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New Exact Traveling Wave Solutions for Some Coupled BBM Equations

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Abstract

The present paper deals with results of explicit traveling wave solutions for some coupled BBM equations. By detailed computation and using the $\left(\frac{G'}{G}\right)$-expansion method, many traveling wave solutions are given. These traveling waves are in the form of hyperbolic functions, the trigonometric functions and the rational functions, which show the reliability and efficiency of the used method.

Keywords

Traveling Waves, $\left(\frac{G'}{G}\right)$-Expansion Method, Coupled BBM Equation

1. Introduction

The study of the traveling wave solutions for nonlinear PDEs plays an important role in the study of nonlinear physical phenomena. Therefore, finding explicit solutions of physics equations is an important and interesting subject. In this paper, we discuss the exact traveling wave solutions for the following nonlinear evolution equations which can be used to describe small-amplitude long waves on the surface of water in a channel.

$$\begin{cases} \eta_t + u_x + (u\eta)_x - \eta_{xxt} = 0, \\ u_t + \eta_x + uu_x + \eta\eta_x - u_{xxt} = 0. \end{cases} \tag{1}$$

Some previous works on the existence and orbital stability of bell-shaped solitary waves with zero and nonzero asymptotic value have been obtained in [1] [2]. Our interest in the present work is to seek many new solutions for the coupled Equations (1). The method we used here is $\left(\frac{G'}{G}\right)$-expansion method which is proposed by Wang *et al.* in [3]. This method assumed that the traveling wave solutions can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$, where $G=G(\xi)$ satisfies the second-order ordinary differential equation $G''(\xi)+\lambda G'(\xi)+\mu G(\xi)=0$. The solutions obtained are expressed by hyperbolic functions, the trigonometric functions and the rational functions. We note that the solutions obtained in this paper extend the existence results in [1] [2]. Until now this method is widely used by many authors [4]-[7], and exact solutions for a variety of nonlinear equations are obtained. Especially, Akbar, Norhashidah and Zayed [6] proposed the extended $\left(\frac{G'}{G}\right)$-expansion method in which the solutions are presented in the form $\sum_{n=-m}^{m}\frac{e_{-n}}{\left(d+(G'/G)\right)^{n}}$. Recently, Taha, Noorani and Hashim [7] apply this method to provide closed-form traveling wave solutions of the generalized thin film equations and stand thin film equation, in which the related balance numbers are not the usual positive integers.

Since every nonlinear equation has its own physically significant rich structure, still much work has to be done. In this paper, we propose some new exact traveling wave solutions for Equations (1), and which stresses its power of $\left(\frac{G'}{G}\right)$-expansion method in handling nonlinear equations.

2. Exact Solutions for Equations (1)

Substituting the solution $\left(\eta(x,t)=\varphi_c(\xi),u(x,t)=\psi_c(\xi)\right)$ into (1), where $\xi=x-ct$ and c represents the wave speed, we obtain

$$\begin{cases}\left(-c\varphi_c+\psi_c+\varphi_c\psi_c\right)'+c\varphi_c'''=0,\\ \left(-c\psi_c+\varphi_c+\frac{1}{2}\varphi_c^2+\frac{1}{2}\psi_c^2\right)'+c\psi_c'''=0.\end{cases}\tag{2}$$

Integrating the equations and the integration constants are chosen as zero, it can be converted to the ODEs

$$\begin{cases}-c\varphi_c+\psi_c+\varphi_c\psi_c+c\varphi_c''=0,\\ -c\psi_c+\varphi_c+\frac{1}{2}\varphi_c^2+\frac{1}{2}\psi_c^2+c\psi_c''=0.\end{cases}\tag{3}$$

