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Asymptotic Estimates for Second-Order Parameterized Singularly Perturbed Problem

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Abstract

The boundary value problem (BVP) for parameterized singularly perturbed second order nonlinear ordinary differential equation is considered. The boundary layer behavior of the solution and its first and second derivatives have been established. An example supporting the theoretical analysis is presented.

Keywords

Parameterized Problem, Asymptotic Bounds, Singular Perturbation, Boundary Layer

1. Introduction

In this paper, we are going to obtain the asymptotıc bounds for the following parameterized singularly perturbed boundary value problem (BVP):

$$Lu := \varepsilon u''(t) + a(t)u'(t) - f(t, u(t), \lambda) = 0, \qquad 0 < t < T, \tag{1}$$

$$u(0) = \mu_0, \quad u'(0) = \frac{\mu_1}{\varepsilon}, \quad u(T) = \mu_2, \tag{2}$$

where $0 < \varepsilon \leq 1$ is a perturbation parameter, $\mu_i (i = 0, 1, 2)$ are given constants and $0 < \alpha \leq a(t) \leq a^*$ is a sufficiently smooth function in $[0, T]$. Further , the function $f(t, u(t), \lambda)$ is assumed to be sufficiently continuously differentiable for our purpose function in $\{0 \leq t \leq T, -\infty < u < \infty, -\infty < \lambda < \infty\}$ and

$$0 \leq \frac{\partial f}{\partial u} \leq M_i^*, \quad 0 < m_1 \leq \left| \frac{\partial f}{\partial \lambda} \right| \leq M_1 < \infty . \tag{3}$$

By a solution of (1), (2) we mean pair $\{u(t),\lambda\}\in C^1[0,T]\times\mathbb{R}$ for which problem (1), (2) is satisfied.

An overview of some existence and uniqueness results and applications of parameterized equations may be obtained, for example, in [1]-[10]. In [11]-[14], some approximating aspects of this kind of problems have also been considered. The qualitative analysis of singular perturbation situations have always been far from trivial because of the boundary layer behavior of the solution. In singular perturbation cases, problems depend on a small parameter ε in such a way that the solution exhibits a multiscale character, *i.e.*, there are thin transition layers where the solution varies rapidly while away from layers it behaves regularly and varies slowly [15] [16]. In this note we establish the boundary layer behaviour for $u(t)$ of the solution of (1)-(2) and its first and second derivatives. Example that agree with the analytical results is given.

Theorem 1.1. For $\rho=1-\alpha^{-1}m_1^{-1}M_1^{-1}T>0$ the solution $\{u(t),\lambda\}$ of the problem (1), (2) satisfies,

$$|\lambda|\leq c_0, \tag{4}$$

$$|u|\leq c_1, \tag{5}$$

where

$$c_0=\rho^{-1}\left[\frac{m_1a^*}{T-\left(a^*\right)^{-1}\left(1-e^{-a^*T}\right)}\left(|\mu_0|+|\mu_2|+\alpha^{-1}T\|F\|_\infty+\alpha^{-1}M_1T\right)\right],$$

$$c_1=|\mu_0|+|\mu_2|+\alpha^{-1}T\|F\|_\infty+\alpha^{-1}c_0M_1T,\quad F(t)=f(t,0,0)$$

and

$$\left|u^{(k)}(t)\right|\leq C\left(1+\frac{1}{\varepsilon^k}e^{-\frac{\alpha t}{\varepsilon}}\right),\quad k=1,2,\ t\in[0,T] \tag{6}$$

provided $a\in C^1[0,T]$ and $\left|\dfrac{\partial f}{\partial t}\right|\leq C$ for $t\in[0,T]$ and $|u|\leq c_1,\ |\lambda|\leq c_0$.

