_n - x_n, p \rangle \\ &= \langle x_n - p, p \rangle + \alpha_n [\langle Tx_n - p + p - x_n, -x_n + p \rangle + \langle Tx_n - x_n, x_n \rangle] \\ &= \langle x_n - p, p \rangle - \alpha_n \langle Tx_n - p, x_n - p \rangle + \alpha_n \langle x_n - p, x_n - p \rangle + \alpha_n \langle Tx_n - x_n, x_n \rangle \\ &\geq \langle x_n - p, p \rangle - \alpha_n [\|x_n - p\|^2 - \lambda \|x_n - Tx_n\|^2] + \alpha_n \|x_n - p\|^2 + \alpha_n \langle Tx_n - x_n, x_n \rangle \\ &= \langle x_n - p, p \rangle + \lambda \alpha_n \|x_n - Tx_n\|^2 + \alpha_n \langle Tx_n - x_n, x_n \rangle \\ &\geq \langle x_n - p, p \rangle \geq \epsilon_0 \|x_n - p\|^2 \geq \epsilon_0 \|x_{n+1} - p\|^2 \end{aligned}$$

Since  $x_n \rightharpoonup p$  from Theorem 2, then  $x_n \rightarrow p$ .

**Example:** Let  $H = \mathbb{R}$  (reals) and  $C = [0, 2]$  be a nonempty closed convex subset of  $H$ . Define  $T : C \rightarrow C$  by

$$Tx = \begin{cases} 2, & \text{if } x < \frac{1}{3} \\ x - \frac{5}{2}(x-1), & \text{if } \frac{1}{3} \leq x < 1 \\ x, & \text{if } x \geq 1 \end{cases}$$

Then it is easily verifiable that  $T$  is demicontractive and satisfies condition (ii) of our theorem for

$$0 < \lambda \leq \frac{2}{5}.$$

**Remark 1:** In [4] Maruster and Maruster noted that if  $T$  satisfies the positivity type condition  $\langle Tx, x \rangle \geq \|x\|^2$ , then it is sufficient to find a non-zero solution of the variational inequality (1.4). This motivates the condition in our theorem. As a matter of fact, our theorem is a necessity result.

**Remark 2:** We note that one of the ways of choosing  $x_0$  is as follows: For any  $\beta > 1$ , choose  $x_0 = P_C(\beta p)$ , where  $p \in F(T)$  and  $P: H \rightarrow C$  is the metric projection from  $H$  into  $C$ . This follows since it is well known (see Definition 1) that  $P$  is firmly nonexpansive (i.e. satisfies condition (ii) of Definition 1), so that

$$\begin{aligned} \|x_0 - p\|^2 &= \|P_C(\beta p) - P(p)\|^2 \leq \langle P_C(\beta p) - P_C(p), \beta p - p \rangle \\ &= \langle x_0 - p, (\beta - 1)p \rangle = (\beta - 1)\langle x_0 - p, p \rangle \end{aligned}$$

This implies  $\langle x_0 - p, p \rangle \geq \varepsilon_0 \|x_0 - p\|^2$ , where  $\varepsilon_0 = \frac{1}{\beta - 1}$ .

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