Next, we apply the $\left(\frac{G'}{G}\right)$-expansion method to solve Equations (3). Firstly, considering the homogeneous balance between the highest order derivative and the non-linear term, we suppose the solutions of (3) can be written in the form

$$\begin{aligned}\varphi_c(\xi)&=a_0+a_1\left(\frac{G'}{G}\right)+a_2\left(\frac{G'}{G}\right)^2,\\ \psi_c(\xi)&=b_0+b_1\left(\frac{G'}{G}\right)+b_2\left(\frac{G'}{G}\right)^2.\end{aligned}\tag{4}$$

where $a_2\neq 0, b_2\neq 0$ and $G=G(\xi)$ satisfies the second order linear ODE

$$G''(\xi)+\lambda G'(\xi)+\mu G(\xi)=0.\tag{5}$$

λ and μ are real constants. Using the general solutions of ODE (5), it is easy to obtain

$$\frac{G'}{G}=\begin{cases}-\frac{\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{c_1\sinh\left\{\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right\}+c_2\cosh\left\{\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right\}}{c_1\cosh\left\{\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right\}+c_2\sinh\left\{\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right\}}\right), & \lambda^2-4\mu>0;\\ -\frac{\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-c_1\sin\left\{\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right\}+c_2\cos\left\{\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right\}}{c_1\cos\left\{\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right\}+c_2\sin\left\{\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right\}}\right), & \lambda^2-4\mu<0;\\ \left(\frac{c_2}{c_1+c_2\xi}\right)-\frac{\lambda}{2}, & \lambda^2-4\mu=0.\end{cases} \tag{6}$$

By Euqations (4) and (5) we derive

$$\begin{aligned}\varphi_c''(\xi)&=6a_2\left(\frac{G'}{G}\right)^4+\left(2a_1+10a_2\lambda\right)\left(\frac{G'}{G}\right)^3\\&+\left(4a_2\lambda^2+8a_2\mu+3a_1\lambda\right)\left(\frac{G'}{G}\right)^2\\&+\left(6a_2\lambda\mu+a_1\lambda^2+2a_1\mu\right)\left(\frac{G'}{G}\right)\\&+\left(2a_2\mu^2+a_1\lambda\mu\right),\end{aligned} \tag{7}$$

$$\begin{aligned}\varphi_c(\xi)\psi_c(\xi)&=a_2b_2\left(\frac{G'}{G}\right)^4+\left(a_1b_2+a_2b_1\right)\left(\frac{G'}{G}\right)^3\\&+\left(a_0b_2+a_1b_1+a_2b_0\right)\left(\frac{G'}{G}\right)^2\\&+\left(a_0b_1+a_1b_0\right)\left(\frac{G'}{G}\right)+a_0b_0.\end{aligned} \tag{8}$$

Substituting (4) (7) and (8) into (3), collecting all terms with the same powers of $\left(\frac{G'}{G}\right)$ and setting each coefficient to zero, we get two sets of algebraic equations for a_2,a_1,a_0, b_2,b_1,b_0 and c,λ,μ.

$$\begin{cases}\left(\frac{G'}{G}\right)^4: 6a_2c+a_2b_2=0,\\ \left(\frac{G'}{G}\right)^3: 2a_1c+10a_2\lambda c+a_1b_2+a_2b_1=0\\ \left(\frac{G'}{G}\right)^2: 4a_2\lambda^2c+8a_2\mu c+3a_1\lambda c+a_0b_2+a_1b_1+a_2b_0-a_2c+b_2=0\\ \left(\frac{G'}{G}\right)^1: 6a_2\lambda\mu c+a_1\lambda^2c+2a_1\mu c+a_0b_1+a_1b_0-a_1c+b_1=0\\ \left(\frac{G'}{G}\right)^0: 2a_2\mu^2c+a_1\lambda\mu c+a_0b_0-a_0c+b_0=0\end{cases}$$

