

A Semi-Analytical Method for Solutions of a Certain Class of Second Order Ordinary Differential Equations

S. O. Edeki^{1*}, H. I. Okagbue¹, A. A. Opanuga¹, S. A. Adeosun²

¹Department of Mathematics, College of Science & Technology, Covenant University, Ota, Nigeria

²Department of Mathematical Sciences, Crescent University, Abeokuta, Nigeria

Email: *soedeki@yahoo.com

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Abstract

This paper presents the theory and applications of a new computational technique referred to as Differential Transform Method (DTM) for solving second order linear ordinary differential equations, for both homogeneous and nonhomogeneous cases. For the robustness and efficiency of the method, four examples are considered. The results indicate that the DTM is reliable and accurate when compared to the exact solutions of the solved problems.

Keywords

Differential Transform Method, Second Order ODEs, Linear Systems, Series Solution

1. Introduction

Most of the problems encountered in applied sciences, management and economics take the forms of second order linear ordinary differential equations. Sometimes, obtaining exact solutions through the direct methods (analytical methods) for these systems seems difficult even if the exact solutions exist, hence the need for numerical techniques for approximate solutions. Some of these numerical methods involve linearization, discretization and perturbation, and they only permit the solutions to a given ODE at a certain interval. In addition, they are intensive in terms of computation and as such, lead to the situations where some basic phenomena are technically avoided.

The notion of DTM was first introduced by Zhou [1] while solving linear and nonlinear initial value problems in electric circuit analysis. The Method provides an analytical approximate solution to linear and nonlinear

*Corresponding author.

systems of differential equations. This involves the construction of a semi-analytical technique with the aid of a Taylor series expansion for the solutions in polynomial forms. The DTM has been proven by many researchers [1]-[7] to be very effective and efficient in handling differential equations with point boundary value problems, kvd and mkdv differential-algebraic equations, and so on. It also reduces the size of computational work while maintaining high accuracy even with a fast convergence rate compared to the theoretical solutions [8].

Some other analytical methods like the variational iterative method and the Adomian decomposition method witness difficulties in handling functions involving complicated integrals; since successive component using those methods depends on the previous component [9]-[13] but DTM overcomes this by solving an algebraic recursive equation.

2. The Differential Transform Method

This section introduces the basic concepts and theorems of DTM needed for applications in the remaining sections.

Definition 1. Let $f(x)$ be a given function of one variable defined at a point $x = x_0$, then the one-dimensional k^{th} differential transform of the $f(x)$ defined as $F(k)$ is given as:

$$F(k) = \frac{1}{k!} \left(\frac{d^k f(x)}{dx^k} \right) \Big|_{x=x_0} \tag{1}$$

Equation (1) is the transformed function of $f(x)$.

Definition 2. The differential inverse transform of $F(k)$ is a Taylor series expansion of the function $f(x)$ about $x = x_0 = 0$, defined as :

$$f(x) = \sum_{k=0}^{\infty} F(k) x^k \tag{2}$$

Combining (1) and (2) yields:

$$f(x) = \sum_{k=0}^{\infty} \left(\frac{d^k f(x)}{dx^k} \right) \frac{x^k}{k!} \tag{3}$$

2.1. Some Basic Theorems of the Differential Transform Method

The following theorems and properties of the DTM are stated below for the sake of applications, their proofs can be found in [14] and [15].

Let $f_1(x)$, $f_2(x)$ and $f_*(x)$ be differentiable functions with differential transforms $F_1(k)$, $F_2(k)$ and $F_*(k)$ respectively, with, $\alpha \in \mathbb{R}$ and δ a Kronecker delta, then the following theorems hold:

Theorem 2.1 If $y = f_1(x) \pm f_2(x)$ then $Y(k) = F_1(k) \pm F_2(k)$

Theorem 2.2 If $y = \alpha f_*(x)$ then $Y(k) = \alpha F_*(k)$

Theorem 2.3 If $y = x^n$ then $Y(k) = \delta(k - n)$ such that:

$$Y(k) = \delta(k - n) = \begin{cases} 1, & k = n \\ 0, & \text{otherwise} \end{cases}$$

Theorem 2.4 If $y = e^{\alpha x}$ then $Y(k) = \frac{\alpha^k}{k!}$, where α is a constant.

Theorem 2.5 If $y = \frac{d^n}{dx^n} [f_*(x)]$ then $Y(k) = \frac{(k+n)!}{n!} F_*(k+n)$.

In particular, we have:

(a) If $y = \frac{d}{dx} [f_*(x)]$ then $Y(k) = (k+1)F_*(k+1)$

(b) If $y = \frac{d^2}{dx^2} [f_*(x)]$ then $Y(k) = (k+1)(k+2)F_*(k+2)$

2.2. The DTM and the Second order Linear Ordinary Differential Equations (ODEs)

In this section, we present clearly how a second order linear ODE with constant coefficients is transformed using the DTM.

