

# Study of Stability Analysis for a Class of Fourth Order Boundary Value Problems

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## Abstract

Fourth order differential equations are considered to develop the class of methods for the numerical solution of boundary value problems. In this paper, we have discussed the regions of absolute stability of fourth order boundary value problems. Methods proposed and derived in this paper are applied to solve a fourth-order boundary value problem. Numerical results are given to illustrate the efficiency of our methods and compared with exact solution.

## Keywords

Numerical Differentiation, Initial Value Problem, Boundary Value Problem, Absolute Stability, Multistep Methods

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## 1. Introduction

The determination process for the numerical solution of initial value problems in ordinary differential equations can be classified into two categories-single step methods and multistep methods. Single step methods are those in which the approximation for the point  $x = x_{n+1}$  involves information from only one of the previous points  $x = x_n$ . Methods using the approximation at more than one previous points to determine the approximation at the next point are called multistep methods. Thus a  $k$ -step method requires information about the solution at  $k$  points  $x_n, x_{n-1}, \dots, x_{n-k+1}$  to compute the solution at the point  $x_{n+1}$ . Finite difference methods for boundary value problems are discussed in [1]. Linear multi step methods of second order differential equations are discussed in [2]. The methods based on numerical differentiation for first-order differential equations have been shown to be stiffly stable by Gear [3]. A detailed study of the single step and multistep methods has been carried out by Gear [3], Gragg and Statter [4] and Henrici [5]. Gear [3] and Peter Henrici [5] have derived special multistep methods based on numerical integration and numerical differentiation for solving first-order differential equations. Jain [6]

has considered high order stiffly stable methods. Further information can be had from [7] and [8]. Special multistep methods based on numerical differentiation for solving the initial value problem have been derived in Rama Chandra Rao [9]. The methods now to be discussed are based on replacing the function  $f(x, y(x))$  which is unknown, by an interpolating polynomial having the values  $f_n = f(x_n, y_n)$  on a set of points  $x_n$  where  $y_n$  has already been computed. The methods discussed in this paper are essentially based on the idea that the solution is best approximated by polynomials. The motivation for the work carried out in this paper arises from the methods based on numerical differentiation for the first-order differential equations, special multistep methods based on numerical integration for the solution of the special second-order differential equations by Henrici [5] and Special multistep methods based on numerical differentiation for solving the initial value problem by Rama Chandra Rao [9]. In Henrici [5] methods based on Numerical Integration have been derived by integrating  $y'' = f(x, y)$  twice and replacing the function  $f(x, y)$  by an interpolating polynomial. Special multistep methods have been derived by replacing  $y(x)$  on the left hand side of  $y^{iv}(x) = f(x, y)$  by an interpolating polynomial and differentiating it four times. We have investigated a class of implicit methods. It is found that the implicit methods have order  $(k-3)$ . Some local truncation errors are provided. The regions of absolute stability of the methods are derived. Numerical tests of the performance of the methods are established by solving differential equation and compared with the exact solution. The numerical results reported show the validity of our methods.

## 2. General Linear Multistep Methods for Special Fourth-Order Differential Equations

The special fourth order differential equation

$$y^{iv} = f(x, y), \quad y(0) = y_0, \quad y'(0) = y'_0, \quad y''(0) = y''_0, \quad y'''(0) = y'''_0 \tag{1}$$

occurs frequently in many number of problems of science and engineering.

A general linear multistep method of step number  $k$  for the numerical solution of equation (1) is given by

$$y_{n+1} = \sum_{j=1}^k a_j y_{n+1-j} + h^4 \sum_{j=0}^k b_j y_{n+1-j} \tag{2}$$

where  $a_j, b_j$  are constants and “ $h$ ” is the step length.

Introducing the polynomials

$$\rho(\xi) = \xi^k - \sum_{j=1}^k a_j \xi^{k-j} \quad \text{and} \quad \sigma(\xi) = \sum_{j=0}^k b_j \xi^{k-j} \tag{3}$$

Equation (2) can be written as

$$\rho(E)y_{n-k+1} - h^4 \sigma(E)y_{n-k+1}^{iv} = 0 \tag{4}$$

In Equation (4), “ $E$ ” is the shift operator defined by  $E(y_n) = y_{n+1}$

Applying (4) to  $y^{iv} = \lambda y$ , we get the characteristic equation

$$\rho(\xi) - \bar{h}\sigma(\xi) = 0, \quad \text{where } \bar{h} = \lambda h^4 \tag{5}$$