$$\begin{cases} \left(\frac{G'}{G}\right)^4 : 6b_2c + \frac{1}{2}\left(a_2^2 + b_2^2\right) = 0, \\ \left(\frac{G'}{G}\right)^3 : 2b_1c + 10b_2\lambda c + a_1a_2 + b_1b_2 = 0 \\ \left(\frac{G'}{G}\right)^2 : 4b_2\lambda^2 c + 8b_2\mu c + 3b_1\lambda c + \frac{1}{2}\left(a_1^2 + b_1^2\right) + a_0a_2 + b_0b_2 + a_2 - b_2c = 0 \\ \left(\frac{G'}{G}\right)^1 : 6b_2\lambda\mu c + b_1\lambda^2 c + 2b_1\mu c + a_0a_1 + b_0b_1 + a_1 - b_1c = 0 \\ \left(\frac{G'}{G}\right)^0 : 2b_2\mu^2 c + b_1\lambda\mu c + \frac{1}{2}\left(a_0^2 + b_0^2\right) + a_0 - b_0c = 0 \end{cases}$$

Solving the above algebraic equations yields eight groups of values of unknowns.

1. $$\begin{cases} a_2 = b_2 = -6c, \ a_1 = b_1 = -6\lambda c, \ \mu = \dfrac{\lambda^2 c + (c-1)}{4c}; \\ a_0 = b_0 = -\dfrac{3}{2}\lambda^2 c - \dfrac{1}{2}(c-1). \end{cases} \tag{9}$$

2. $$\begin{cases} a_2 = b_2 = -6c, \ a_1 = b_1 = -6\lambda c, \ \mu = \dfrac{\lambda^2 c + (c-1)}{4c}; \\ a_0 = -\dfrac{3}{2}\lambda^2 c - \dfrac{3}{2}c - \dfrac{1}{2}, \ b_0 = -\dfrac{3}{2}\lambda^2 c + \dfrac{1}{2}c + \dfrac{3}{2}. \end{cases} \tag{10}$$

3. $$\begin{cases} a_2 = b_2 = -6c, \ a_1 = b_1 = -6\lambda c, \ \mu = \dfrac{\lambda^2 c - (c-1)}{4c}; \\ a_0 = b_0 = -\dfrac{3}{2}\lambda^2 c + \dfrac{3}{2}(c-1). \end{cases} \tag{11}$$

4. $$\begin{cases} a_2 = b_2 = -6c, \ a_1 = b_1 = -6\lambda c, \ \mu = \dfrac{\lambda^2 c - (c-1)}{4c}; \\ a_0 = -\dfrac{3}{2}\lambda^2 c + \dfrac{1}{2}c - \dfrac{5}{2}, \ b_0 = -\dfrac{3}{2}\lambda^2 c + \dfrac{5}{2}c - \dfrac{1}{2}. \end{cases} \tag{12}$$

5. $$\begin{cases} a_2 = -b_2 = 6c, \ a_1 = -b_1 = 6\lambda c, \ \mu = \dfrac{\lambda^2 c + (c+1)}{4c}; \\ a_0 = -b_0 = \dfrac{3}{2}\lambda^2 c + \dfrac{1}{2}(c+1). \end{cases} \tag{13}$$

6. $$\begin{cases} a_2 = -b_2 = 6c, \ a_1 = -b_1 = 6\lambda c, \ \mu = \dfrac{\lambda^2 c + (c+1)}{4c}; \\ a_0 = \dfrac{3}{2}\lambda^2 c + \dfrac{3}{2}c - \dfrac{1}{2}, \ b_0 = -\dfrac{3}{2}\lambda^2 c + \dfrac{1}{2}c - \dfrac{3}{2}. \end{cases} \tag{14}$$

7. $$\begin{cases} a_2 = -b_2 = 6c, \ a_1 = -b_1 = 6\lambda c, \ \mu = \dfrac{\lambda^2 c - (c+1)}{4c}; \\ a_0 = -b_0 = \dfrac{3}{2}\lambda^2 c - \dfrac{3}{2}(c+1). \end{cases} \tag{15}$$

