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Dividend Payments with a Hybrid Strategy in the Compound Poisson Risk Model

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Abstract

In this paper, a hybrid dividend strategy in the compound Poisson risk model is considered. In the absence of dividends, the surplus of an insurance company is modelled by a compound Poisson process. Dividends are paid at a constant rate whenever the modified surplus is in a interval; the premium income no longer goes into the surplus but is paid out as dividends whenever the modified surplus exceeds the upper bound of the interval, otherwise no dividends are paid. Integro-differential equations with boundary conditions satisfied by the expected total discounted dividends until ruin are derived; for example, closed-form solutions are given when claims are exponentially distributed. Accordingly, the moments and moment-generating functions of total discounted dividends until ruin are considered. Finally, the Gerber-Shiu function and Laplace transform of the ruin time are discussed.

Keywords

Hybrid Dividend Strategy, Compound Poisson Risk Model, Moment-Generating Function, Gerber-Shiu Function

1. Introduction

The dividends problem was first proposed by Finetti [1], who considered a discrete time risk model and found that the optimal dividend strategy is a barrier strategy, that is, any surplus above a certain level would be paid as dividend. Nowadays, this problem still attracts a lot of research interest. For example, [2] [3] considered the compound Poisson risk model. [4] studied the continuous counterpart of Finetti [1], and it is assumed that the surplus is a Brownian motion with a positive drift. Jeanblanc-Picque and Shiryaev [5] and Asmussen and Taksar

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[6] postulated a modified version of barrier strategy called threshold strategy, that is, dividends are paid at a constant rate whenever the surplus is above a threshold level; however, when the surplus is below the threshold level, no dividends are paid. Some calculations for the classical risk model and Brownian motion model are given in [7] [8]. For recent publications on this topic, see, for example, [9]-[14].

Recently, the multi-layer dividend strategy as an extension of the threshold dividend strategy has drawn many authors' attention. Under such a dividend strategy, premiums will be collected at different rates whenever the surplus is in different layers. The modified surplus process is obtained from the original surplus process by refraction at each threshold level. Within this framework, many authors have studied the Gerber-Shiu expected discounted penalty function, see, for instance, [15]-[17] and the references therein.

Under such framework, Ng [18] combined barrier strategy and threshold strategy for the first time and then proposed a hybrid dividend strategy, who considered a dual risk model with phase-type gains under a hybrid dividend strategy and derived the explicit formula for the expected total discounted dividends until ruin and the Laplace transform of the time of ruin. In this paper, we consider the hybrid dividend strategy for the classical risk model. Let $b_2 > b_1$ be two positive constants, under a hybrid strategy, no dividends are paid whenever the modified surplus is below the level b_1; dividends are paid at a constant rate α $(\alpha > 0)$ whenever the modified surplus is in interval (b_1, b_2); the premium income no longer goes into the surplus but is paid out as dividends whenever the modified surplus exceeds the level b_2. The modified surplus is obtained from the original surplus process by refraction at the level b_1 and reflection at the level b_2. The hybrid dividend strategy introduced above is a generalization of a pure barrier strategy and a pure threshold strategy. Apparently the hybrid strategy is more realistic than a pure barrier strategy, because it is inflexible for companies to use a switching mechanism of either paying nothing or paying all excess surplus as dividends. In the meantime, it is more practical than a pure threshold, because it is the ideal for a surplus of a company to be allowed to grow infinitely.

The rest of the paper is organized as follows. In Section 2, we find the integro-differential equations and boundary conditions for the expected discounted dividend payments until ruin. The integro-differential equations with boundary conditions satisfied by the moments and the moment-generating function are given in Section 3. Section 4 discussed the integro-differential equations with boundary conditions for the Gerber-Shiu function, and Section 5 presents the integro-differential equations with boundary conditions satisfied by the Laplace transform of ruin time.

2. The Model

We consider the compound Poisson model of risk theory with initial surplus $u > 0$. In the absence of dividends, the surplus process U_t at time *t* is given by

$$U_t = u + \mu t - S_t \equiv u + \mu t - \sum_{i=1}^{N(t)} Y_i, \quad t \geq 0,$$

where μ is the premium rate, and $\{S_t\}$ representing the aggregate claims up to time *t*, $N(t)$ is a Poisson process with intensity λ, and $Y_i, i = 1, \cdots,$ independent of $\{N(t); t > 0\}$, are positive i.i.d. random variables with distribution function $P(y)$ and density function $p(y)$.

Unlike the dividend strategies in [4] [8], we assume the company will pay dividends to its shareholders according to a hybrid dividend strategy with parameters $b_2 > b_1 > 0$. The dividends consists of two parts. The first part of dividends are paid at a constant rate $\alpha \in (0, \mu)$ whenever the modified surplus between the level b_1 and the level b_2. The second part, whenever the modified surplus reaches the level b_2, the overflow will be paid as dividends. For $t > 0$, let $D(t) = D_1(t) + D_2(t)$ denote the aggregate dividends paid by time *t*, where $D_1(t)$ and $D_2(t)$ caused by the two parts of dividends, respectively. Thus

$$\tilde{U}_t = U_t - D(t), \tag{2.1}$$

is the company's modified surplus at time *t*.

Let *T* be the ruin time of $\{\tilde{U}_t; t \geq 0\}$, namely

$$T = \inf\{t \geq 0 | \tilde{U}_t < 0\},$$

and let $\delta > 0$ be the force of interest for valuation, we denote

$$D=\int_0^T e^{-\delta t}\mathrm{d}D(t),\ D_1=\int_0^T e^{-\delta t}\mathrm{d}D_1(t),\ D_2=\int_0^T e^{-\delta t}\mathrm{d}D_2(t).$$

We use the symbols $V(u;b_1,b_2), V_d(u;b_1,b_2), V_r(u;b_1,b_2)$ to denote the expectations of D, D_1, D_2, *i.e.*

$$V(u;b_1,b_2)=E_u[D],\ V_d(u;b_1,b_2)=E_u[D_1],\ V_r(u;b_1,b_2)=E_u[D_2].$$

Define the moment-generating function of D by

$$M(u,z;b_1,b_2)=E\left[e^{zD}\middle|\tilde{U}_0=u\right]\equiv E_u\left[e^{zD}\right],\ u\ge 0,$$

and kth moment by

$$V_k(u;b_1,b_2)=E\left[D^k\middle|\tilde{U}_0=u\right],\ u\ge 0,\ k\in N,$$

with $V_0(u;b_1,b_2)=1$, and the Gerber-Shiu functions by

$$\Phi(u;b_1,b_2)=E\left[e^{-\delta T}\omega\left(\tilde{U}_{T-},\left|\tilde{U}_T\right|\right)I(T<\infty)\middle|\tilde{U}_0=u\right],\ u\ge 0, \tag{2.2}$$

where $\tilde{U}_{T-}$ is the surplus immediately before ruin, $\left|\tilde{U}_T\right|$ is the deficit at ruin and the penalty $\omega(x,y)$ is a nonnegative bounded measurable function of $x>0, y>0$, and the Laplace transform of ruin time by

$$L(u;b_1,b_2)=E\left[e^{-\delta T}\middle|\tilde{U}_0=u\right]\equiv E_u\left[e^{-\delta T}\right],\ u\ge 0. \tag{2.3}$$

3. Expected Discounted Dividend Payments

In this section, we consider the hybrid dividend strategy for dividend payments in a compound Poisson risk model. We write

$$V_d(u;b_1,b_2)=\begin{cases}V_{d1}(u;b_1,b_2), & 0\le u<b_1,\\ V_{d2}(u;b_1,b_2), & b_1\le u\le b_2.\end{cases}$$

$$V_r(u;b_1,b_2)=\begin{cases}V_{r1}(u;b_1,b_2), & 0\le u<b_1,\\ V_{r2}(u;b_1,b_2), & b_1\le u\le b_2.\end{cases}$$

Then, we have

$$V(u;b_1,b_2)=\begin{cases}V_{1*}(u;b_1,b_2)=V_{d1}(u;b_1,b_2)+V_{r1}(u;b_1,b_2), & 0\le u<b_1,\\ V_{2*}(u;b_1,b_2)=V_{d2}(u;b_1,b_2)+V_{r2}(u;b_1,b_2), & b_1\le u\le b_2.\end{cases}$$

In the following, we first derive the integro-differential equations and boundary conditions satisfied by $V_d(u;b_1,b_2)$ and $V_r(u;b_1,b_2)$.