The roots  $\xi_i$  of the characteristic Equation (5) and  $\bar{h}$  are in general, complex and the region of absolute stability is defined to be the region of the complex  $\bar{h}$ -plane such that the roots of the characteristic Equation (5) lie within the unit circle whenever  $\bar{h}$  lies in the interior of the region. Denoting the region of absolute stability of  $R$  and its boundary by  $\partial R$ , the locus of  $\partial R$  is given by

$$\bar{h}(\theta) = \rho(e^{i\theta}) / \sigma(e^{i\theta}), \quad 0 \leq \theta \leq 2\pi \tag{6}$$

## 3. Derivation of the Methods

Let  $p(x)$  be the backward difference interpolating polynomial of  $y(x)$  at  $(k+1)$  abscissas  $x_{n+1}, x_n, \dots, x_{n-k+1}$ . Then  $p(x)$  is given by

$$p(x) = \sum_{m=0}^k (-1)^m \binom{-s}{m} \nabla^m y_{n+1}, \quad s = \frac{(x - x_{n+1})}{h} \tag{7}$$

Differentiating (7) four times with respect to  $x$ , we get

$$p^{iv}(x) = \left(\frac{1}{h^4}\right) \sum_{m=0}^k \frac{d^4}{ds^4} \left[ (-1)^m \binom{-s}{m} \right] \nabla^m y_{n+1}.$$

Replacing  $y^{iv}(x)$  by  $p^{iv}(x)$  in Equation (1) and putting  $x = x_{n+1-r}$  i.e.  $s = -r$ , we get,

$$\sum_{m=0}^k \delta_{r,m} \nabla^m y_{n+1} = h^4 f_{n+1-r} \tag{8}$$

where

$$\delta_{r,m} = \frac{d^4}{ds^4} \left[ (-1)^m \binom{-s}{m} \right] \tag{9}$$

Taking  $r = 0$  in (8), a class of methods can be attained which are given by

$$\sum_{m=0}^k \delta_{0,m} \nabla^m y_{n+1} = h^4 f_{n+1} \tag{10}$$

The coefficients  $\delta_{0,m}$  are shown in **Table 1**.

Differences in (10) are expressed in terms of function values.

After simplification, the Equation (10) will turn out into the form

$$\sum_{j=0}^k a_j y_{n+1-j} = h^4 f_{n+1} \tag{11}$$

The coefficients  $a_j$  are shown in **Table 2**.

The local truncation error of the formula (11) is given by

$$LTE = \delta_{0,k+1} h^{k+1} y^{k+1}(\eta) \tag{12}$$

**Table 1.** Coefficients of  $\delta_{0,m}$ ;  $m = 0(1)9$ .

M	0	1	2	3	4	5	6	7	8	9
$\delta_{0,m}$	0	0	0	0	1	2	$\frac{17}{6}$	$\frac{7}{2}$	$\frac{967}{240}$	$\frac{4523}{945}$

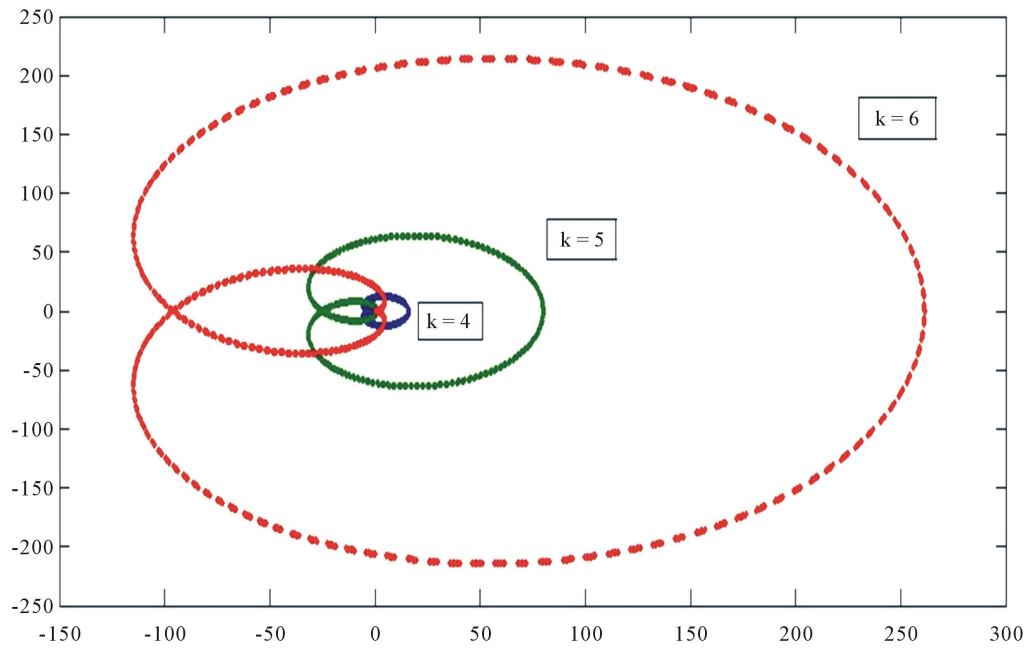
**Table 2.** Coefficients of  $a_j$ ;  $j = 0(1)k$ ,  $k = 4(1)9$ .

