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On Relations between the General Recurrence Formula of the Extension of Murase-Newton's Method (the Extension of Tsuchikura*-Horiguchi's Method) and Horner's Method

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Abstract

In 1673, Yoshimasu Murase made a cubic equation to obtain the thickness of a hearth. He introduced two kinds of recurrence formulas of square x_k^2 and the deformation (Ref. [1]). We find that the three formulas lead to the extension of Newton-Raphson's method and Horner's method at the same time. This shows originality of Japanese native mathematics (Wasan) in the Edo era (1600-1867). Suzuki (Ref. [2]) estimates Murase to be a rare mathematician in not only the history of Wasan but also the history of mathematics in the world. Section 1 introduces Murase's three solutions of the cubic equation of the hearth. Section 2 explains the Horner's method. We give the generalization of three formulas and the relation between these formulas and Horner's method. Section 3 gives definitions of Murase-Newton's method (Tsuchikura-Horiguchi's method), general recurrence formula of Murase-Newton's method (Tsuchikura-Horiguchi's method), and general recurrence formula of the extension of Murase-Newton's method (the extension of Tsuchikura-Horiguchi's method) concerning *n*-degree polynomial equation. Section 4 is contents of the title of this paper.

Keywords

Recurrence Formula; Newton-Raphson's Method (Newton's Method); Extensions of Murase-Newton's Method; Horner's Method

*Tsuchikura is Tamotsu Tsuchikura, the professor emeritus of Tohoku University.

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1. Introduction

All the references are written in Japanese. We wrote this paper from two kinds of recurrence formulas of the square x_k^2 and the deformation of a cubic equation written in Ref. [1], and a hint of Tsuchikura. Therefore, it is enough for readers to know these three formulas. But it is very difficult even for Japanese people to read the Murase's book written in the Japanese ancient writing. Therefore, the readers do not need to read the book. Furthermore, the readers do not need to mind Japanese references. From now on, we explain the Murase's three formulas as introduction. The readers can know the origin of this paper.

Murase made the cubic equation for the next problem in 1673.

There is a rectangular solid (base is a square). We put it together four and make the hearth as **Figure 1**.

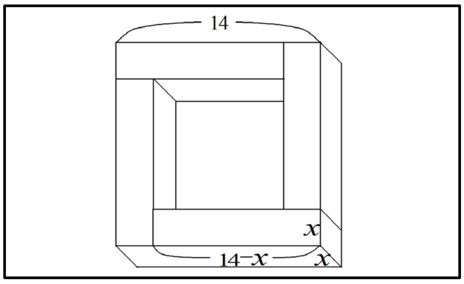

Figure 1. Hearth.

We claim one side of length of the square that one side is 14, and a volume becomes 192 of the hearth. Let one side of length of the square be x then the next cubic equation is obtained.

$$4x^2(14-x)=192 \tag{1.1}$$

that is

$$f(x)=x^3-14x^2+48=0 \tag{1.2}$$

This has three solutions of real number 2, $6\pm 2\sqrt{15}$.

Murase derived two following recurrence formulas and deformed equation from (1.2).

The first method

$$x_{k+1}^2=\frac{48+x_k^3}{14}\left(k=0,1,2,\cdots\right) \tag{1.3}$$

Using on an abacus, Murase calculates to $x_0 = 0$ (initial value), $x_1 = 1.85$, $x_2 = 1.97$, $x_3 = 1.9936$, and decides a solution with 2.

The second method

$$x_{k+1}^2=\frac{48}{14-x_k}\left(k=0,1,2,\cdots\right) \tag{1.4}$$

here he calculates to $x_0 = 0$, $x_1 = 1.85$, $x_2 = 1.976$, $x_3 = 1.9989$, $x_4 = 1.9999907$, and decides a solution with 2. An expression (1.4) has better precision than that (1.3), and convergence becomes fast.

The third method was nonrecurring in spite of a short sentence for many years. However, Yasuo Fujii (Takakazu Seki Mathematics Research Institute of Yokkaichi University) succeeds in decoding in May, 2009. It is the next equation.

The third method

$$(14-2x)x^2=48-x^3 \tag{1.5}$$

The studies of three formulas of Murase progress by the third method (already decoded).