Theorem 3.1 *Assume that* $V_d(u;b_1,b_2)$ *is continuously differentiable in u on* $(0,b_1)\cup(b_1,b_2)$. *Then,* $V_d(u;b_1,b_2)$ *satisfies the following integro-differential equations, when* $0<u<b_1$,

$$\mu\frac{\partial V_{d1}(u;b_1,b_2)}{\partial u}-(\lambda+\delta)V_{d1}(u;b_1,b_2)+\lambda\int_0^u V_{d1}(u-y;b_1,b_2)p(y)\mathrm{d}y=0, \tag{3.1}$$

and, when $b_1<u<b_2$,

$$\begin{aligned}&\alpha+(\mu-\alpha)\frac{\partial V_{d2}(u;b_1,b_2)}{\partial u}-(\lambda+\delta)V_{d2}(u;b_1,b_2)\\ &+\lambda\left[\int_0^{u-b_1}V_{d2}(u-y;b_1,b_2)p(y)\mathrm{d}y+\int_{u-b_1}^{u}V_{d1}(u-y;b_1,b_2)p(y)\mathrm{d}y\right]=0,\end{aligned} \tag{3.2}$$

with boundary conditions

$$V_{d1}(b_1-;b_1,b_2)=V_{d2}(b_1+;b_1,b_2), \tag{3.3}$$

$$\left.\frac{\partial V_{d2}\left(u;b_1,b_2\right)}{\partial u}\right|_{u=b_2}=0, \tag{3.4}$$

$$\mu\left.\frac{\partial V_{d1}\left(u;b_1,b_2\right)}{\partial u}\right|_{u=b_1-}=\alpha+\left(\mu-\alpha\right)\left.\frac{\partial V_{d2}\left(u;b_1,b_2\right)}{\partial u}\right|_{u=b_1+}. \tag{3.5}$$

Proof. When $0\le u<b_1$, consider $t>0$ such that the modified surplus can not reach level b_1 by time *t*, *i.e.* $u+\mu t<b_1$. In view of the strong Markov property of the surplus process $\{\tilde{U},t\ge 0\}$, we have

$$\begin{aligned}V_{d1}\left(u;b_1,b_2\right)&=E_u\left[\int_t^T \mathrm{e}^{-\delta s}\mathrm{d}D_1\left(t\right)\right]+o\left(t\right)\\&=E_u\left[\int_0^{T-t}\mathrm{e}^{-\delta(s+t)}\mathrm{d}D_1\left(t+s\right)\right]+o\left(t\right)\\&=\mathrm{e}^{-\delta t}E_u\left[\int_0^T \mathrm{e}^{-\delta s}\mathrm{d}D_1\left(s\right)\circ\theta_t\right]+o\left(t\right)\\&=\mathrm{e}^{-\delta t}E_u\left\{E_{\tilde{U}_t}\left[\int_0^T \mathrm{e}^{-\delta s}\mathrm{d}D_1\left(s\right)\right]\right\}+o\left(t\right)\\&=\mathrm{e}^{-\delta t}E_u\left[V_{d1}\left(\tilde{U}_t;b_1,b_2\right)\right]+o\left(t\right),\end{aligned} \tag{3.6}$$

where θ_t is the shift operator. By conditioning on the time and amount of the first claim and whether the claim causes ruin or not, and using (3.6), we get

$$V_{d1}\left(u;b_1,b_2\right)=\mathrm{e}^{-\delta t}\left(1-\lambda t\right)V_{d1}\left(u+\mu t;b_1,b_2\right)+\mathrm{e}^{-\delta t}\lambda t\int_0^{u+\mu t}V_{d1}\left(u+\mu t-y;b_1,b_2\right)p\left(y\right)\mathrm{d}y+o\left(t\right). \tag{3.7}$$

By Taylor's expansion,

$$V_{d1}\left(u+\mu t;b_1,b_2\right)=V_{d1}\left(u;b_1,b_2\right)+\mu t\frac{\partial V_{d1}\left(u;b_1,b_2\right)}{\partial u}+o\left(t\right).$$

Substituting the above expressions into (3.7), and dividing both sides of (3.7) by *t* and letting $t\to 0$, we can get (3.1).

When $b_1<u<b_2$, we still consider a small time interval $\left[0,t\right]$, with $t\left(>0\right)$ being sufficiently small so that the modified surplus will not reach b_2 in the time interval. In view of the strong Markov property of the surplus process $\{\tilde{U}_t,t\ge 0\}$, we have

$$\begin{aligned}V_{d2}\left(u;b_1,b_2\right)&=\alpha t+E_u\left[\int_t^T \mathrm{e}^{-\delta s}\mathrm{d}D_1\left(t\right)\right]+o\left(t\right)\\&=\alpha t+\mathrm{e}^{-\delta t}E_u\left[V_{d2}\left(\tilde{U}_t;b_1,b_2\right)\right]+o\left(t\right).\end{aligned} \tag{3.8}$$

By conditioning on the time and amount of the first claim and whether the claim causes ruin or not, and using (3.8), we get

$$\begin{aligned}V_{d2}\left(u;b_1,b_2\right)=&\,\alpha t+\mathrm{e}^{-\delta t}\left(1-\lambda t\right)V_{d2}\left(u+\left(\mu-\alpha\right)t;b_1,b_2\right)\\&+\mathrm{e}^{-\delta t}\lambda t\left[\int_0^{u+(\mu-\alpha)t-b_1}V_{d2}\left(u+\left(\mu-\alpha\right)t-y;b_1,b_2\right)p\left(y\right)\mathrm{d}y\right.\\&\left.+\int_{u+(\mu-\alpha)t-b_1}^{u+(\mu-\alpha)t}V_{d1}\left(u+\left(\mu-\alpha\right)t-y;b_1,b_2\right)p\left(y\right)\mathrm{d}y\right]+o\left(t\right).\end{aligned} \tag{3.9}$$

By Taylor's expansion,

$$V_{d2}\left(u+\left(\mu-\alpha\right)t;b_1,b_2\right)=V_{d2}\left(u;b_1,b_2\right)+\left(\mu-\alpha\right)t\frac{\partial V_{d2}\left(u;b_1,b_2\right)}{\partial u}+o\left(t\right).$$

Substituting the above expressions into (3.9), and dividing both sides of (3.9) by *t* and letting $t\to 0$, we can get (3.2).

Next we prove the condition (3.3). It follows from

$$e^{\frac{-(\delta+\lambda)\epsilon}{\mu}} V_{d2}(b_1;b_1,b_2) \le V_{d1}(b_1-\epsilon;b_1,b_2) \le V_{d2}(b_1;b_1,b_2),\ \ 0<\epsilon\le b_1,$$

let $\epsilon \downarrow 0$, we have

$$V_{d1}(b_1-;b_1,b_2) = V_{d2}(b_1;b_1,b_2).$$

Similarly,

$$e^{\frac{-(\delta+\lambda)\epsilon}{\mu}} V_{d2}(b_1+\epsilon;b_1,b_2) \le V_{d2}(b_1;b_1,b_2) \le V_{d2}(b_1+\epsilon;b_1,b_2),\ \ 0<\epsilon\le b_2-b_1,$$

let $\epsilon \downarrow 0$, we obtain

$$V_{d2}(b_1+;b_1,b_2) = V_{d2}(b_1;b_1,b_2).$$

So we get (3.3).

Furthermore, when the initial surplus is b_2, we can mimic the derivation of (3.9) to obtain

$$\begin{aligned} V_{d2}(b_2;b_1,b_2) &= \alpha t + e^{-\delta t}(1-\lambda t)V_{d2}(b_2;b_1,b_2) \\ &\quad + e^{-\delta t}\lambda t\left[\int_0^{b_2-b_1} V_{d2}(b_2-y;b_1,b_2)\,p(y)\,dy \right. \\ &\quad \left. + \int_{b_2-b_1}^{b_2} V_{d1}(b_2-y;b_1,b_2)\,p(y)\,dy\right] + o(t). \end{aligned} \tag{3.10}$$

Dividing both sides of (3.10) by *t* and letting $t \to 0$, we can obtain

$$(\delta+\lambda)V_{d2}(b_2;b_1,b_2) = \alpha + \lambda\left[\int_0^{b_2-b_1} V_{d2}(b_2-y;b_1,b_2)\,p(y)\,dy + \int_{b_2-b_1}^{b_2} V_{d1}(b_2-y;b_1,b_2)\,p(y)\,dy\right]. \tag{3.11}$$

Letting $u \uparrow b_2$ in (3.2) and comparing it to (3.11), we obtain

$$\left.\frac{\partial V_{d2}(u;b_1,b_2)}{\partial u}\right|_{u=b_2-} = 0.$$

When $u > b_2$, we have

$$V_d(u;b_1,b_2) = V_{d2}(b_2;b_1,b_2),$$

thus,

$$\left.\frac{\partial V_d(u;b_1,b_2)}{\partial u}\right|_{u=b_2+} = 0.$$

So we get (3.4).