K	J	0	1	2	3	4	5	6	7	8	9
4	1	-4	6	-4	1						
5	3	-14	26	-24	11	-2					
6	$\frac{35}{6}$	$\frac{-186}{6}$	$\frac{411}{6}$	$\frac{-484}{6}$	$\frac{321}{6}$	$\frac{-114}{6}$	$\frac{17}{6}$				
7	$\frac{56}{6}$	$\frac{-333}{6}$	$\frac{852}{6}$	$\frac{-1009}{6}$	$\frac{846}{6}$	$\frac{-555}{6}$	$\frac{164}{6}$	$\frac{-21}{6}$			
8	$\frac{3207}{240}$	$\frac{-21056}{240}$	$\frac{61156}{240}$	$\frac{-94512}{240}$	$\frac{101530}{240}$	$\frac{-76352}{240}$	$\frac{33636}{240}$	$\frac{-8576}{240}$	$\frac{967}{240}$		
9	$\frac{4275}{240}$	$\frac{-30668}{240}$	$\frac{99604}{240}$	$\frac{-184224}{240}$	$\frac{236098}{240}$	$\frac{-210920}{240}$	$\frac{123348}{240}$	$\frac{-47024}{240}$	$\frac{10579}{240}$	$\frac{-1068}{240}$	

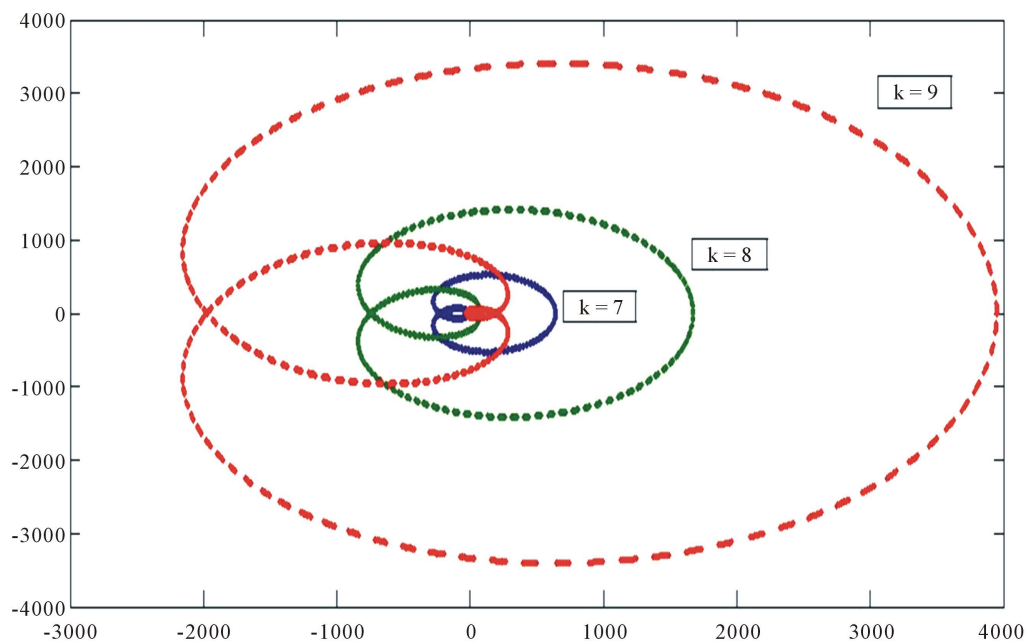
It follows that the  $k$ -step method (14) has the order  $k - 3$ , which is absolutely stable for  $h \in [-4, 0]$   
 For the method (13), we have

$$\rho(\xi) = \sum_{j=0}^k a_j \xi^{k-j} \text{ and } \sigma(\xi) = \xi^k. \tag{13}$$

The regions of absolute stability of the method for  $k = 4, 5, 6, 7, 8$  and  $9$  are shown in **Figure 1** and **Figure 2** (Taking real part on  $x$ -axis and imaginary part on  $y$ -axis). The region of absolute stability is the region lying outside the boundary.



**Figure 1.** The region of absolute stability of the method (13) for  $k = 4, 5$  and  $6$ .



**Figure 2.** The region of absolute stability of the method (13) for  $k = 7, 8$  and  $9$ .

## 4. Numerical Example

In this section, we have applied ND methods to solve the differential equation

$$y^{iv} = y + \sin x, \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = -1, \quad y'''(0) = 0 \quad (14)$$

in the interval  $[0, 4]$  with  $h = 0.01$  and  $h = 0.02$  and the results are shown in **Table 3** and **Table 4**.