8. $$\begin{cases} a_2 = -b_2 = 6c, \ a_1 = -b_1 = 6\lambda c, \ \mu = \dfrac{\lambda^2 c - (c+1)}{4c}; \\ a_0 = \dfrac{3}{2}\lambda^2 c - \dfrac{1}{2}c - \dfrac{5}{2}, \ b_0 = -\dfrac{3}{2}\lambda^2 c + \dfrac{5}{2}c + \dfrac{1}{2}. \end{cases} \tag{16}$$

Substituting (9)-(16) into (5), using the expression (6), three types of traveling wave solutions of (1) are given as follows.

Case a. When $\lambda^2 - 4\mu > 0$,

$$\varphi_c^1(\xi) = \psi_c^1(\xi) = \frac{1}{2}(1-c) - \frac{3}{2}(1-c)\left(\frac{c_1 \sinh\left\{\frac{1}{2}\sqrt{\frac{1-c}{c}}\xi\right\} + c_2 \cosh\left\{\frac{1}{2}\sqrt{\frac{1-c}{c}}\xi\right\}}{c_1 \cosh\left\{\frac{1}{2}\sqrt{\frac{1-c}{c}}\xi\right\} + c_2 \sinh\left\{\frac{1}{2}\sqrt{\frac{1-c}{c}}\xi\right\}}\right)^2. \tag{17}$$

$$\varphi_c^2(\xi) = -\frac{3}{2}c - \frac{1}{2} - \frac{3}{2}(1-c)\left(\frac{c_1 \sinh\left\{\frac{1}{2}\sqrt{\frac{1-c}{c}}\xi\right\} + c_2 \cosh\left\{\frac{1}{2}\sqrt{\frac{1-c}{c}}\xi\right\}}{c_1 \cosh\left\{\frac{1}{2}\sqrt{\frac{1-c}{c}}\xi\right\} + c_2 \sinh\left\{\frac{1}{2}\sqrt{\frac{1-c}{c}}\xi\right\}}\right)^2,$$
$$\psi_c^2(\xi) = \frac{1}{2}c + \frac{3}{2} - \frac{3}{2}(1-c)\left(\frac{c_1 \sinh\left\{\frac{1}{2}\sqrt{\frac{1-c}{c}}\xi\right\} + c_2 \cosh\left\{\frac{1}{2}\sqrt{\frac{1-c}{c}}\xi\right\}}{c_1 \cosh\left\{\frac{1}{2}\sqrt{\frac{1-c}{c}}\xi\right\} + c_2 \sinh\left\{\frac{1}{2}\sqrt{\frac{1-c}{c}}\xi\right\}}\right)^2. \tag{18}$$

$$\varphi_c^3(\xi) = \psi_c^3(\xi) = \frac{3}{2}(c-1) - \frac{3}{2}(c-1)\left(\frac{c_1 \sinh\left\{\frac{1}{2}\sqrt{\frac{c-1}{c}}\xi\right\} + c_2 \cosh\left\{\frac{1}{2}\sqrt{\frac{c-1}{c}}\xi\right\}}{c_1 \cosh\left\{\frac{1}{2}\sqrt{\frac{c-1}{c}}\xi\right\} + c_2 \sinh\left\{\frac{1}{2}\sqrt{\frac{c-1}{c}}\xi\right\}}\right)^2. \tag{19}$$