Finally, letting $u \uparrow b_1$ in (3.1) and $u \downarrow b_1$ in (3.2), we can get (3.5). This completes the proof of Theorem 3.1.

Remark 3.1 *Letting* $b_2 \to \infty$ *in Theorem* 3.1, *then* (3.1) *and* (3.2) *reduce, respectively, to* (5.1) *and* (5.2) *of* [7].

Theorem 3.2 *Assume that* $V_r(u;b_1,b_2)$ *is continuously differentiable in u on* $(0,b_1)\cup(b_1,b_2)$. *Then,* $V_r(u;b_1,b_2)$ *satisfies the following integro-differential equations, when* $0<u<b_1$,

$$\mu\frac{\partial V_{r1}(u;b_1,b_2)}{\partial u} - (\lambda+\delta)V_{r1}(u;b_1,b_2) + \lambda\int_0^u V_{r1}(u-y;b_1,b_2)\,p(y)\,dy = 0, \tag{3.12}$$

and, when $b_1<u<b_2$,

$$\begin{aligned} &(\mu-\alpha)\frac{\partial V_{r2}(u;b_1,b_2)}{\partial u} - (\lambda+\delta)V_{r2}(u;b_1,b_2) \\ &+\lambda\left[\int_0^{u-b_1} V_{r2}(u-y;b_1,b_2)\,p(y)\,dy + \int_{u-b_1}^{u} V_{r1}(u-y;b_1,b_2)\,p(y)\,dy\right] = 0, \end{aligned} \tag{3.13}$$

with boundary conditions

$$V_{r1}\left(b_1-;b_1,b_2\right)=V_{r2}\left(b_1+;b_1,b_2\right), \tag{3.14}$$

$$\left.\frac{\partial V_{r2}\left(u;b_1,b_2\right)}{\partial u}\right|_{u=b_2}=1, \tag{3.15}$$

$$\mu\left.\frac{\partial V_{d1}\left(u;b_1,b_2\right)}{\partial u}\right|_{u=b_1-}=\left(\mu-\alpha\right)\left.\frac{\partial V_{d2}\left(u;b_1,b_2\right)}{\partial u}\right|_{u=b_1+}. \tag{3.16}$$

Proof. In view of the strong Markov property of the surplus process $\left\{\tilde{U}_t,t\geq 0\right\}$, we have

$$V_r\left(u;b_1,b_2\right)=\mathrm{e}^{-\delta t}E_u\left[V_r\left(\tilde{U}_t;b_1,b_2\right)\right]+o\left(t\right). \tag{3.17}$$

When $0<u<b_1$, we consider a small time interval $\left[0,t\right]$, with $t>0$ being sufficiently small so that the modified surplus will not reach b_1 in the interval. By conditioning on the time and amount of the first claim and whether the claim causes ruin or not, and using (3.17), we get

$$\begin{aligned}V_{r1}\left(u;b_1,b_2\right)&=\mathrm{e}^{-\delta t}\left(1-\lambda t\right)V_{r1}\left(u+\mu t;b_1,b_2\right)\\&\quad+\mathrm{e}^{-\delta t}\lambda t\int_0^{u+\mu t}V_{r1}\left(u+\mu t-y;b_1,b_2\right)p\left(y\right)\mathrm{d}y+o\left(t\right).\end{aligned} \tag{3.18}$$

By Taylor's expansion,

$$V_{r1}\left(u+\mu t;b_1,b_2\right)=V_{r1}\left(u;b_1,b_2\right)+\mu t\frac{\partial V_{r1}\left(u;b_1,b_2\right)}{\partial u}+o\left(t\right).$$

Substituting the above expression into (3.18), and dividing both sides of (3.18) by *t* and letting $t\to 0$, we can get (12).

When $b_1<u<b_2$, we still consider a small time interval $\left[0,t\right]$, with $t>0$ being sufficiently small so that the modified surplus will not reach b_2 in the interval. Similar to the derivation of (3.12), we can obtain Equation (3.13).

The condition (3.14) can be obtained similar to (3.3).

When the initial surplus is b_2, we have

$$\begin{aligned}V_{r2}\left(b_2;b_1,b_2\right)&=\mathrm{e}^{-\delta t}\left(1-\lambda t\right)\left[\left(\mu-\alpha\right)t+V_{r2}\left(b_2;b_1,b_2\right)\right]\\&\quad+\mathrm{e}^{-\delta t}\lambda t\left[\int_0^{b_2-b_1}V_{r2}\left(b_2-y;b_1,b_2\right)p\left(y\right)\mathrm{d}y\right.\\&\quad\left.+\int_{b_2-b_1}^{b_2}V_{r1}\left(b_2-y;b_1,b_2\right)p\left(y\right)\mathrm{d}y\right]+o\left(t\right).\end{aligned} \tag{3.19}$$

Dividing both sides of (3.19) by *t* and letting $t\to 0$, we can obtain

$$\begin{aligned}\left(\delta+\lambda\right)V_{r2}\left(b_2;b_1,b_2\right)&=\left(\mu-\alpha\right)+\lambda\left[\int_0^{b_2-b_1}V_{r2}\left(b_2-y;b_1,b_2\right)p\left(y\right)\mathrm{d}y\right.\\&\quad\left.+\int_{b_2-b_1}^{b_2}V_{r1}\left(b_2-y;b_1,b_2\right)p\left(y\right)\mathrm{d}y\right].\end{aligned} \tag{3.20}$$

Letting $u\uparrow b_2$ in (3.13) and comparing it to (3.20), we obtain

$$\left.\frac{\partial V_{r2}\left(u;b_1,b_2\right)}{\partial u}\right|_{u=b_2-}=1.$$

When $u>b_2$, we have

$$V_r\left(u;b_1,b_2\right)=u-b_2+V_{r2}\left(b_2;b_1,b_2\right),$$

thus,

$$\left.\frac{\partial V_r\left(u;b_1,b_2\right)}{\partial u}\right|_{u=b_2+}=1.$$

So we get (3.15).

Finally, letting $u \uparrow b_1$ in (3.12) and $u \downarrow b_1$ in (3.13), we can get (3.16). This completes the proof of Theorem 3.2.

According to the definition of $V(u;b_1,b_2)$, from Theorems 3.1 and 3.2, we can lead to the integro-differential equations and the boundary conditions satisfied by $V(u;b_1,b_2)$.