2. Horner's Method

Let

$$f(x)=a_1x^n+a_2x^{n-1}+\cdots+a_nx+a_{n+1} \tag{2.1}$$

be a n-th degree polynomial where $a_1,\cdots,a_{n+1}$ are real numbers.

The Horner's method is an algorithm to calculate $f(\alpha)$. Dividing $f(x)$ by $x-\alpha$, we obtain the next formula.

$$f(x)=(x-\alpha)\left(b_1x^{n-1}+b_2x^{n-2}+\cdots+b_{n-1}x+b_n\right)+R_1 \tag{2.2}$$

Therefore, $f(\alpha)$ becomes R_1. Next, dividing $b_1x^{n-1}+b_2x^{n-2}+\cdots+b_{n-1}x+b_n$ by $x-\alpha$, we obtain the next formula.

$$f(x)=(x-\alpha)\left\{(x-\alpha)\left(c_1x^{n-2}+c_2x^{n-3}+\cdots+c_{n-2}x+c_{n-1}\right)+R_2\right\}+R_1 \tag{2.3}$$

Here, R_2 is the differential coefficient $f'(\alpha)$.

Furthermore, dividing $c_1x^{n-2}+c_2x^{n-3}+\cdots+c_{n-2}x+c_{n-1}$ by $x-\alpha$, we obtain the next formula.

$$f(x)=(x-\alpha)\left[(x-\alpha)\left\{(x-\alpha)\left(d_1x^{n-3}+d_2x^{n-4}+\cdots+d_{n-3}x+d_{n-2}\right)+R_3\right\}+R_2\right]+R_1 \tag{2.4}$$

Comparing the coefficients of formula (2.1) and (2.2), (2.2) and (2.3), (2.3) and (2.4) respectively, we obtain the next calculating formula of b_i, R_1, c_i, R_2, d_i, R_3.

$$\begin{cases} b_1=a_1 \\ b_i=a_i+b_{i-1}\cdot\alpha\,(i=2,3,\cdots,n+1) \end{cases} \tag{2.5}$$

$$\begin{cases} c_1=b_1 \\ c_i=b_i+c_{i-1}\cdot\alpha\,(i=2,3,\cdots,n) \end{cases} \tag{2.6}$$

$$\begin{cases} d_1=c_1 \\ d_i=c_i+d_{i-1}\cdot\alpha\,(i=2,3,\cdots,n-1) \end{cases} \tag{2.7}$$

$$b_{n+1}=R_1=f(\alpha),c_n=R_2=f'(\alpha),d_{n-1}=R_3 \tag{2.8}$$

Similarly, we can continue calculating.

The indication of calculating formula by synthetic division is next **Table 1**.

Table 1. Synthetic division for an expression (2.1) (Ref. [3]).

	a_1	a_2	a_3	………	a_{n-1}	a_n	a_{n+1}
+)		$b_1\alpha$	$b_2\alpha$	………	$b_{n-2}a$	$b_{n-1}a$	$b_n\alpha$
	b_1	b_2	b_3	………	b_{n-1}	b_n	$R_1=f(\alpha)$
+)		$c_1\alpha$	$c_2\alpha$	………	$c_{n-2}a$	$c_{n-1}a$	
	c_1	c_2	c_3	………	c_{n-1}	$R_2=f'(\alpha)$	
+)		$d_1\alpha$	$d_2\alpha$	………	$d_{n-2}a$		
	d_1	d_2	d_3	………	R_3		

Example 2.1. If we apply Horner's method to an expression (1.2) in case of solution $x=2$, then it is calculated in **Table 2**.

Table 2. Synthetic division for an expression (1.2).

	1	−14	0	48
+)		2	−24	−48
	1	−12	−24	0
+)		2	−20	
	1	−10	−44	

We obtain the next theorem from the Murase's three formulas and **Table 2** of Horner's method.

Theorem 2.2. (1) We expand the first, second, third method of Murase, and obtain the next recurrence formulas where *m* is a real number.

$$x_{k+1}^2 = \frac{48-(m-1)x_k^3}{14-mx_k} \tag{2.9}$$

(2) 14, $14-x_k$ of denominator of formula (1.3), (1.4) respectively, and $14-2x$ of formula (1.5) change 14→12→10 if $x_k = x = 2$. Therefore, these changes correspond to the second line × −1 of the **Table 2** of the calculation of Horner's method in Example 2.1 (Ref. [4]).

