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Application of Classification of Traveling Wave Solutions to the Zakhrov-Kuznetsov-Benjamin-Bona-Mahony Equation

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Abstract

In order to get the traveling wave solutions of the Zakharov-Kuznetsov-Benjamin-Bona-Mahony (ZK-BBM) equation, it is reduced to an ordinary differential equation (ODE) under the travelling wave transformation first. Then complete discrimination system for polynomial is applied to the ZK-BBM equation. The traveling wave solutions of the equation can be obtained.

Keywords

The Nonlinear Partial Differential Equation, The Zakharov-Kuznetsov-Benjamin-Bona-Mahony Equation, Traveling Wave Transform, Complete Discrimination System for Polynomial, The Traveling Wave Solution

1. Introduction

The nonlinear partial differential equation (PDE) is widely used to describe physical phenomena in various fields of sciences, especially in fluid mechanics, solid state physics, plasma physics, plasma waves, biology and so on. During the past few decades, various methods have been developed by researchers to find the solutions for the NLEEs.

In this article, we will use complete discrimination system for polynomial proposed by Liu [1]-[4] to study the traveling wave solutions of the ZK-BBM equation. The generalised form of the (2 + 1) dimensional ZK-BBM equation is given as:

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$$u_t+u_x-\alpha\left(u^2\right)_x-\left(\beta u_{xt}+\gamma u_{yt}\right)_x=0. \tag{1}$$

where α,β and γ are arbitrary constants.

Equation (1) arises as a description of gravity water waves in the long-wave regime [5] [6]. The solutions of Equation (1) have been studied in various aspects. For example, Sadaf Bibi [7] used the Sine-cosine method to obtain the travelling wave solutions of Equation (1). Rajesh Kumar Gupta [8] used the $\left(\frac{G'}{G}\right)$-expansion method to find some hyperbolic, trigonometric and rational solutions, and so on. It is worth mentioning that Wazwaz [9] [10] made a detailed study for compact and noncompact physical structures and calculated the exact solutions of compact and noncompact structures by the extended tanh method for the ZK-BBM equation.

2. Classification

Taking the traveling wave transformation $u=u(\xi_1)$ and $\xi_1=k_1x+k_2y+\omega t$, we can obtain the corresponding reduced ODE of Equation (1).

$$(\omega+k_1)u'-2\alpha k_1uu'-\left(\beta k_1^2\omega+\gamma k_1k_2\omega\right)u'''=0. \tag{2}$$

Integrating Equation (2) with respect to ξ_1 once, we yield

$$(\omega+k_1)u-\alpha k_1u^2-\left(\beta k_1^2\omega+\gamma k_1k_2\omega\right)u''=c_1. \tag{3}$$

where c_1 is an integral constant.

Equation (3) can be written as

$$u''=a_0+a_1u+a_2u^2. \tag{4}$$

where $a_0=-\dfrac{c_1}{\beta k_1^2\omega+\gamma k_1k_2\omega},a_1=\dfrac{\omega+k_1}{\beta k_1^2\omega+\gamma k_1k_2\omega},a_2=-\dfrac{\alpha k_1}{\beta k_1^2\omega+\gamma k_1k_2\omega}$.

From Equation (4) we have

$$(u')^2=\frac{2}{3}a_2u^3+a_1u^2+2a_0u+c_0. \tag{5}$$

where c_0 is an integral constant.

We use the complete discrimination system for the third order polynomial and have the following solving process.