Theorem 3.3 *Assume that* $V(u;b_1,b_2)$ *is continuously differentiable in u on* $(0,b_1)\cup(b_1,b_2)$. *Then,* $V(u;b_1,b_2)$ *satisfies the following integro-differential equations, when* $0<u<b_1$,

$$\mu\frac{\partial V_{1*}(u;b_1,b_2)}{\partial u}-(\lambda+\delta)V_{1*}(u;b_1,b_2)+\lambda\int_0^u V_{1*}(u-y;b_1,b_2)\,p(y)\,\mathrm{d}y=0, \tag{3.21}$$

and, when $b_1<u<b_2$,

$$\begin{aligned}&\alpha+(\mu-\alpha)\frac{\partial V_{2*}(u;b_1,b_2)}{\partial u}-(\lambda+\delta)V_{2*}(u;b_1,b_2)+\lambda\Big[\int_0^{u-b_1}V_{2*}(u-y;b_1,b_2)\,p(y)\,\mathrm{d}y\\&+\int_{u-b_1}^{u}V_{1*}(u-y;b_1,b_2)\,p(y)\,\mathrm{d}y\Big]=0,\end{aligned} \tag{3.22}$$

with boundary conditions

$$V_{1*}(b_1-;b_1,b_2)=V_{2*}(b_1+;b_1,b_2), \tag{3.23}$$

$$\left.\frac{\partial V_{2*}(u;b_1,b_2)}{\partial u}\right|_{u=b_2}=1, \tag{3.24}$$

$$\mu\left.\frac{\partial V_{1*}(u;b_1,b_2)}{\partial u}\right|_{u=b_1-}=\alpha+(\mu-\alpha)\left.\frac{\partial V_{2*}(u;b_1,b_2)}{\partial u}\right|_{u=b_1+}. \tag{3.25}$$

Example 3.1. Now we assume that the individual claim amounts are exponentially distributed with mean $1/\beta$, *i.e.*

$$p(y)=\beta\mathrm{e}^{-\beta y},\quad y>0.$$

Then, we have

$$p'(y)+\beta p(y)=0. \tag{3.26}$$

Applying the operator $(\mathrm{d}/\mathrm{d}u+\beta)$ on (3.21) and (3.22) respectively, and using (3.26) and rearranging them, we get

$$\mu\frac{\partial V_{1*}^2(u;b_1,b_2)}{\partial u^2}+(\beta\mu-\lambda-\delta)\frac{\partial V_{1*}(u;b_1,b_2)}{\partial u}-\beta\delta V_{1*}(u;b_1,b_2)=0, \tag{3.27}$$

for $0<u<b_1$, and for $b_1<u<b_2$

$$(\mu-\alpha)\frac{\partial V_{2*}^2(u;b_1,b_2)}{\partial u^2}+\big[\beta(\mu-\alpha)-\lambda-\delta\big]\frac{\partial V_{2*}(u;b_1,b_2)}{\partial u}-\beta\delta V_{2*}(u;b_1,b_2)+\beta\alpha=0. \tag{3.28}$$

We can obtain the solutions of Equation (3.27) as follows

$$V_{1*}(u;b_1,b_2)=A\mathrm{e}^{ru}+B\mathrm{e}^{su},\quad 0\le u\le b_1, \tag{3.29}$$

with the coefficients A and B being independent of u, and r and s being the roots of the characteristic equation

$$\mu\xi^2+(\beta\mu-\lambda-\delta)\xi-\beta\delta=0.$$

We let r denote the positive root and s the negative root, *i.e.*

$$r=\frac{-(\beta\mu-\lambda-\delta)+\sqrt{(\beta\mu-\lambda-\delta)^2+4\mu\beta\delta}}{2\mu},$$

$$s=\frac{-(\beta\mu-\lambda-\delta)-\sqrt{(\beta\mu-\lambda-\delta)^2+4\mu\beta\delta}}{2\mu}.$$

Substituting (3.29) in Equation (3.21) and equating the coefficient of $e^{-\beta u}$ with 0, we have

$$\lambda\beta\left(\frac{A}{r+\beta}+\frac{B}{s+\beta}\right)=0. \quad (3.30)$$

From (3.29) and (3.30), we can rewrite

$$V_{1*}(u;b_1,b_2)=\gamma\left[(r+\beta)e^{ru}-(\beta+s)e^{su}\right],\quad 0<u<b_1, \quad (3.31)$$

where γ dose not depend on u. A particular solution of (3.28) is α/δ. Hence, the solutions of Equation (3.28) are given by

$$V_{2*}(u;b_1,b_2)=\frac{\alpha}{\delta}+Ce^{vu}+Ge^{wu},\quad b_1<u\le b_2, \quad (3.32)$$

where the coefficients *C* and *G* are independent of *u*, and $w<0$ and $v>0$ are the roots of the characteristic equation

$$(\mu-\alpha)\xi^2+\left[\beta(\mu-\alpha)-\lambda-\delta\right]\xi-\delta\beta=0,$$

namely,

$$v=\frac{-\left[\beta(\mu-\alpha)-\lambda-\delta\right]+\sqrt{\left[\beta(\mu-\alpha)-\lambda-\delta\right]^2+4(\mu-\alpha)\beta\delta}}{2(\mu-\alpha)},$$

$$w=\frac{-\left[\beta(\mu-\alpha)-\lambda-\delta\right]-\sqrt{\left[\beta(\mu-\alpha)-\lambda-\delta\right]^2+4(\mu-\alpha)\beta\delta}}{2(\mu-\alpha)}.$$

From (3.31) and (3.32), we observe that the convolution integral in Equation (3.22) is

$$\begin{aligned}&\int_0^{b_1}V_{1*}(y;b_1,b_2)\,p(u-y)\,dy+\int_{b_1}^{u}V_{2*}(y;b_1,b_2)\,p(u-y)\,dy\\&=\gamma\int_0^{b_1}\left[(r+\beta)e^{ry}-(s+\beta)e^{sy}\right]\beta e^{-\beta(u-y)}dy+\int_{b_1}^{u}\left(\frac{\alpha}{\delta}+Ce^{vy}+Ge^{wy}\right)\beta e^{-\beta(u-y)}dy\\&=\beta e^{-\beta u}\left\{\gamma\left[e^{(r+\beta)b_1}-e^{(s+\beta)b_1}\right]+\frac{\alpha}{\delta\beta}\left(e^{\beta u}-e^{\beta b_1}\right)+\frac{C}{\beta+v}\left[e^{(\beta+v)u}-e^{(\beta+v)b_1}\right]+\frac{G}{\beta+w}\left[e^{(\beta+w)u}-e^{(\beta+w)b_1}\right]\right\}.\end{aligned}$$

By setting the coefficient of $e^{-\beta u}$ to 0, we have

$$\gamma\left(e^{rb_1}-e^{sb_1}\right)=\frac{\alpha}{\delta\beta}+\frac{C}{\beta+v}e^{vb_1}+\frac{G}{\beta+w}e^{wb_1}. \quad (3.33)$$

From (23) and (24), we have the conditions

$$\gamma\left[(r+\beta)e^{rb_1}-(s+\beta)e^{sb_1}\right]=\frac{\alpha}{\delta}+Ce^{vb_1}+Ge^{wb_1}, \quad (3.34)$$

and

$$Cve^{vb_2}+Gwe^{wb_2}=1. \quad (3.35)$$

It follows from (33) and (34) that

$$C=\frac{\beta+v}{w-v}\left\{\gamma\left[(w-r)e^{rb_1}-(w-s)e^{sb_1}\right]-\frac{\alpha w}{\delta\beta}\right\}e^{-vb_1}. \quad (3.36)$$

$$G=\frac{\beta+w}{v-w}\left\{\gamma\left[(v-r)e^{rb_1}-(v-s)e^{sb_1}\right]-\frac{\alpha v}{\delta\beta}\right\}e^{-wb_1}. \tag{3.37}$$

Substitution of (3.36) and (3.37) into (3.35), thus we get the closed-form solution of γ as follows,

$$\gamma=\frac{\alpha wv\left[(\beta+w)e^{w(b_2-b_1)}+(\beta+v)e^{v(b_2-b_1)}\right]+\delta\beta(w-v)}{E}, \tag{3.38}$$

where

$$\begin{aligned}E=\delta\beta\Big\{&v(\beta+v)\left[(w-r)e^{rb_1}-(w-s)e^{sb_1}\right]e^{v(b_2-b_1)}\\&-w(\beta+w)\left[(v-r)e^{rb_1}-(v-s)e^{sb_1}\right]e^{w(b_2-b_1)}\Big\}.\end{aligned} \tag{3.39}$$

We can get C and G by substituting γ into (3.36) and (3.37).

Hence

$$V_{1*}(u;b_1,b_2)=\gamma\left[(r+\beta)e^{ru}-(s+\beta)e^{su}\right],\ \text{if } 0\le u<b_1, \tag{3.40}$$

and

$$\begin{aligned}V_{2*}(u;b_1,b_2)=&\frac{\alpha}{\delta}+\frac{\beta+v}{w-v}\left\{\gamma\left[(w-r)e^{rb_1}-(w-s)e^{sb_1}\right]-\frac{\alpha w}{\delta\beta}\right\}e^{v(u-b_1)}\\&+\frac{\beta+w}{v-w}\left\{\gamma[(v-r)e^{rb_1}-(v-s)e^{sb_1}]-\frac{\alpha v}{\delta\beta}\right\}e^{w(u-b_1)},\quad \text{if } b_1\le u\le b_2.\end{aligned} \tag{3.41}$$

Remark 3.2 *Let us compare our results with known results.*

1) *When* $b_1=b_2,\alpha=\mu$, *the hybrid dividend becomes a barrier dividend strategy, the condition* (3.25) *is the same as* (3.24), *from* (3.31) *and* (3.24), *we have*

$$\gamma=\frac{1}{(\beta+r)re^{rb_1}-(\beta+s)se^{sb_1}},$$

which agrees with formula (7.8) *in* [2].