The corresponding ODE is of the form:

$$ay''(x) + by'(x) + cy(x) = f(x), \quad a \neq 0 \tag{4}$$

with initial conditions $y(0) = \alpha_1$ and $y'(0) = \alpha_2$

Equation (4) is re-expressed in the form of

$$y''(x) = g(x, y(x), y'(x)) \tag{5}$$

Thus, (4) becomes

$$y''(x) = \frac{1}{a} [f(x) - cy(x) - by'(x)] \tag{6}$$

We will take the differential transform (DT) of (6) by applying theorems (2.1-2.6) as follows:

$$DT \left[y''(x) = \frac{1}{a} [f(x) - by'(x) - cy(x)] \right] \tag{7}$$

$$(k+1)(k+2)Y(k+2) = \frac{1}{a} [F(k) - b(k+1)Y(k+1) - cY(k)] \tag{8}$$

$$Y(k+2) = \frac{1}{a(k+1)(k+2)} [F(k) - b(k+1)Y(k+1) - cY(k)] \tag{9}$$

subject to the initial conditions $Y(0) = Y_0 = \alpha_1$ and $Y'(0) = Y(1) = \alpha_2$.

Equation (9) is a recursive formula for the computation of coefficient terms in the series solution of the problem. Therefore, using (2) and (9) gives the approximate solution of (4) as:

$$y(x) = \sum_{k=0}^{\infty} Y(k)x^k \tag{10}$$

3. Applications and Numerical Results

In this section, we will apply the discussed DTM to solve some problems whose results will be compared with the theoretical (exact) solutions. Two cases with two examples each are considered. Case I and Case II for homogeneous and nonhomogeneous respectively.

Case I Example 1: Consider the ODE

$$y''(x) - 4y'(x) + 3y(x) = 0 \tag{11}$$

subject to $y(0) = 1, y'(0) = 1$ with a theoretical solution

$$y(x) = e^x \tag{12}$$

Procedure

We rewrite (11) in a standard form and take the differential transform (DT) as follows:

$$DT [y''(x) = 4y'(x) - 3y(x)] \tag{13}$$

$$\Rightarrow (k+1)(k+2)Y(k+2) = 4(k+1)Y(k+1) - 3Y(k) \tag{14}$$

$$\therefore Y(k+2) = \frac{1}{(k+1)(k+2)} [4(k+1)Y(k+1) - 3Y(k)] \tag{15}$$

with the initial conditions $Y(0) = 1, Y(1) = 1$.

By using the recursive relation in (15) with $k \geq 0$, we obtain values for $Y(2), Y(3), Y(4), \dots$, as showed below:

For $k = 0, Y(2) = \frac{1}{2!}$; for $k = 1, Y(3) = \frac{1}{3!}$; for $k = 2, Y(4) = \frac{1}{4!}$; for $k = 3, Y(5) = \frac{1}{5!}, \dots$; for $k = n, Y(n) = \frac{1}{n!}$

But from (10),

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} Y(k)x^k \\ y(x) &= Y(0) + Y(1)x + Y(2)x^2 + Y(3)x^3 + Y(4)x^4 + \dots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots = e^x \end{aligned} \quad (16)$$

Equation (16) is the same with the theoretical solution in (12).

Case I Example 2: Consider the ODE

$$y''(x) + y(x) = 0 \quad (17)$$

subject to $y(0) = 1$ and $y'(0) = 0$, with a theoretical solution

$$y(x) = \cos x \quad (18)$$

Procedure

We re-write (16) in a standard form and take the differential transform as follows:

$$\begin{aligned} DT[y''(x) + y(x)] &= 0 \\ \therefore (k+1)(k+2)Y(k+2) &= -Y(k) \\ \Rightarrow Y(k+2) &= \frac{-Y(k)}{(k+1)(k+2)} \end{aligned} \quad (19)$$

with the initial conditions $Y(0) = 1, Y(1) = 0$.

Computing $Y(2), Y(3), Y(4), \dots$, for $k \geq 0$ using (19) gives the following:

For $k = 0, Y(2) = \frac{-1}{2!}$; for $k = 1, Y(3) = 0$; for $k = 2, Y(4) = \frac{1}{4!}$; for $k = 3, Y(5) = 0$, for $k = 4, Y(6) = \frac{-1}{6!}$,
 \dots ; for $k = n, Y(2n+2) = \frac{(-1)^n}{(2n+2)!}$

We observed that $Y(1) = Y(3) = Y(5) = \dots = Y(2n+1) = 0$ for $k \geq 0$. Hence, the solution of (17) by (10) is reformed as:

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} Y(2k)x^{2k} = Y(0) + Y(2)x^2 + Y(4)x^4 + Y(6)x^6 \dots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \cos x \end{aligned} \quad (20)$$

Equation (20) agrees with the theoretical solution in (18).