$$\varphi_c^4(\xi) = \frac{1}{2}c - \frac{5}{2} - \frac{3}{2}(c-1)\left(\frac{c_1 \sinh\left\{\frac{1}{2}\sqrt{\frac{c-1}{c}}\xi\right\} + c_2 \cosh\left\{\frac{1}{2}\sqrt{\frac{c-1}{c}}\xi\right\}}{c_1 \cosh\left\{\frac{1}{2}\sqrt{\frac{c-1}{c}}\xi\right\} + c_2 \sinh\left\{\frac{1}{2}\sqrt{\frac{c-1}{c}}\xi\right\}}\right)^2,$$
$$\psi_c^4(\xi) = \frac{5}{2}c - \frac{1}{2} - \frac{3}{2}(c-1)\left(\frac{c_1 \sinh\left\{\frac{1}{2}\sqrt{\frac{c-1}{c}}\xi\right\} + c_2 \cosh\left\{\frac{1}{2}\sqrt{\frac{c-1}{c}}\xi\right\}}{c_1 \cosh\left\{\frac{1}{2}\sqrt{\frac{c-1}{c}}\xi\right\} + c_2 \sinh\left\{\frac{1}{2}\sqrt{\frac{c-1}{c}}\xi\right\}}\right)^2. \tag{20}$$

$$\varphi_c^5(\xi) = -\psi_c^5(\xi) = \frac{1}{2}(c+1) - \frac{3}{2}(c+1)\left(\frac{c_1 \sinh\left\{\frac{1}{2}\sqrt{\frac{-1-c}{c}}\xi\right\} + c_2 \cosh\left\{\frac{1}{2}\sqrt{\frac{-1-c}{c}}\xi\right\}}{c_1 \cosh\left\{\frac{1}{2}\sqrt{\frac{-1-c}{c}}\xi\right\} + c_2 \sinh\left\{\frac{1}{2}\sqrt{\frac{-1-c}{c}}\xi\right\}}\right)^2. \tag{21}$$

$$\varphi_c^6(\xi)=\frac{3}{2}c-\frac{1}{2}-\frac{3}{2}(c+1)\left(\frac{c_1\sinh\left\{\frac{1}{2}\sqrt{\frac{-c-1}{c}}\xi\right\}+c_2\cosh\left\{\frac{1}{2}\sqrt{\frac{-c-1}{c}}\xi\right\}}{c_1\cosh\left\{\frac{1}{2}\sqrt{\frac{-c-1}{c}}\xi\right\}+c_2\sinh\left\{\frac{1}{2}\sqrt{\frac{-c-1}{c}}\xi\right\}}\right)^2,$$

$$\psi_c^6(\xi)=\frac{1}{2}c-\frac{3}{2}+\frac{3}{2}(c+1)\left(\frac{c_1\sinh\left\{\frac{1}{2}\sqrt{\frac{-c-1}{c}}\xi\right\}+c_2\cosh\left\{\frac{1}{2}\sqrt{\frac{-c-1}{c}}\xi\right\}}{c_1\cosh\left\{\frac{1}{2}\sqrt{\frac{-c-1}{c}}\xi\right\}+c_2\sinh\left\{\frac{1}{2}\sqrt{\frac{-c-1}{c}}\xi\right\}}\right)^2. \tag{22}$$

$$\varphi_c^7(\xi)=-\psi_c^7(\xi)=-\frac{3}{2}(c+1)+\frac{3}{2}(c+1)\left(\frac{c_1\sinh\left\{\frac{1}{2}\sqrt{\frac{c+1}{c}}\xi\right\}+c_2\cosh\left\{\frac{1}{2}\sqrt{\frac{c+1}{c}}\xi\right\}}{c_1\cosh\left\{\frac{1}{2}\sqrt{\frac{c+1}{c}}\xi\right\}+c_2\sinh\left\{\frac{1}{2}\sqrt{\frac{c+1}{c}}\xi\right\}}\right)^2. \tag{23}$$

$$\varphi_c^8(\xi)=-\frac{1}{2}c-\frac{5}{2}+\frac{3}{2}(c+1)\left(\frac{c_1\sinh\left\{\frac{1}{2}\sqrt{\frac{c+1}{c}}\xi\right\}+c_2\cosh\left\{\frac{1}{2}\sqrt{\frac{c+1}{c}}\xi\right\}}{c_1\cosh\left\{\frac{1}{2}\sqrt{\frac{c+1}{c}}\xi\right\}+c_2\sinh\left\{\frac{1}{2}\sqrt{\frac{c+1}{c}}\xi\right\}}\right)^2,$$