2) *Letting* $b_2\to\infty$, *the hybrid dividend strategy becomes a threshold dividend strategy, we get*

$$\lim_{b_2\to\infty}\gamma=\frac{\alpha w}{\delta\beta}\frac{1}{(w-r)e^{rb_1}-(w-s)e^{sb_1}}. \tag{3.42}$$

From (3.36), (3.37) *and* (3.42), *we have*

$$\lim_{b_2\to\infty}G=\frac{\alpha(\beta+w)}{\delta\beta}\frac{re^{rb_1}-se^{sb_1}}{(w-r)e^{rb_1}-(w-s)e^{sb_1}}e^{-wb_1}. \tag{3.43}$$

$$\lim_{b_2\to\infty}C=0. \tag{3.44}$$

It follows from (3.40) *to* (3.44) *that*

$$\lim_{b_2\to\infty}V_{1*}(u;b_1,b_2)=\frac{\alpha w}{\delta\beta}\frac{(\beta+r)e^{ru}-(\beta-s)e^{su}}{(w-r)e^{rb_1}-(w-s)e^{sb_1}},\ \text{if } 0\le u\le b_1,$$

and

$$\begin{aligned}\lim_{b_2\to\infty}V_{2*}(u;b_1,b_2)&=\frac{\alpha}{\delta}+\frac{\alpha(\beta+w)}{\delta\beta}\frac{re^{rb_1}-se^{sb_1}}{(w-r)e^{rb_1}-(w-s)e^{sb_1}}e^{w(u-b_1)}\\&=\frac{\alpha}{\delta}\left[1-e^{w(u-b_1)}\right]+\lim_{b_2\to\infty}V_{1*}(b_1;b_1,b_2)e^{w(u-b_1)},\quad \text{if } b_1\le u,\end{aligned}$$

which are (6.14) *and* (6.15) *in* [7].

4. The Moment-Generating Function

In this section, we study the moment-generating function $M(u,z;b_1,b_2)$ which has been discussed in various models, for example, see [8] [19]. We can analyze the moments of D through $M(u,z;b_1,b_2)$. Since $M(u,z;b_1,b_2)$ has different paths for $0 \le u < b_1$ and $b_1 \le u \le b_2$, we define

$$M(u,z;b_1,b_2) = \begin{cases} M_1(u,z;b_1,b_2), & 0 \le u < b_1, \\ M_2(u,z;b_1,b_2), & b_1 \le u \le b_2. \end{cases}$$

We first derive the integro-differential equations and boundary conditions for $M(u,z;b_1,b_2)$.

Theorem 4.1 *Assume that* $M(u,z;b_1,b_2)$ *is continuously differentiable in* u *on* $(0,b_1)\cup(b_1,b_2)$ *and in* $y \ge 0$. *Then,* $M(u,z;b_1,b_2)$ *satisfies the following integro-differential equations, when* $0 < u < b_1$,

$$\mu \frac{\partial M_1(u,z;b_1,b_2)}{\partial u} = \delta z \frac{\partial M_1(u,z;b_1,b_2)}{\partial z} + \lambda M_1(u,z;b_1,b_2) - \lambda\left[\int_0^u M_1(u-y,z;b_1,b_2)\,p(y)\,\mathrm{d}y + 1 - P(u)\right], \tag{4.1}$$

and, when $b_1 < u < b_2$,

$$(\mu-\alpha)\frac{\partial M_2(u,z;b_1,b_2)}{\partial u} = \delta z \frac{\partial M_2(u,z;b_1,b_2)}{\partial z} + (\lambda - z\alpha) M_2(u,z;b_1,b_2) - \lambda\left[\int_0^{u-b_1} M_2(u-y,z;b_1,b_2)\,p(y)\,\mathrm{d}y + \int_{u-b_1}^{u} M_1(u-y,z;b_1,b_2)\,p(y)\,\mathrm{d}y + 1 - P(u)\right], \tag{4.2}$$

with boundary conditions

$$M_1(b_1-,z;b_1,b_2) = M_2(b_1+,z;b_1,b_2), \tag{4.3}$$

$$\left.\frac{\partial M_2(u,z;b_1,b_2)}{\partial u}\right|_{u=b_2-} = zM_2(b_2,z;b_1,b_2), \tag{4.4}$$

$$\mu \left.\frac{\partial M_1(u,z;b_1,b_2)}{\partial u}\right|_{u=b_1-} = (\mu-\alpha)\left.\frac{\partial M_2(u,z;b_1,b_2)}{\partial u}\right|_{u=b_1+} + z\alpha M_2(b_1,z;b_1,b_2). \tag{4.5}$$

Proof. In view of the strong Markov property of the surplus process $\{\tilde{U}_t, t \ge 0\}$, we have

$$M_1(u,z;b_1,b_2) = E\left[M_1\left(\tilde{U}_t, z\mathrm{e}^{-\delta t}; b_1,b_2\right)\right] + o(t), \tag{4.6}$$

when $0 \le u < b_1$, consider $t > 0$ being sufficiently small so that the modified surplus can not reach level b_1 by time t. By conditioning on the time and amount of the first claim and whether the claim causes ruin or not, and using (4.6), we get

$$M_1(u,z;b_1,b_2) = (1-\lambda t) M_1\left(u+\mu t, z\mathrm{e}^{-\delta t}; b_1,b_2\right) + \lambda t\left[\int_0^{u+\mu t} M_1\left(u+\mu t - y, z\mathrm{e}^{-\delta t}; b_1,b_2\right) p(y)\,\mathrm{d}y\right] + \lambda t\left[1 - P(u+\mu t)\right] + o(t). \tag{4.7}$$

By Taylor's expansion,

$$M_1\left(u+\mu t, z\mathrm{e}^{-\delta t}; b_1,b_2\right) = M_1(u,z;b_1,b_2) + \mu t \frac{\partial M_1(u,z;b_1,b_2)}{\partial u} - \delta z t \frac{\partial M_1(u,z;b_1,b_2)}{\partial z} + o(t).$$

Substituting the above expression into (4.7), and dividing both sides of (4.7) by t and letting $t \to 0$, we can get (4.1).

When $b_1 < u < b_2$, we still consider a small time interval $[0,t]$, with $t > 0$ being sufficiently small so that the modified surplus will not reach b_2 in the interval. In view of the strong Markov property of the surplus process $\{\tilde{U}_t, t \geq 0\}$, we have

$$M_2(u,z;b_1,b_2) = \mathrm{e}^{z\alpha t} E_u\left[M_2\left(\tilde{U}_t, z\mathrm{e}^{-\delta t}; b_1, b_2\right)\right] + o(t). \tag{4.8}$$

By conditioning on the time and amount of the first claim and whether the claim causes ruin or not, and using (4.8), we yield

$$\begin{aligned} M_2(u,z;b_1,b_2) = {} & (1-\lambda t)\mathrm{e}^{z\alpha t} M_2\left(u+(\mu-\alpha)t, z\mathrm{e}^{-\delta t}; b_1, b_2\right) \\ & + \lambda t \mathrm{e}^{z\alpha t}\left[\int_0^{u+(\mu-\alpha)t-b_1} M_2\left(u+(\mu-\alpha)t-y, z\mathrm{e}^{-\delta t}; b_1,b_2\right)p(y)\mathrm{d}y\right. \\ & \left. + \int_{u+(\mu-\alpha)t-b_1}^{u+(\mu-\alpha)t} M_1\left(u+(\mu-\alpha)t-y, z\mathrm{e}^{-\delta t}; b_1,b_2\right)p(y)\mathrm{d}y\right] \\ & + \lambda t \mathrm{e}^{z\alpha t}\left[1-P\left(u+(\mu-\alpha)t\right)\right] + o(t). \end{aligned} \tag{4.9}$$

Since

$$\begin{aligned} & M_2\left(u+(\mu-\alpha)t, z\mathrm{e}^{-\delta t}; b_1, b_2\right) \\ & = M_2(u,z;b_1,b_2) + (\mu-\alpha)t\frac{\partial M_2(u,z;b_1,b_2)}{\partial u} - \delta z t\frac{\partial M_2(u,z;b_1,b_2)}{\partial z} + o(t), \end{aligned}$$

using the similar arguments as above, we get (4.2) from (4.9).