Proof. We rewrite Equation (1) in the form

$$\varepsilon u''(t)+a(t)u'(t)-b(t)u(t)-\lambda c(t)-F(t)=0,\ t\in[0,T], \tag{7}$$

where,

$$b(t)=\frac{\partial f}{\partial u}(t,\tilde{u},\tilde{\lambda}),\ c(t)=\frac{\partial f}{\partial\lambda}(t,\tilde{u},\tilde{\lambda}),\ \tilde{u}=\gamma u,\ \tilde{\lambda}=\gamma\lambda\quad(0<\gamma<1)\text{—intermediate values.}$$

From (7) for the first derivate, we have

$$u'(t)=\frac{\mu_1}{\varepsilon}e^{-\frac{1}{\varepsilon}\int_0^t a(\tau)d\tau}+\frac{1}{\varepsilon}\int_0^t b(\tau)u(\tau)e^{-\frac{1}{\varepsilon}\int_\tau^t a(\eta)d\eta}d\tau+\frac{\lambda}{\varepsilon}\int_0^t c(\tau)e^{-\frac{1}{\varepsilon}\int_\tau^t a(\eta)d\eta}d\tau$$

$$+\frac{1}{\varepsilon}\int_0^t F(\tau)e^{-\frac{1}{\varepsilon}\int_\tau^t a(\eta)d\eta}d\tau. \tag{8}$$

Integrating this equality over $(0,t)$ we get

$$u(t)=\mu_0+\frac{\mu_1}{\varepsilon}\int_0^t e^{-\frac{1}{\varepsilon}\int_0^\xi a(\tau)d\tau}d\xi+\frac{1}{\varepsilon}\int_0^t d\tau b(\tau)u(\tau)\int_\tau^t e^{-\frac{1}{\varepsilon}\int_\tau^\xi a(\eta)d\eta}d\xi$$

$$+\frac{1}{\varepsilon}\int_0^t d\tau F(\tau)\int_\tau^t e^{-\frac{1}{\varepsilon}\int_\tau^\xi a(\eta)d\eta}d\xi+\frac{\lambda}{\varepsilon}\int_0^t d\tau c(\tau)\int_\tau^t e^{-\frac{1}{\varepsilon}\int_\tau^\xi a(\eta)d\eta}d\xi, \tag{9}$$

from which by setting the boundary condition $u(T)=\mu_2$, we obtain,

$$\lambda = \frac{\varepsilon(\mu_0 - \mu_2)}{\int_0^T d\tau c(\tau) \int_\tau^T e^{-\frac{1}{\varepsilon}\int_\tau^\xi a(\eta)d\eta} d\xi} + \frac{\int_0^T e^{-\frac{1}{\varepsilon}\int_0^\xi a(\tau)d\tau} d\xi}{\int_0^T d\tau c(\tau) \int_\tau^T e^{-\frac{1}{\varepsilon}\int_\tau^\xi a(\eta)d\eta} d\xi} + \frac{\int_0^T d\tau b(\tau)u(\tau) \int_\tau^T e^{-\frac{1}{\varepsilon}\int_\tau^\xi a(\eta)d\eta} d\xi}{\int_0^T d\tau c(\tau) \int_\tau^T e^{-\frac{1}{\varepsilon}\int_\tau^\xi a(\eta)d\eta} d\xi}$$

$$+ \frac{\int_0^t d\tau F(\tau) \int_\tau^t e^{-\frac{1}{\varepsilon}\int_\tau^\xi a(\eta)d\eta} d\xi}{\int_0^T d\tau c(\tau) \int_\tau^T e^{-\frac{1}{\varepsilon}\int_\tau^\xi a(\eta)d\eta} d\xi},$$

(10)

Applying the mean value theorem for integrals, we deduce that,

$$\frac{\left| \int_0^t d\tau F(\tau) \int_\tau^t e^{-\frac{1}{\varepsilon}\int_\tau^\xi a(\eta)d\eta} d\xi \right|}{\left| \int_0^T d\tau c(\tau) \int_\tau^T e^{-\frac{1}{\varepsilon}\int_\tau^\xi a(\eta)d\eta} d\xi \right|} \leq \frac{\|F\|_\infty \left| \int_0^t d\tau \int_\tau^t e^{-\frac{1}{\varepsilon}\int_\tau^\xi a(\eta)d\eta} d\xi \right|}{m_1 \left| \int_0^T d\tau \int_\tau^T e^{-\frac{1}{\varepsilon}\int_\tau^\xi a(\eta)d\eta} d\xi \right|} \leq \|F\|_\infty m_1^{-1},$$