3. Expansions Recurrence Formula of Murase-Newton

In 2009, we found the extension of Newton-Raphson's method from the Murase's three formulas and a hint of Tamotsu Tsuchikura, and called it the Murase-Newton's method or the Tsuchikura-Horiguchi's method. We obtained the extension of Newton-Raphson's method as follows.

Let $x^q = t$ where *q* is a real number that is not 0. We define the function $g(t)$ such as

$$g(t) := f\left(t^{1/q}\right) = f(x) \tag{3.1}$$

Applying the Newton-Raphson's method to $g(t)$ and express it again in $f(x_k)$, we have the next definition.

Definition 3.1. For equation $f(x)=0$, we call the next recurrence formula the Murase-Newton's method or the Tsuchikura-Horiguchi's method (2009) where *q* is a real number that is not 0.

$$x_{k+1}^q = x_k^q - qx_k^{q-1}\frac{f(x_k)}{f'(x_k)} \left(q \neq 0, q \in \boldsymbol{R}\right) \tag{3.2}$$

here, if *q* = 1, then the Formula (3.2) becomes Newton-Raphson's method. Furthermore, we call the next formula general recurrence formula of the Murase-Newton's method or general recurrence formula of the Tsuchikura-Horiguchi's method. Here *q* and *λ* are real numbers that are not 0.

$$x_{k+1}^q = x_k^q - \lambda x_k^r \frac{f(x_k)}{f^{(i)}(x_k)} \left(q, \lambda \neq 0, q, \lambda \in \boldsymbol{R}\right) \tag{3.3}$$

The Formula (3.3) switches in various recurrence formula by *q*, *λ*, *r*, *i*. In particular, if $\lambda = q, r = q-1, i = 1$, then (3.3) becomes Tsuchikura-Horiguchi's method (Ref. [5]).

Let $a_i \left(i = 1,\cdots,n+1\right)$ be a real number. The *j*-th term of polynomial of *n*-th degree

$$f(x) = a_1x^n + a_2x^{n-1} + a_3x^{n-2} + \cdots + a_{n-1}x^2 + a_nx + a_{n+1} \tag{3.4}$$

is $a_jx^{n-j+1}\left(j=1,\cdots,n+1\right)$. The *j*-th term of *i*-th derived function $f^{(i)}(x)\left(i=1,\cdots,n\right)$ of $f(x)$ is this.

$$(n-j+1)(n-j)(n-j-1)\cdots\left(n-j-(i-2)\right)a_jx^{n-j-(i-1)}\left(j=1,2,\cdots,n-(i-1)\right) \tag{3.5}$$

We replace the coefficient $(n-j+1)(n-j)(n-j-1)\cdots\left(n-j-(i-2)\right)a_j$ of $x^{n-j-(i-1)}$ with a formula including a real variable *m* or constant, and denote such $f^{(i)}(x)$ in $f^{(i)}(x)_m$. We can associate $f^{(i)}(x)_m$ with the line of the indication of calculating formula of the Horner's method, and it becomes $f^{(i)}(x)$ when we substitute a certain real number in *m*. For an understanding of the notation $f^{(i)}(x)_m$, see $f^{(i)}(x)_m$ in the next example and Formulas (4.2)-(4.5) in Section 4.

Example 3.2. In Murase's formula $f(x) = x^3 - 14x^2 + 48$, let $f'(x)_m := 2mx^2 - 28x$. If we take *m* in 3/2, then $f'(x)_m$ becomes $f'(x)$. Furthermore if we take *m* in 33/4, 34/4, 35/4 and *x* in 2, then $f'(x)_m$ becomes 10, 12, 14, respectively. These correspond to the second line × −1 of the **Table 2** of the calculation of Horner's method in Example 2.1.