Next we prove the condition (4.3). For $0 \leq u < b_1$, let $\tau_{b_1} = \inf\{t : \tilde{U}_t = b_1, 0 \leq U_0 < b_1\}$, and t_0 is the time that the modified surplus reaches b_1 for the first time from $0 \leq u < b_1$ with no claims, *i.e.* $u + \mu t_0 = b_1$. Then τ_{b_1} is a stopping time, and by the strong Markov property, we have

$$\begin{aligned} M_1(u,z;b_1,b_2) & = E_u\left[\mathrm{e}^{zD} I\left(\tau_{b_1} < T\right)\right] + E_u\left[\mathrm{e}^{zD} I\left(\tau_{b_1} \geq T\right)\right] \\ & = E_u[M_2\left(b_1, z\mathrm{e}^{-\delta\tau_{b_1}}; b_1, b_2\right) I\left(\tau_{b_1} < T\right)] + P\left(\tau_{b_1} \geq T\right) \\ & \leq M_2\left(b_1, z\mathrm{e}^{-\delta\tau_{b_1}}; b_1, b_2\right) + P\left(\tau_{b_1} \geq T\right). \end{aligned} \tag{4.10}$$

On the other hand, we have

$$\begin{aligned} M_1(u,z;b_1,b_2) & \geq E_u\left[\mathrm{e}^{zD} I\left(\tau_{b_1} < T, \tau_{b_1} = t_0\right)\right] + E_u\left[\mathrm{e}^{zD} I\left(\tau_{b_1} \geq T\right)\right] \\ & = E_u\left[M_2\left(b_1, z\mathrm{e}^{-\delta\tau_{b_1}}; b_1, b_2\right) I\left(\tau_{b_1} < T, \tau_{b_1} = t_0\right)\right] + P\left(\tau_{b_1} \geq T\right) \\ & \geq M_2\left(b_1, z\mathrm{e}^{-\delta t_0}; b_1, b_2\right) P\left(T_1 > t_0\right) + P\left(\tau_{b_1} \geq T\right), \end{aligned} \tag{4.11}$$

where T_1 is the first time that the claim happens. When $u \uparrow b_1$, τ_{b_1} and t_0 both go into zero, and $\lim_{u\uparrow b_1} P\left(\tau_{b_1} \geq T\right) = 0$, letting $u \uparrow b_1$ in (4.10) and (4.11), we obtain

$$M_1(b_1-, z; b_1, b_2) = M_2(b_1, z; b_1, b_2).$$

When $u = b_1$, we consider an infinitesimal time interval $[0,t]$, then

$$\begin{aligned} M_2(b_1,z;b_1,b_2) = {} & \mathrm{e}^{z\int_0^t \alpha \mathrm{e}^{-\delta s} I_{(b_1<\tilde{U}_s\leq b_2)}\mathrm{d}s}\left\{\mathrm{e}^{-\lambda t} E_{b_1}\left[M_2\left(b_1+(\mu-\alpha)t, z\mathrm{e}^{-\delta t}; b_1, b_2\right)\right]\right. \\ & \left. + E_{b_1}\left[M\left(\tilde{U}_t, z\mathrm{e}^{-\delta t}; b_1, b_2\right) I\left(N(t)=1, T>t\right)\right]\right\} + O(t). \end{aligned}$$

From this formula we get

$$M_2(b_1,z;b_1,b_2) \geq \mathrm{e}^{z\int_0^t \alpha \mathrm{e}^{-\delta s} I_{(b_1<\tilde{U}_s\leq b_2)}\mathrm{d}s} \mathrm{e}^{-\lambda t} E_{b_1}\left[M_2\left(b_1+(\mu-\alpha)t, z\mathrm{e}^{-\delta t}; b_1, b_2\right)\right],$$

and

$$M_2(b_1,z;b_1,b_2) \le e^{\left|z\int_0^t \alpha e^{-\delta s}ds\right|}\Big\{e^{-\lambda t}E_{b_1}\left[M_2\left(b_1+(\mu-\alpha)t, ze^{-\delta t};b_1,b_2\right)\right]$$
$$+E_{b_1}\left[M\left(\tilde{U}_t, ze^{-\delta t};b_1,b_2\right)I\left(N(t)=1,T>t\right)\right]\Big\}+\left|O(t)\right|.$$

Let $t\downarrow 0$, we obtain

$$M_2(b_1+,z;b_1,b_2)=M_2(b_1,z;b_1,b_2).$$

So we obtain (4.3).

Furthermore, when the initial surplus is b_2, we can mimic the derivation of (4.9) to obtain

$$\begin{aligned} M_2(b_2,z;b_1,b_2) &= (1-\lambda t)e^{z(\mu-\alpha)t}e^{z\alpha t}M_2\left(b_2, ze^{-\delta t};b_1,b_2\right) \\ &+\lambda te^{z\alpha t}\left[\int_0^{b_2-b_1}M_2\left(b_2-y, ze^{-\delta t};b_1,b_2\right)p(y)dy\right. \\ &\left.+\int_{b_2-b_1}^{b_2}M_1\left(b_2-y, ze^{-\delta t};b_1,b_2\right)p(y)dy\right] \\ &+\lambda te^{z\alpha t}\left[1-P(b_2)\right]+o(t). \end{aligned} \tag{4.12}$$

Using

$$M_2\left(b_2, ze^{-\delta t};b_1,b_2\right)=M_2(b_2,z;b_1,b_2)-\delta zt\frac{\partial M_2(b_2,z;b_1,b_2)}{\partial z}+o(t).$$

Substituting the above expression into (4.12), and dividing both sides of (4.12) by t and letting $t\to 0$, we can obtain

$$\begin{aligned} \delta z\frac{\partial M_2(b_2,z;b_1,b_2)}{\partial z} &= \left[(\mu-\alpha)z+z\alpha-\lambda\right]M_2(b_2,z;b_1,b_2) \\ &+\lambda\left[\int_0^{b_2-b_1}M_2(b_2-y,z;b_1,b_2)p(y)dy\right. \\ &\left.+\int_{b_2-b_1}^{b_2}M_1(b_2-y,z;b_1,b_2)p(y)dy+1-P(b_2)\right]. \end{aligned} \tag{4.13}$$

Letting $u\uparrow b_2$ in (4.2) and comparing it to (4.13), we obtain

$$\left.\frac{\partial M_2(u,z;b_1,b_2)}{\partial u}\right|_{u=b_2-} = zM_2(b_2,z;b_1,b_2).$$

Finally, letting $u\uparrow b_1$ in (4.1) and $u\downarrow b_2$ in (4.2), we can get (4.5). This completes the proof of Theorem 4.1.

Remark 4.1 1) *In the case of* $b_1=b_2$, (4.1) *is corresponding to* (3.1) *of* [20] *by letting* $a=0$, $\sigma=0$ *and* μ *substitute c there.*

2) *In the case of* $b_2\to\infty$, (4.1) *and* (4.2) *are corresponding to* (2.10) *and* (2.11) *of* [21] *by letting* $\sigma=0$, $r=0$ *and* μ *substitute c there.*

By the definitions of $M(u,z;b_1,b_2)$ and $V_k(u;b_1,b_2)$, we obtain

$$M(u,z;b_1,b_2)=1+\sum_{k=1}^{\infty}\frac{z^k}{k!}V_k(u;b_1,b_2). \tag{4.14}$$

We denote

$$V_k(u;b_1,b_2)=\begin{cases} V_{k,1}(u;b_1,b_2), & 0\le u<b_1, \\ V_{k,2}(u;b_1,b_2), & b_1\le u\le b_2. \end{cases}$$

Substituting (4.14) into (4.1) and (4.2) respectively and comparing the coefficients of z^k yields the following integro-differential equations and corresponding boundary conditions.