(11)

and

$$\frac{\left| \int_0^t d\tau b(\tau)u(\tau) \int_\tau^t e^{-\frac{1}{\varepsilon}\int_\tau^\xi a(\eta)d\eta} d\xi \right|}{\left| \int_0^T d\tau c(\tau) \int_\tau^T e^{-\frac{1}{\varepsilon}\int_\tau^\xi a(\eta)d\eta} d\xi \right|} \leq \frac{\|u\|_\infty \left| \int_0^t d\tau b(\tau) \int_\tau^t e^{-\frac{1}{\varepsilon}\int_\tau^\xi a(\eta)d\eta} d\xi \right|}{m_1 \left| \int_0^T d\tau \int_\tau^T e^{-\frac{1}{\varepsilon}\int_\tau^\xi a(\eta)d\eta} d\xi \right|} \leq m_1^{-1} M_i^* \|u\|_\infty.$$

(12)

Also, for first and second terms in right side of (10), for $\varepsilon \leq 1$ values, we have

$$\frac{\varepsilon|\mu_0 - \mu_2|}{\left| \int_0^T d\tau c(\tau) \int_\tau^T e^{-\frac{1}{\varepsilon}\int_\tau^\xi a(\eta)d\eta} d\xi \right|} + \frac{|\mu_1| \left| \int_0^T e^{-\frac{1}{\varepsilon}\int_0^\xi a(\tau)d\tau} d\xi \right|}{\left| \int_0^T d\tau c(\tau) \int_\tau^T e^{-\frac{1}{\varepsilon}\int_\tau^\xi a(\eta)d\eta} d\xi \right|}$$

$$\leq \frac{m_1^{-1}(a^*)|\mu_0 - \mu_2|}{T - \varepsilon(a^*)^{-1}\left(1 - e^{-\frac{a^*T}{\varepsilon}}\right)} + \frac{\alpha^{-1} m_1^{-1}(a^*)|\mu_1|}{T - \varepsilon(a^*)^{-1}\left(1 - e^{-\frac{a^*T}{\varepsilon}}\right)}$$

(13)

$$\leq \frac{m_1^{-1}(a^*)}{T - \varepsilon(a^*)^{-1}\left(1 - e^{-\frac{a^*T}{\varepsilon}}\right)}\left[|\mu_0 - \mu_2| + \alpha^{-1}|\mu_1|\right],$$

It then follows from (11)-(13)

$$|\lambda| \leq \frac{m_1^{-1}(a^*)}{T - \varepsilon(a^*)^{-1}\left(1 - e^{-a^*T}\right)}\left[|\mu_0 - \mu_2| + \alpha^{-1}|\mu_1|\right] + m_1^{-1}\|F\|_\infty + m_1^{-1} M_1^* \|u\|_\infty$$

(14)

Next from (9), we see that

$$\|u\|_\infty \le |\mu_0| + |\mu_2| + \alpha^{-1} T \|F\|_\infty + \alpha^{-1} |\lambda| T,$$

Under the conditions $0 < \alpha \le a(t) \le a^*$ and $0 \le b(t) \le M_1^*$ the operator $Lv := \varepsilon v''(t) + a(t) v'(t) - b(t) v(t)$ admits the following maximum principle: Suppose $v(t) \in C^2[0,T]$ be any function satisfiying $Lv \le 0\,(0 < t < T)$, $v(0) \ge 0$ and $v(T) \ge 0$, then $v(t) \ge 0\,(0 < t < T)$.