Let

$$v=\left(\frac{2}{3}a_2\right)^{\frac{1}{3}}u,\xi=\left(\frac{2}{3}a_2\right)^{\frac{1}{3}}\xi_1,d_2=a_1\left(\frac{2}{3}a_2\right)^{-\frac{2}{3}},d_1=2a_0\left(\frac{2}{3}a_2\right)^{-\frac{1}{3}},d_0=c_0. \tag{6}$$

Then Equation (5) becomes

$$(v')^2=v^3+d_2v^2+d_1v+d_0. \tag{7}$$

where v is a function of ξ. The integral form of Equation (7) is

$$\pm(\xi-\xi_0)=\int\frac{\mathrm{d}v}{\sqrt{v^3+d_2v^2+d_1v+d_0}}. \tag{8}$$

Denote

$$F(v)=v^3+d_2v^2+d_1v+d_0. \tag{9}$$

$$\Delta=-27\left(\frac{2d_2^3}{27}+d_0-\frac{d_1d_3}{3}\right)^2-4\left(d_1-d_2^2\right)^3,D_1=d_1-\frac{d_2^2}{3}. \tag{10}$$

According to the complete discrimination system, we give the corresponding single traveling wave solutions to Equation (1).

Case 1. $\Delta = 0, D_1 < 0. F(v) = 0$ has a double real root and a simple real root. Then we have

$$F(v) = (v - \lambda_1)^2 (v - \lambda_2), \lambda_1 \neq \lambda_2. \tag{11}$$

when $v > \lambda_2$, the corresponding solutions are

$$u_1 = \left(\frac{2}{3}a_2\right)^{-\frac{1}{3}} \left\{ (\lambda_1 - \lambda_2) \tanh^2 \left[\frac{\sqrt{\lambda_1 - \lambda_2}}{2} \left(\frac{2}{3}a_2\right)^{\frac{1}{3}} (k_1 x + k_2 y + \omega t - \xi_0) \right] + \lambda_2 \right\}, (\lambda_1 > \lambda_2); \tag{12}$$

$$u_2 = \left(\frac{2}{3}a_2\right)^{-\frac{1}{3}} \left\{ (\lambda_1 - \lambda_2) \coth^2 \left[\frac{\sqrt{\lambda_1 - \lambda_2}}{2} \left(\frac{2}{3}a_2\right)^{\frac{1}{3}} (k_1 x + k_2 y + \omega t - \xi_0) \right] + \lambda_2 \right\}, (\lambda_1 > \lambda_2); \tag{13}$$

$$u_3 = \left(\frac{2}{3}a_2\right)^{-\frac{1}{3}} \left\{ (-\lambda_1 + \lambda_2) \sec^2 \left[\frac{\sqrt{-\lambda_1 + \lambda_2}}{2} \left(\frac{2}{3}a_2\right)^{\frac{1}{3}} (k_1 x + k_2 y + \omega t - \xi_0) \right] + \lambda_1 \right\}, (\lambda_1 < \lambda_2). \tag{14}$$

Case 2. $\Delta = 0, D_1 = 0. F(v) = 0$ has a triple root. Then we have

$$F(v) = (v - \lambda)^3. \tag{15}$$

The corresponding solution is

$$u_4 = 4\left(\frac{2}{3}a_2\right)^{-\frac{2}{3}} (k_1 x + k_2 y + \omega t - \xi_0)^{-2} + \lambda. \tag{16}$$

Case 3. $\Delta > 0, D_1 < 0. F(v) = 0$ has three different real roots. Then we have

$$F(v) = (v - \lambda_1)(v - \lambda_2)(v - \lambda_3), \lambda_1 < \lambda_2 < \lambda_3. \tag{17}$$

when $\lambda_1 < v < \lambda_2$, we take the transformation as follows

$$v = \lambda_1 + (\lambda_2 - \lambda_1)\sin^2 \varphi. \tag{18}$$

According to the Equation (8), we have

$$\pm(\xi - \xi_0) = \int \frac{\mathrm{d}v}{\sqrt{F(v)}} = \frac{2}{\sqrt{\lambda_3 - \lambda_1}} \int \frac{\mathrm{d}\varphi}{\sqrt{1 - m^2 \sin^2 \varphi}}. \tag{19}$$

where $m^2 = \dfrac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_1}$.