Theorem 4.2 *For each* $k\ge 1$, *we assume that* $V_k(u;b_1,b_2)$ *is continuously differentiable in u on*

$(0,b_1)\cup(b_1,b_2)$. *Then,* $V_k(u;b_1,b_2)$ *satisfies the following integro-differential equations*:

$$\mu\frac{\partial V_{k,1}(u;b_1,b_2)}{\partial u}=(k\delta+\lambda)V_{k,1}(u;b_1,b_2)-\lambda\int_0^u V_{k,1}(u-y;b_1,b_2)p(y)\mathrm{d}y,\ \ 0<u<b_1,$$

and

$$(\mu-\alpha)\frac{\partial V_{k,2}(u;b_1,b_2)}{\partial u}=(k\delta+\lambda)V_{k,2}(u;b_1,b_2)+\alpha kV_{k-1,2}(u;b_1,b_2)-\lambda\Big[\int_0^{u-b_1}V_{k,2}(u-y;b_1,b_2)p(y)\mathrm{d}y$$
$$+\int_{u-b_1}^{u}V_{k,1}(u-y;b_1,b_2)p(y)\mathrm{d}y\Big],\quad b_1<u<b_2,$$

with boundary conditions

$$V_{k,1}(b_1-;b_1,b_2)=V_{k,2}(b_1+;b_1,b_2),$$
$$\left.\frac{\partial V_{k,2}(u;b_1,b_2)}{\partial u}\right|_{u=b_2-}=kV_{k-1,2}(b_2;b_1,b_2),\tag{4.15}$$
$$\mu\left.\frac{\partial V_{k,1}(u;b_1,b_2)}{\partial u}\right|_{u=b_1-}=(\mu-\alpha)\left.\frac{\partial V_{k,2}(u;b_1,b_2)}{\partial u}\right|_{u=b_1+}+\alpha kV_{k-1,2}(b_1;b_1,b_2).$$

Remark 4.2 *Letting* $k=1$, *we have* $V_1(u;b_1,b_2)=V(u;b_1,b_2)$, *Theorem* 3.3 *can be reduced by Theorem* 4.2. *From* (4.15).

$$\left.\frac{\partial V_{1,2}(u;b_1,b_2)}{\partial u}\right|_{u=b_2-}=1$$

is an obvious result since $V_{01}(b_2;b_1,b_2)=1$.

5. The Gerber-Shiu Functions

In the following we will discuss the famous Gerber-Shiu expected discounted penalty function $\Phi(u;b_1,b_2)$. We also write

$$\Phi(u;b_1,b_2)=\begin{cases}\Phi_1(u;b_1,b_2), & 0\le u<b_1,\\ \Phi_2(u;b_1,b_2), & b_1\le u\le b_2.\end{cases}$$

By a similar derivation to Theorem 4.1, we get the integro-differential equations and boundary conditions for $\Phi(u;b_1,b_2)$.

Theorem 5.1 *Assume that* $\Phi(u;b_1,b_2)$ *is continuously differentiable in* u *on* $(0,b_1)\cup(b_1,b_2)$. *Then,* $\Phi(u;b_1,b_2)$ *satisfies the following integro-differential equations, when* $0<u<b_1$,

$$\mu\frac{\partial\Phi_1(u;b_1,b_2)}{\partial u}=(\lambda+\delta)\Phi_1(u;b_1,b_2)-\lambda\Big[\int_0^u\Phi_1(u-y;b_1,b_2)p(y)\mathrm{d}y+A(u)\Big],\tag{5.1}$$

and, when $b_1<u<b_2$,

$$(\mu-\alpha)\frac{\partial\Phi_2(u;b_1,b_2)}{\partial u}=(\lambda+\delta)\Phi_2(u;b_1,b_2)-\lambda\Big[\int_0^{u-b_1}\Phi_2(u-y;b_1,b_2)p(y)\mathrm{d}y$$
$$+\int_{u-b_1}^{u}\Phi_1(u-y;b_1,b_2)p(y)\mathrm{d}y+A(u)\Big],\tag{5.2}$$

where $A(u)=\int_u^\infty\omega(u,y-u)p(y)\mathrm{d}y$ *and with boundary conditions*

$$\Phi_1(b_1-;b_1,b_2)=\Phi_2(b_1+;b_1,b_2),\tag{5.3}$$

$$\left.\frac{\partial\Phi_2(u;b_1,b_2)}{\partial u}\right|_{u=b_2}=0,\tag{5.4}$$

$$\mu \frac{\partial \Phi_1(u;b_1,b_2)}{\partial u}\bigg|_{u=b_1-} = (\mu-\alpha)\frac{\partial \Phi_2(u;b_1,b_2)}{\partial u}\bigg|_{u=b_1+}. \tag{5.5}$$

Proof. We can mimic the derivation of (4.1), (4.2), (4.3) and (4.5) to obtain (5.1), (5.2), (5.3) and (5.5).

Next we prove the condition (5.4). In view of the strong Markov property of the surplus process $\{\tilde{U}_t, t \geq 0\}$, we have

$$\Phi(u;b_1,b_2) = \mathrm{e}^{-\delta t} E_u\left[\Phi\left(\tilde{U}_t;b_1,b_2\right)\right]. \tag{5.6}$$

When the initial surplus is b_2,

$$\Phi_2(b_2;b_1,b_2) = (1-\lambda t)\mathrm{e}^{-\delta t}\Phi_2(b_2;b_1,b_2) + \mathrm{e}^{-\delta t}\lambda t\left[\int_0^{b_2-b_1}\Phi_2(b_2-y;b_1,b_2)p(y)\mathrm{d}y \right.$$
$$\left. + \int_{b_2-b_1}^{b_2}\Phi_1\left(b_2-y, z\mathrm{e}^{-\delta t};b_1,b_2\right)p(y)\mathrm{d}y + \int_{b_2}^{\infty}\omega(b_2,y-b_2)p(y)\mathrm{d}y\right],$$

dividing t on both sides of the above expression, letting $t \to 0$, we can obtain

$$(\delta+\lambda)\Phi_2(b_2;b_1,b_2) = \lambda\left[\int_0^{b_2-b_1}\Phi_2(b_2-y;b_1,b_2)p(y)\mathrm{d}y + \int_{b_2-b_1}^{b_2}\Phi_1(b_2-y,z;b_1,b_2)p(y)\mathrm{d}y \right.$$
$$\left. + \int_{b_2}^{\infty}\omega(b_2,y-b_2)p(y)\mathrm{d}y\right]. \tag{5.7}$$

Letting $u \uparrow b_2$ in (5.2) and comparing it to (5.7), we obtain

$$\frac{\partial \Phi_2(u;b_1,b_2)}{\partial u}\bigg|_{u=b_2-} = 0.$$

When $u > b_2$, we have

$$\Phi(u;b_1,b_2) = \Phi_2(b_2;b_1,b_2),$$

thus,

$$\frac{\partial \Phi(u;b_1,b_2)}{\partial u}\bigg|_{u=b_2+} = 0.$$

So we get (5.4).

This completes the proof of Theorem 5.1.

Remark 5.1 1) *In the case of* $b_1 = b_2$, (5.1) *is corresponding to* (2.6) *of* [3] *by letting* $\mu, A(u)$ *substitute* $c, \zeta(u)$.

2) *Letting* $b_2 \to \infty$, (5.1) *and* (5.2) *are corresponding to* (3.1) *of* [9] *by letting* $\mu, A(u)$ *substitute* $c_1, \zeta(u)$.

6. Explicit Expressions of the Laplace Transform of Ruin Time

In this section, we give the closed form expression for the Laplace transform of ruin time when claim size has exponential distribution with mean $1/\beta$, *i.e.* $p(y) = \beta \mathrm{e}^{-\beta y}, y > 0$. We also write

$$L(u;b_1,b_2) = \begin{cases} L_1(u;b_1,b_2), & 0 \leq u < b_1, \\ L_2(u;b_1,b_2), & b_1 \leq u \leq b_2. \end{cases}$$

By setting $\omega(x,y) \equiv 1$ in (5.1) and (5.2) and letting $L(u;b_1,b_2)$ substitute $\Phi(u;b_1,b_2)$, we obtain the integro-differential equations and the boundary conditions satisfied by $L(u;b_1,b_2)$ from Theorem 5.1.