$$\psi_c^8(\xi)=\frac{5}{2}c+\frac{1}{2}-\frac{3}{2}(c+1)\left(\frac{c_1\sinh\left\{\frac{1}{2}\sqrt{\frac{c+1}{c}}\xi\right\}+c_2\cosh\left\{\frac{1}{2}\sqrt{\frac{c+1}{c}}\xi\right\}}{c_1\cosh\left\{\frac{1}{2}\sqrt{\frac{c+1}{c}}\xi\right\}+c_2\sinh\left\{\frac{1}{2}\sqrt{\frac{c+1}{c}}\xi\right\}}\right)^2. \tag{24}$$

Case b. When $\lambda^2-4\mu<0$,

$$\varphi_c^9(\xi)=\psi_c^9(\xi)=-\frac{1}{2}(c-1)-\frac{3}{2}(c-1)\left(\frac{c_1\sin\left\{\frac{1}{2}\sqrt{\frac{c-1}{c}}\xi\right\}+c_2\cos\left\{\frac{1}{2}\sqrt{\frac{c-1}{c}}\xi\right\}}{c_1\cos\left\{\frac{1}{2}\sqrt{\frac{c-1}{c}}\xi\right\}+c_2\sin\left\{\frac{1}{2}\sqrt{\frac{c-1}{c}}\xi\right\}}\right)^2. \tag{25}$$

$$\varphi_c^{10}(\xi)=-\frac{3}{2}c-\frac{1}{2}-\frac{3}{2}(c-1)\left(\frac{c_1\sin\left\{\frac{1}{2}\sqrt{\frac{c-1}{c}}\xi\right\}+c_2\cos\left\{\frac{1}{2}\sqrt{\frac{c-1}{c}}\xi\right\}}{c_1\cos\left\{\frac{1}{2}\sqrt{\frac{c-1}{c}}\xi\right\}+c_2\sin\left\{\frac{1}{2}\sqrt{\frac{c-1}{c}}\xi\right\}}\right)^2,$$

$$\psi_c^{10}(\xi)=\frac{1}{2}c+\frac{3}{2}-\frac{3}{2}(c-1)\left(\frac{c_1\sin\left\{\frac{1}{2}\sqrt{\frac{c-1}{c}}\xi\right\}+c_2\cos\left\{\frac{1}{2}\sqrt{\frac{c-1}{c}}\xi\right\}}{c_1\cos\left\{\frac{1}{2}\sqrt{\frac{c-1}{c}}\xi\right\}+c_2\sin\left\{\frac{1}{2}\sqrt{\frac{c-1}{c}}\xi\right\}}\right)^2. \tag{26}$$

$$\varphi_c^{11}(\xi)=\psi_c^{11}(\xi)=-\frac{3}{2}(1-c)-\frac{3}{2}(1-c)\left(\frac{c_1\sin\left\{\frac{1}{2}\sqrt{\frac{1-c}{c}}\xi\right\}+c_2\cos\left\{\frac{1}{2}\sqrt{\frac{1-c}{c}}\xi\right\}}{c_1\cos\left\{\frac{1}{2}\sqrt{\frac{1-c}{c}}\xi\right\}+c_2\sin\left\{\frac{1}{2}\sqrt{\frac{1-c}{c}}\xi\right\}}\right)^2. \tag{27}$$

$$\varphi_c^{12}(\xi)=\frac{1}{2}c-\frac{5}{2}-\frac{3}{2}(1-c)\left(\frac{c_1\sin\left\{\frac{1}{2}\sqrt{\frac{1-c}{c}}\xi\right\}+c_2\cos\left\{\frac{1}{2}\sqrt{\frac{1-c}{c}}\xi\right\}}{c_1\cos\left\{\frac{1}{2}\sqrt{\frac{1-c}{c}}\xi\right\}+c_2\sin\left\{\frac{1}{2}\sqrt{\frac{1-c}{c}}\xi\right\}}\right)^2,$$