On the basis of Equation (19) and the definition of the Jacobi elliptic sine function, we have

$$v = \lambda_1 + (\lambda_2 - \lambda_1)\operatorname{sn}^2 \left(\frac{\sqrt{\lambda_3 - \lambda_1}}{2} (\xi_1 - \xi_0), m \right). \tag{20}$$

The corresponding solution is

$$u_5 = \left(\frac{2}{3}a_2\right)^{-\frac{1}{3}} \left[\lambda_1 + (\lambda_2 - \lambda_1)\operatorname{sn}^2 \left(\frac{\sqrt{\lambda_3 - \lambda_1}}{2} \left(\frac{2}{3}a_2\right)^{\frac{1}{3}} (k_1 x + k_2 y + \omega t - \xi_0), m \right) \right]. \tag{21}$$

when $v > \lambda_3$ we take the transformation as follows

$$v = \frac{-\lambda_2 \sin^2\varphi + \lambda_3}{\cos^2\varphi}. \tag{22}$$

The corresponding solutions is

$$u_6=\left(\frac{2}{3}a_2\right)^{-\frac{1}{3}}\left[\frac{\lambda_3-\lambda_2\mathrm{sn}^2\left(\frac{\sqrt{\lambda_3-\lambda_1}}{2}\left(\frac{2}{3}a_2\right)^{\frac{1}{3}}\left(k_1x+k_2y+\omega t-\xi_0\right),m\right)}{\mathrm{cn}^2\left(\frac{\sqrt{\lambda_3-\lambda_1}}{2}\left(\frac{2}{3}a_2\right)^{\frac{1}{3}}\left(k_1x+k_2y+\omega t-\xi_0\right),m\right)}\right]. \tag{23}$$

where $m^2=\frac{\lambda_2-\lambda_1}{\lambda_3-\lambda_1}$.

Case 4. $\Delta<0.F(v)=0$ has only a real root. Then we have

$$F(v)=(v-\lambda)\left(v^2+pv+q\right),\ p^2-4q<0. \tag{24}$$

when $v>\lambda_1$, we take the transformation as follows

$$v=\lambda+\sqrt{\lambda^2+p\lambda+q}\tan^2\frac{\varphi}{2}. \tag{25}$$

According to the Equation (8), we have

$$\xi-\xi_0=\int\frac{\mathrm{d}v}{\sqrt{(v-\lambda)\left(v^2+pv+q\right)}}=\frac{1}{\left(\lambda^2+p\lambda+q\right)^{\frac{1}{4}}}\int\frac{\mathrm{d}\varphi}{\sqrt{1-m^2\sin^2\varphi}}. \tag{26}$$

where $m^2=\frac{1}{2}\left(1-\frac{\lambda+\frac{p}{2}}{\sqrt{\lambda^2+p\lambda+q}}\right)$.

On the basis of Equation (26) and the definition of the Jacobi elliptic cosine function, we have

$$v=\lambda+\frac{2\sqrt{\lambda^2+p\lambda+q}}{1+\mathrm{cn}\left(\left(\lambda^2+p\lambda+q\right)^{\frac{1}{4}}\left(\xi_1-\xi_0\right),m\right)}-\sqrt{\lambda^2+p\lambda+q}. \tag{27}$$

The corresponding solutions is

$$u_7=\left(\frac{2}{3}a_2\right)^{-\frac{1}{3}}\left[\lambda+\frac{2\sqrt{\lambda^2+p\lambda+q}}{1+\mathrm{cn}\left(\left(\lambda^2+p\lambda+q\right)^{\frac{1}{4}}\left(\frac{2}{3}a_2\right)^{\frac{1}{3}}\left(k_1x+k_2y+\omega t-\xi_0\right),m\right)}-\sqrt{\lambda^2+p\lambda+q}\right]. \tag{28}$$

In Equations (12), (13), (14), (16), (21), (23) and (28), the integration constant ξ_0 has been rewritten, but we still use it. The solutions $u_i\left(i=1,\cdots,7\right)$ are all possible exact traveling wave solutions to Equation (1). We can see it is easy to write the corresponding solutions to the ZK-BBM equation.

3. Conclusion

In this article, the traveling wave solutions to ZK-BBM equation were obtained by the complete discrimination system for polynomial and direct integral method. This method has the characteristics of simple steps and clear effectivity. In this way we can solve a lot of other equations.

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