Theorem 6.1 $L(u;b_1,b_2)$ *satisfies the following integro-differential equations, when* $0 < u < b_1$,

$$\mu \frac{\partial L_1(u;b_1,b_2)}{\partial u} = (\lambda+\delta)L_1(u;b_1,b_2) - \lambda\left[\int_0^u L_1(u-y;b_1,b_2)p(y)\mathrm{d}y + 1 - P(u)\right], \tag{6.1}$$

and, when $b_1 < u < b_2$,

$$(\mu-\alpha)\frac{\partial L_2(u;b_1,b_2)}{\partial u}=(\lambda+\delta)L_2(u;b_1,b_2)-\lambda\Big[\int_0^{u-b_1}L_2(u-y;b_1,b_2)p(y)\mathrm{d}y + \int_{u-b_1}^{u}L_1(u-y;b_1,b_2)p(y)\mathrm{d}y+1-P(u)\Big], \tag{6.2}$$

with boundary conditions

$$L_1(b_1-;b_1,b_2)=L_2(b_1+;b_1,b_2), \tag{6.3}$$

$$\left.\frac{\partial L_2(u;b_1,b_2)}{\partial u}\right|_{u=b_2}=0, \tag{6.4}$$

$$\mu\left.\frac{\partial L_1(u;b_1,b_2)}{\partial u}\right|_{u=b_1-}=(\mu-\alpha)\left.\frac{\partial L_2(u;b_1,b_2)}{\partial u}\right|_{u=b_1+}. \tag{6.5}$$

Remark 6.1 *In the case of* $b_2\to\infty$, (6.1) *and* (6.2) *are corresponding to equations* (10.2) *and* (10.3) *in* [7].

Applying $(\mathrm{d}/\mathrm{d}u+\beta)$ to (6.1) and (6.2) in the case of $p(y)=\beta e^{-\beta y}, y>0$ respectively, and using (3.26) and rearranging them, we have that for $0<u<b_1$

$$\mu\frac{\partial L_1^2(u;b_1,b_2)}{\partial u^2}+(\beta\mu-\lambda-\delta)\frac{\partial L_1(u;b_1,b_2)}{\partial u}-\beta\delta L_1(u;b_1,b_2)=0, \tag{6.6}$$

and for $b_1<u<b_2$

$$(\mu-\alpha)\frac{\partial L_2^2(u;b_1,b_2)}{\partial u^2}+\left[\beta(\mu-\alpha)-\lambda-\delta\right]\frac{\partial L_2(u;b_1,b_2)}{\partial u}-\beta\delta L_2(u;b_1,b_2)=0. \tag{6.7}$$

We can obtain the solutions of Equation (6.6) and (6.7) as follows

$$L_1(u;b_1,b_2)=C_0e^{ru}+C_1e^{su},\quad 0\le u\le b_1, \tag{6.8}$$

$$L_2(u;b_1,b_2)=G_0e^{wu}+G_1e^{vu},\quad b_1<u\le b_2, \tag{6.9}$$

with the coefficients C_0, C_1, G_0 and G_1 being independent of u, and r, s, w and v are the same as in Example 3.1. Substituting (5.8) in Equation (5.1) and equating the coefficient of $e^{-\beta u}$ with 0, we have

$$\beta\left(\frac{C_0}{r+\beta}+\frac{C_1}{s+\beta}\right)=1. \tag{6.10}$$

Substitute (5.8) and (5.9) in Equation (5.2) and equating the coefficient of $e^{-\beta u}$ with 0, we have

$$C_0\frac{e^{rb_1}}{r+\beta}+C_1\frac{e^{sb_1}}{s+\beta}=G_0\frac{e^{wb_1}}{\beta+w}+G_1\frac{e^{vb_1}}{\beta+v}. \tag{6.11}$$

From (5.3) and (5.4), we have the conditions

$$C_0e^{rb_1}+C_1e^{sb_1}=G_0e^{wb_1}+G_1e^{vb_1}, \tag{6.12}$$

and

$$G_0we^{wb_2}+G_1ve^{vb_2}=0. \tag{6.13}$$

It follows from (6.11) and (6.12) that

$$C_0=\frac{\beta+r}{r-s}\left(e^{wb_1}\frac{w-s}{\beta+w}G_0+e^{vb_1}\frac{v-s}{\beta+v}G_1\right)e^{-rb_1}, \tag{6.14}$$

$$C_1=\frac{\beta+r}{s-r}\left(e^{wb_1}\frac{w-r}{\beta+w}G_0+e^{vb_1}\frac{v-r}{\beta+v}G_1\right)e^{-sb_1}, \tag{6.15}$$

and from (6.13), we get

$$G_0 = -(v/w)e^{(v-w)b_2}G_1. \tag{6.16}$$

Substituting (6.14), (6.15) into (6.10) and then using (6.16), the constants G_0 and G_1 can be given by

$$G_0 = \frac{r-s}{\beta}\frac{v(\beta+w)(\beta+v)}{I}e^{b_2(v-w)}, \tag{6.17}$$

$$G_1 = \frac{s-r}{\beta}\frac{w(\beta+w)(\beta+v)}{I}, \tag{6.18}$$

where

$$\begin{aligned} I &= (\beta+w)w\left[(v-r)e^{-sb_1} - (v-s)e^{-rb_1}\right]e^{vb_1} \\ &\quad + (\beta+v)v\left[(w-s)e^{-rb_1} - (w-r)e^{-sb_1}\right]e^{wb_1+(v-w)b_2}. \end{aligned} \tag{6.19}$$

Substituting (6.17) and (6.18) into (6.14) and (6.15), the constants C_0 and C_1 can be given by

$$C_0 = \frac{\beta+r}{\beta I}\left[v(\beta+v)(w-s)e^{wb_1+(v-w)b_2} - w(\beta+w)(v-s)e^{vb_1}\right]e^{-rb_1}, \tag{6.20}$$

$$C_1 = \frac{\beta+s}{\beta I}\left[-v(\beta+v)(w-r)e^{wb_1+(v-w)b_2} + w(\beta+w)(v-r)e^{vb_1}\right]e^{-sb_1}. \tag{6.21}$$

From (6.17)-(6.21), we have

$$\begin{aligned} L_1(u;b_1,a) &= \frac{\beta+r}{\beta I}\left[v(\beta+v)(w-s)e^{wb_1+(v-w)b_2} - w(\beta+w)(v-s)e^{vb_1}\right]e^{r(u-b_1)} \\ &\quad + \frac{\beta+s}{\beta I}\left[-v(\beta+v)(w-r)e^{wb_1+(v-w)b_2} + w(\beta+w)(v-r)e^{vb_1}\right]e^{s(u-b_1)}, \quad \text{if } 0 \le u < b_1, \end{aligned} \tag{6.22}$$

$$L_2(u;b_1,a) = \frac{s-r}{w}\beta Iv(\beta+w)(\beta+v)e^{wu+(v-w)b_2} + \frac{r-s}{\beta I}(\beta+w)(\beta+v)e^{vu}, \quad \text{if } b_1 \le u \le b_2. \tag{6.23}$$

Remark 6.2 *Letting* $b_2 \to \infty$, *from* (6.17) *to* (6.21), *we have*

$$\lim_{b_2\to\infty} C_0 = \frac{\beta+r}{\beta}\frac{(w-s)e^{sb_1}}{(w-s)e^{sb_1} - (w-r)e^{rb_1}},$$

$$\lim_{b_2\to\infty} C_1 = \frac{\beta+s}{\beta}\frac{(r-w)e^{rb_1}}{(w-s)e^{sb_1} - (w-r)e^{rb_1}},$$

$$\lim_{b_2\to\infty} G_0 = \frac{r-s}{\beta}\frac{\beta+w}{(w-s)e^{-rb_1} - (w-r)e^{-sb_1}}e^{-wb_1},$$

$$\lim_{b_2\to\infty} G_1 = 0.$$

Thus,

$$\lim_{b_2\to\infty} L_1(u;b_1,b_2) = \frac{1}{\beta}\frac{(\beta+r)(w-s)e^{sb_1+ru} + (\beta+s)(r-w)e^{rb_1+su}}{(w-s)e^{sb_1} - (w-r)e^{rb_1}}, \quad 0 \le u \le b_1,$$

$$\lim_{b_2\to\infty} L_2(u;b_1,b_2) = \frac{r-s}{\beta}\frac{\beta+w}{(w-s)e^{-rb_1} - (w-r)e^{-sb_1}}e^{w(u-b_1)}, \quad u > b_1,$$

which are (10.17) *and* (10.19) *of* [7].

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