$$\psi_c^{12}(\xi)=\frac{5}{2}c-\frac{1}{2}-\frac{3}{2}(1-c)\left(\frac{c_1\sin\left\{\frac{1}{2}\sqrt{\frac{1-c}{c}}\xi\right\}+c_2\cos\left\{\frac{1}{2}\sqrt{\frac{1-c}{c}}\xi\right\}}{c_1\cos\left\{\frac{1}{2}\sqrt{\frac{1-c}{c}}\xi\right\}+c_2\sin\left\{\frac{1}{2}\sqrt{\frac{1-c}{c}}\xi\right\}}\right)^2. \tag{28}$$

$$\varphi_c^{13}(\xi)=-\psi_c^{13}(\xi)=\frac{1}{2}(c+1)+\frac{3}{2}(c+1)\left(\frac{c_1\sin\left\{\frac{1}{2}\sqrt{\frac{c+1}{c}}\xi\right\}+c_2\cos\left\{\frac{1}{2}\sqrt{\frac{c+1}{c}}\xi\right\}}{c_1\cos\left\{\frac{1}{2}\sqrt{\frac{c+1}{c}}\xi\right\}+c_2\sin\left\{\frac{1}{2}\sqrt{\frac{c+1}{c}}\xi\right\}}\right)^2. \tag{29}$$

$$\varphi_c^{14}(\xi)=\frac{3}{2}c-\frac{1}{2}+\frac{3}{2}(c+1)\left(\frac{c_1\sin\left\{\frac{1}{2}\sqrt{\frac{c+1}{c}}\xi\right\}+c_2\cos\left\{\frac{1}{2}\sqrt{\frac{c+1}{c}}\xi\right\}}{c_1\cos\left\{\frac{1}{2}\sqrt{\frac{c+1}{c}}\xi\right\}+c_2\sin\left\{\frac{1}{2}\sqrt{\frac{c+1}{c}}\xi\right\}}\right)^2,$$

$$\psi_c^{14}(\xi)=\frac{1}{2}c-\frac{3}{2}-\frac{3}{2}(c+1)\left(\frac{c_1\sin\left\{\frac{1}{2}\sqrt{\frac{c+1}{c}}\xi\right\}+c_2\cos\left\{\frac{1}{2}\sqrt{\frac{c+1}{c}}\xi\right\}}{c_1\cos\left\{\frac{1}{2}\sqrt{\frac{c+1}{c}}\xi\right\}+c_2\sin\left\{\frac{1}{2}\sqrt{\frac{c+1}{c}}\xi\right\}}\right)^2. \tag{30}$$

$$\varphi_c^{15}(\xi)=-\psi_c^{15}(\xi)=-\frac{3}{2}(c+1)-\frac{3}{2}(c+1)\left(\frac{c_1\sin\left\{\frac{1}{2}\sqrt{\frac{-1-c}{c}}\xi\right\}+c_2\cos\left\{\frac{1}{2}\sqrt{\frac{-1-c}{c}}\xi\right\}}{c_1\cos\left\{\frac{1}{2}\sqrt{\frac{-1-c}{c}}\xi\right\}+c_2\sin\left\{\frac{1}{2}\sqrt{\frac{-1-c}{c}}\xi\right\}}\right)^2. \tag{31}$$

$$\varphi_c^{16}(\xi)=-\frac{1}{2}c-\frac{5}{2}-\frac{3}{2}(c+1)\left(\frac{c_1\sin\left\{\frac{1}{2}\sqrt{\frac{-c-1}{c}}\xi\right\}+c_2\cos\left\{\frac{1}{2}\sqrt{\frac{-c-1}{c}}\xi\right\}}{c_1\cos\left\{\frac{1}{2}\sqrt{\frac{-c-1}{c}}\xi\right\}+c_2\sin\left\{\frac{1}{2}\sqrt{\frac{-c-1}{c}}\xi\right\}}\right)^2,$$