Case II Example 1: Consider the ODE

$$y''(x) - y(x) = x \quad (21)$$

subject to $y(0) = 1$ and $y'(0) = 0$, with a theoretical solution:

$$y(x) = e^x - x \quad (22)$$

Procedure

Equation (21) by the differential transform method becomes;

$$\begin{aligned} DT[y''(x) - y(x) + x] &= 0 \\ \Rightarrow Y(k+2) &= \frac{1}{(k+1)(k+2)} [Y(k) + \delta(k-1)] \end{aligned} \quad (23)$$

with the initial conditions $Y(0) = 1, Y(1) = 0$.

Thus, for $k \geq 0$, the values of $Y(2), Y(3), Y(4), \dots$, obtained using (23) are given below:

For $k = 0, Y(2) = \frac{1}{2!}$; for $k = 1, Y(3) = \frac{1}{3!}$; for $k = 2, Y(4) = \frac{1}{4!}$; for $k = 3, Y(5) = \frac{1}{5!}$, for $k = 4, Y(6) = \frac{1}{6!}$
 \dots ; for $k = n, Y(n) = \frac{1}{n!}$.

As such, from (10), the solution of (21) is:

$$y(x) = \sum_{k=0}^{\infty} Y(k)x^k = Y(0) + Y(1)x + Y(2)x^2 + Y(3)x^3 + Y(4)x^4 + \dots$$

$$= 1 + 0 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots = 1 + \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
(24)

Since

$$\sum_{n=0}^{\infty} \beta^n = \sum_{n=0}^v \beta^n + \sum_{n=v+1}^{\infty} \beta^n$$
(25)

$$\therefore y(x) = 1 + \sum_{n=2}^{\infty} x^n = -x + e^x$$
(26)

Equation (26) is the same with the theoretical solution in (22).

Case II Example 2: Consider the ODE

$$y''(x) - 3y'(x) + 2y(x) = e^x$$
(27)

subject to $y(0) = 1$ and $y'(0) = 0$, with a theoretical solution:

$$y(x) = (1-x)e^x$$
(28)

Procedure

Equation (27) by the differential transform method becomes;

$$DT[y''(x) - 3y'(x) + 2y(x) = e^x]$$

$$\Rightarrow Y(k+2) = \frac{1}{(k+1)(k+2)} \left[3(k+1)Y(k+1) - 2Y(k) + \frac{1}{k!} \right]$$
(29)

subject to the initials $Y(0) = 1, Y(1) = 0$.

Thus, for $k \geq 0$, the values of $Y(2), Y(3), Y(4), \dots$, obtained using (29) are given below:

For $k = 0, Y(2) = \frac{-1}{2!}$; for $k = 1, Y(3) = \frac{-2}{3!}$; for $k = 2, Y(4) = \frac{-3}{4!}$; for $k = 3, Y(5) = \frac{-4}{5!}$, for
 $k = 4, Y(6) = \frac{1}{6!}$; \dots ; for $k = n, Y(n) = \frac{-(n-1)}{n!} = \frac{1}{n!} - \frac{1}{(n-1)!}$.

Hence, from (10), the solution of (27) is:

$$y(x) = \sum_{k=0}^{\infty} Y(k)x^k = Y(0) + Y(1)x + Y(2)x^2 + Y(3)x^3 + Y(4)x^4 + \dots$$

$$= 1 + 0 - \frac{x^2}{2!} - \frac{2x^3}{3!} - \frac{3x^4}{4!} - \frac{4x^5}{5!} - \frac{5x^6}{6!} \dots = 1 - \sum_{n=2}^{\infty} (n-1) \frac{x^n}{n!}$$

$$= -\sum_{n=0}^{\infty} (n-1) \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} - x \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} - x \sum_{m=0}^{\infty} \frac{x^m}{(m)!}, \text{ for } m = n-1$$

$$\therefore y(x) = e^x - xe^x$$
(30)

Equation (30) corresponds with the theoretical solution in (28).

Remark 3.1:

We present numerical comparisons between the exact, and the numerical solutions based on a 5-iterate DTM using the coefficient terms from **Table 1**, **Table 2**, **Table 3**, and **Table 4** as shown in **Table 5**, **Table 6**, **Table 7**, and **Table 8** respectively.

4. Discussion of Results and Conclusion

In this paper, we have presented a semi-analytical method (DTM) for solving a certain class of ODEs. The DTM has advantages over other numerical techniques as it does not involve linearization, discretization or perturbation of a given problem; hence it has no effect of computational round off error. The DTM also provides a closed-form solution; therefore, it is very powerful and effective in finding both analytical and numerical solutions of second order linear ODEs with constant coefficients.