The fourth order numerical differentiation method derived in this paper for  $k = 6$  is

$$y_{n+1} = \frac{186}{35}y_n - \frac{411}{35}y_{n-1} + \frac{484}{35}y_{n-2} - \frac{321}{35}y_{n-3} + \frac{114}{35}y_{n-4} - \frac{17}{35}y_{n-5} + \frac{6}{35}h^4 f_{n+1} \quad (15)$$

## 5. Discussion and Conclusion

The methods based on numerical integration are found to be closed regions of absolute stability; the methods

**Table 3.** Solution by fifth order ND with  $h = 0.01$ .

X	Exact Solution	Numerical Solution by fifth order ND	Absolute Error
0.10	9.500016525794E-02	9.500016525794E-02	7.813194535800E-15
0.20	1.800052419090E-01	1.800052419090E-01	5.329070518201E-15
0.30	2.550394442682E-01	2.550394442681E-01	5.662137425588E-15
0.40	3.201646548346E-01	3.201646548346E-01	3.719247132494E-15
0.50	3.754975976986E-01	3.754975976986E-01	3.774758283726E-15
0.60	4.212257271813E-01	4.212257271813E-01	2.775557561563E-15
0.70	4.576217499812E-01	4.576217499812E-01	2.720046410332E-15
0.80	4.850567100313E-01	4.850567100313E-01	8.881784197001E-16
0.90	5.040115785530E-01	5.040115785530E-01	1.221245327088E-15
1.00	5.150873072263E-01	5.150873072263E-01	2.997602166488E-15
1.10	5.190133197868E-01	5.190133197868E-01	1.887379141863E-15
1.20	5.166544364783E-01	5.166544364783E-01	4.662936703426E-15
1.30	5.090162464214E-01	5.090162464214E-01	5.440092820663E-15
1.40	4.972489648573E-01	4.972489648573E-01	5.440092820663E-15
1.50	4.826498351587E-01	4.826498351587E-01	8.826273045770E-15
1.60	4.666641592316E-01	4.666641592316E-01	9.992007221626E-15
1.70	4.508850642096E-01	4.508850642096E-01	1.304512053935E-14
1.80	4.370521379547E-01	4.370521379547E-01	1.637578961322E-14
1.90	4.270490905788E-01	4.270490905789E-01	1.471045507628E-14
2.00	4.229006237963E-01	4.229006237963E-01	1.737499033538E-14

**Table 4.** Solution by fifth order ND with  $h = 0.02$ .

X	Exact Solution	Numerical Solution by fifth order ND	Absolute Error
0.10	9.500016525794E-02	9.500016525718E-02	7.650130529058E-13
0.20	1.800052419090E-01	1.800052419082E-01	7.430722703816E-13
0.30	2.550394442682E-01	2.550394442675E-01	6.964984144986E-13
0.40	3.201646548346E-01	3.201646548340E-01	6.305511668359E-13
0.50	3.754975976986E-01	3.754975976981E-01	5.463962615693E-13
0.60	4.212257271813E-01	4.212257271809E-01	4.425348976156E-13
0.70	4.576217499812E-01	4.576217499809E-01	3.217981436876E-13
0.80	4.850567100313E-01	4.850567100311E-01	1.886824030350E-13
0.90	5.040115785530E-01	5.040115785529E-01	4.274358644807E-14
1.00	5.150873072263E-01	5.150873072265E-01	1.179056852152E-13
1.10	5.190133197868E-01	5.190133197871E-01	2.902122986370E-13
1.20	5.166544364783E-01	5.166544364787E-01	4.607425552194E-13
1.30	5.090162464214E-01	5.090162464221E-01	6.451505996097E-13
1.40	4.972489648573E-01	4.972489648581E-01	8.351652702743E-13
1.50	4.826498351587E-01	4.826498351598E-01	1.023958695612E-12
1.60	4.666641592316E-01	4.666641592328E-01	1.214361944335E-12
1.70	4.508850642096E-01	4.508850642110E-01	1.404265592697E-12
1.80	4.370521379547E-01	4.370521379563E-01	1.590561016229E-12
1.90	4.270490905788E-01	4.270490905806E-01	1.771249813487E-12
2.00	4.229006237963E-01	4.229006237983E-01	1.950384298510E-12

based on numerical differentiation are found to be absolutely stable outside some closed boundaries. We have obtained the solution by numerical differentiation methods which are derived in this paper and are more accurate. The absolute errors are very small.

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