$$\psi_c^{16}(\xi)=\frac{5}{2}c+\frac{1}{2}+\frac{3}{2}(c+1)\left(\frac{c_1\sin\left\{\frac{1}{2}\sqrt{\frac{-c-1}{c}}\xi\right\}+c_2\cos\left\{\frac{1}{2}\sqrt{\frac{-c-1}{c}}\xi\right\}}{c_1\cos\left\{\frac{1}{2}\sqrt{\frac{-c-1}{c}}\xi\right\}+c_2\sin\left\{\frac{1}{2}\sqrt{\frac{-c-1}{c}}\xi\right\}}\right)^2. \tag{32}$$

Case c. When $\lambda^2 - 4\mu = 0$,

$$\varphi_c^{17}(\xi) = \psi_c^{17}(\xi) = -6\left(\frac{c_2}{c_1 + c_2\xi}\right)^2. \quad (33)$$

$$\varphi_c^{18}(\xi) = -2 - 6\left(\frac{c_2}{c_1 + c_2\xi}\right)^2, \ \psi_c^{18}(\xi) = 2 - 6\left(\frac{c_2}{c_1 + c_2\xi}\right)^2. \quad (34)$$

$$\varphi_c^{19}(\xi) = -\psi_c^{19}(\xi) = -6\left(\frac{c_2}{c_1 + c_2\xi}\right)^2. \quad (35)$$

$$\varphi_c^{20}(\xi) = -2 - 6\left(\frac{c_2}{c_1 + c_2\xi}\right)^2, \ \psi_c^{20}(\xi) = -2 + 6\left(\frac{c_2}{c_1 + c_2\xi}\right)^2. \quad (36)$$

Remark If $c_1 \neq 0, c_2 = 0$ or $c_1 = 0, c_2 \neq 0$, the waves obtained in the form of (17)-(24) are just the waves given in [1] [2].

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References

[1] Cui, L.W. and Zhao, Y. (2012) Orbital Stability of Solitary Waves for Coupled BBM Equations. *Advances in Mathematics* (*Chinese*), **41**, 341-346.

[2] Zhao, Y. and Xu, Q. (to appear) Existence and Instability of Solitary Waves with Nonzero Asymptotic Value for Some BBM Equations.

[3] Wang, M., Li, X. and Zhang, J. (2008) The $\left(\frac{G'}{G}\right)$-Expansion Method and Traveling Wave Solutions of Nonlinear Evolution Equations in Mathematical Physics. *Physics Letters A*, **372**, 417-423. http://dx.doi.org/10.1016/j.physleta.2007.07.051

[4] Bekir, A. (2008) Application of the $\left(\frac{G'}{G}\right)$-Expansion Method for Nonlinear Evolution Equations. *Physics Letters A*, **372**, 3400-3406. http://dx.doi.org/10.1016/j.physleta.2008.01.057

[5] Zayed, E.M.E. and Al-Joudi, S. (2010) Application of an Extended $\left(\frac{G'}{G}\right)$-Expansion Method to Find Exact Solutions of Nonlinear PDEs in Mathematical Physics. *Mathematical Problems in Engineering*, **2010**, Article ID: 768573.

[6] Ali Akbar, M., Ali, N.H.M. and Zayed, E.M.E. (2012) A Generalized and Improved $\left(\frac{G'}{G}\right)$-Expansion Method for Nonlinear Evolution Equations. *Mathematical Problems in Engineering*, **2012**, Article ID: 459879.

[7] Taha, W.M., Noorani, M.S.M. and Hashim, I. (2014) New Exact Solutions of Sixth-Order Thin-Film Equation. *Journal of King Saud University—Science*, **26**, 75-78.