Table 1. Coefficient terms from Equation (15).

k	0	1	2	3	4	...	n
$Y(k+2)$	$Y(2) = \frac{1}{2!}$	$Y(3) = \frac{1}{3!}$	$Y(4) = \frac{1}{4!}$	$Y(5) = \frac{1}{5!}$	$Y(6) = \frac{1}{6!}$...	$Y(n) = \frac{1}{n!}$

Table 2. Coefficient terms from Equation (19).

k	0	1	2	3	4	...	n
$Y(k+2)$	$Y(2) = \frac{-1}{2!}$	$Y(3) = 0$	$Y(4) = \frac{1}{4!}$	$Y(5) = 0$	$Y(6) = \frac{-1}{6!}$...	$Y(2n+2) = \frac{(-1)^n}{(2n+2)!}$

Table 3. Coefficient terms from Equation (23).

k	0	1	2	3	4	...	n
$Y(k+2)$	$Y(2) = \frac{1}{2!}$	$Y(3) = \frac{1}{3!}$	$Y(4) = \frac{1}{4!}$	$Y(5) = \frac{1}{5!}$	$Y(6) = \frac{1}{6!}$...	$Y(n) = \frac{1}{n!}$

Table 4. Coefficient terms from Equation (29).

k	0	1	2	3	4	...	n
$Y(k+2)$	$Y(2) = \frac{-1}{2!}$	$Y(3) = \frac{-2}{3!}$	$Y(4) = \frac{-3}{4!}$	$Y(5) = \frac{-4}{5!}$	$Y(6) = \frac{1}{6!}$...	$Y(n) = \frac{-(n-1)}{n!}$

Table 5. Numerical comparisons for Case I Example 1.

x	<i>Exact solution</i>	<i>5-Iterate DTM</i>	<i>Absolute Error</i>
0.0	1.000000000	1.000000000	0.000000000
0.1	1.105170918	1.105170917	1.40898E-09
0.2	1.221402758	1.221402667	9.14935E-08
0.3	1.349858808	1.349857750	1.05758E-06
0.4	1.491824698	1.491818667	6.03097E-06
0.5	1.648721271	1.648697917	2.3354E-05
0.6	1.822118800	1.822048000	7.08004E-05
0.7	2.013752707	2.013571417	0.000181291
0.8	2.225540928	2.225130667	0.000410262
0.9	2.459603111	2.458758250	0.000844861
1.0	2.718281828	2.716666667	0.001615162

Table 6. Numerical comparisons for *Case I* Example 2.

x	<i>Exact solution</i>	<i>5-Iterate DTM</i>	<i>Absolute Error</i>
0.0	1.000000000	1.000000000	0.000000000
0.1	0.995004165	0.99500418	1.52775E-08
0.2	0.980066578	0.98006711	5.3327E-07
0.3	0.955336489	0.95534088	4.38587E-06
0.4	0.921060994	0.92108089	1.98949E-05
0.5	0.877582562	0.87764757	6.50076E-05
0.6	0.825335615	0.82550800	0.000172385
0.7	0.764842187	0.76523760	0.000395410
0.8	0.696706709	0.69752178	0.000815068
0.9	0.621609968	0.62315763	0.001547657
1.0	0.540302306	0.54305556	0.002753250

Table 7. Numerical comparisons for *Case II* Example 1.

x	<i>Exact solution</i>	<i>5-Iterate DTM</i>	<i>Absolute Error</i>
0.0	1.000000000	1.000000000	0.000000000
0.1	1.005170918	1.005170917	1.40898E-09
0.2	1.021402758	1.021402667	9.14935E-08
0.3	1.049858808	1.049857750	1.05758E-06
0.4	1.091824698	1.091818667	6.03097E-06
0.5	1.148721271	1.148697917	2.3354E-05
0.6	1.222118800	1.222048000	7.08004E-05
0.7	1.313752707	1.313571417	0.000181291
0.8	1.425540928	1.425130667	0.000410262
0.9	1.559603111	1.558758250	0.000844861
1.0	1.718281828	1.716666667	0.001615162

Table 8. Numerical comparisons for *Case II* Example 2.

x	<i>Exact solution</i>	<i>5-Iterate DTM</i>	<i>Absolute Error</i>
0.0	1.000000000	1.000000000	0.000000000
0.1	0.994653826	0.994653840	1.40097E-08
0.2	0.977122207	0.977123111	9.04583E-07
0.3	0.944901165	0.944911563	1.03972E-05
0.4	0.895094819	0.895153778	5.89592E-05
0.5	0.824360635	0.824587674	0.000227038
0.6	0.728847520	0.729532000	0.000684480
0.7	0.604125812	0.605868840	0.001743028
0.8	0.445108186	0.449031111	0.003922925
0.9	0.245960311	0.253995063	0.008034751
1.0	0.000000000	0.015277778	0.015277778

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