Using the maximum principle whith barrier functions $\psi_\pm(t) = \pm v(t) + \alpha^{-1}(T-t)\|Lv\|_\infty + |v(0)| + |v(T)|$ we have the inequality

$$\|u\|_\infty \le |\mu_0| + |\mu_2| + \alpha^{-1} T \|F\|_\infty + \alpha^{-1} |\lambda| T. \tag{15}$$

The inequlities (14), (15) immediately leads to (4), (5). After taking into consideration the uniformly boundnees in ε of $u(t)$ and λ, it then follows from (8) that,

$$u'(t) = \frac{|\mu_1|}{\varepsilon} e^{-\frac{\alpha t}{\varepsilon}} + \frac{1}{\varepsilon} c_1 M_1^* \int_0^t e^{-\frac{\alpha(t-\tau)}{\varepsilon}} \, d\tau + \frac{1}{\varepsilon} \|F\|_\infty \int_0^t e^{-\frac{\alpha(t-\tau)}{\varepsilon}} \, d\tau + \frac{1}{\varepsilon} c_0 M_1 \int_0^t e^{-\frac{\alpha(t-\tau)}{\varepsilon}} \, d\tau,$$

which proves (6) for $k = 1$. To obtain (6) for $k = 2$, first from Equation (1) we have

$$|u''(0)| \le \frac{|a(0) u'(0) - f(0, u(0), \lambda)|}{\varepsilon},$$

from which after taking into consideration here $u'(0) = \dfrac{\mu_1}{\varepsilon}$ and (4)

$$|u''(0)| \le \frac{C}{\varepsilon^2}. \tag{16}$$

Next, differentiation (1) gives

$$\varepsilon v''(t) + a(t) v'(t) + \varphi(t) = 0, \quad 0 < t < T, \tag{17}$$

$$v'(0) = 0\left(\varepsilon^{-2}\right) \tag{18}$$

with

$$v(t) = u'(t),$$

$$\varphi(t) = a'(t) v(t) - \frac{\partial f}{\partial t}(t, u(t), \lambda) - \frac{\partial f}{\partial u}(t, u(t), \lambda),$$

and due to our assumptions clearly,

$$|\varphi(t)| \le C\left(1 + \frac{1}{\varepsilon^k} e^{-\frac{\alpha t}{\varepsilon}}\right).$$

Consequently, from (17), (18) we have

$$|v'(t)| \le \frac{C}{\varepsilon^2} e^{-\frac{\alpha t}{\varepsilon}} + \frac{1}{\varepsilon} \int_0^t |\varphi(\tau)| e^{-\frac{\alpha(t-\tau)}{\varepsilon}} \, d\tau$$

$$= \frac{C}{\varepsilon^2} e^{-\frac{\alpha t}{\varepsilon}} + C\left(1 + \frac{1}{\varepsilon^2} e^{-\frac{\alpha t}{\varepsilon}}\right) \int_0^t e^{-\frac{\alpha(t-\tau)}{\varepsilon}} \, d\tau$$

$$= \frac{C}{\varepsilon^2} e^{-\frac{\alpha t}{\varepsilon}} + C\left(1 + \frac{1}{\varepsilon^2} e^{-\frac{\alpha t}{\varepsilon}}\right)$$

which proves (6) for $k = 2$.

Example. Consider the particular problem

$$\varepsilon u''(t) + 2 u'(t) + \lambda + \tanh(t + \lambda) = f(t), \quad 0 < t < 1,$$

$$u(0) = 1, \ u'(0) = \frac{\mu_1}{\varepsilon}, \ u(1) = 0,$$

where, μ_1 and $f(t)$ selected so that the solution is

$$u(t) = \gamma_1 + \gamma_2 e^{-\frac{2}{\varepsilon}t} + \frac{1}{2+\varepsilon} e^t, \quad \lambda = 0.5$$

with

$$\gamma_1 = -\frac{e + (1+\varepsilon)e^{-\frac{2}{\varepsilon}}}{(2+\varepsilon)\left(1 - e^{-\frac{2}{\varepsilon}}\right)}, \quad \gamma_2 = \frac{1 + \varepsilon + e}{(2+\varepsilon)\left(1 - e^{-\frac{2}{\varepsilon}}\right)}.$$

First and second derivatives have the form

$$u^{(k)}(t) = \left(-\frac{2}{\varepsilon}\right)^k \gamma_2 e^{-\frac{2}{\varepsilon}t} + \frac{1}{2+\varepsilon} e^t, \ k = 1, 2.$$

Therefore we observe here the accordance in our theoretical results described above.

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