We make next recurrence formula.

$$x_{k+1}^2 = x_k^2 - 2x_k \frac{f(x_k)}{f'(x_k)_m} = x_k^2 - 2x_k \frac{x_k^3 - 14x_k^2 + 48}{2mx_k^2 - 28x_k} \tag{3.6}$$

This is equal to Formula (2.9) in Section 2.

Definition 3.3. The formula

$$x_{k+1}^q = x_k^q - \lambda x_k^r \frac{f(x_k)}{f^{(i)}(x_k)_m} \left(\lambda \neq 0, q \neq 0, q, \lambda \in \boldsymbol{R}\right) \tag{3.7}$$

is called general recurrence formula of the extension of Murase-Newton's method or general recurrence formula of the extension of Tsuchikura-Horiguchi's method concerning of *n*-th degree polynomial equation.

4. On Relations between General Recurrence Formula of the Extension of Tsuchikura-Horiguchi's Method (the Extension of Murase-Newton's Method) and Horner's Method

We easily explain by the next fifth-degree equation. Here $a_2, \cdots, a_6$ are real numbers.

$$f(x) = x^5 + a_2x^4 + a_3x^3 + a_4x^2 + a_5x + a_6 = 0 \tag{4.1}$$

4.1. Horner's Method for an Expression (4.1)

Let α be a real number. If we apply the Horner's method to polynomial $f(x)$, then we obtain the calculation in **Table 3**.

Table 3. Synthetic division for an expression (4.1).

1	a_2	a_3	a_4	a_5	a_6
	α	$(a_2+\alpha)\alpha$	$(a_3+(a_2+\alpha)\alpha)\alpha$	$(a_4+(a_3+(a_2+\alpha)\alpha)\alpha)\alpha$	$(a_5+(a_4+(a_3+(a_2+\alpha)\alpha)\alpha)\alpha)\alpha$
1	$a_2+\alpha$	$a_3+(a_2+\alpha)\alpha$	$a_4+(a_3+(a_2+\alpha)\alpha)\alpha$	$a_5+(a_4+(a_3+(a_2+\alpha)\alpha)\alpha)\alpha$	$a_6+(a_5+(a_4+(a_3+(a_2+\alpha)\alpha)\alpha)\alpha)\alpha$
	α	$(a_2+2\alpha)\alpha$	$(a_3+(2a_2+3\alpha)\alpha)\alpha$	$(a_4+(2a_3+(3a_2+4\alpha)\alpha)\alpha)\alpha$	
1	$a_2+2\alpha$	$a_3+(2a_2+3\alpha)\alpha$	$a_4+(2a_3+(3a_2+4\alpha)\alpha)\alpha$	$a_5+(2a_4+(3a_3+(4a_2+5\alpha)\alpha)\alpha)\alpha$	
	α	$(a_2+3\alpha)\alpha$	$(a_3+(3a_2+6\alpha)\alpha)\alpha$		
1	$a_2+3\alpha$	$a_3+(3a_2+6\alpha)\alpha$	$a_4+(3a_3+(6a_2+10\alpha)\alpha)\alpha$		
	α	$(a_2+4\alpha)\alpha$			
1	$a_2+4\alpha$	$a_3+(4a_2+10\alpha)\alpha$			
	α				
1	$a_2+5\alpha$				

4.2. In the Case of General Fifth-Degree Equation (4.1)

Theorem 4.1. There exists $f^{(i)}(x)_m \ (i=1,\cdots,4)$, so that it equals to $f^{(i)}(x)$ if $m=i+1$. Furthermore, if $x=\alpha$ and $m=1,\cdots,i+1$, then $f^{(i)}(x)_m$ corresponds to the $(6-i)$-th line × *i*! of **Table 3** of Horner's method, respectively.

Proof. We should define the formulas $f'(x)_m$, $f''(x)_m$, $f'''(x)_m$, $f^{(4)}(x)_m$ as follows.

$$f'(x)_m := \left((3m-2)(m-1)+1\right)x^4 + \left(\frac{3m}{2}(m-1)+1\right)a_2x^3 + \frac{m}{2}(m+1)a_3x^2 + ma_4x + a_5 \tag{4.2}$$

$$f''(x)_m := 2!\left(\left(\frac{3m}{2}(m-1)+1\right)x^3 + \frac{m}{2}(m+1)a_2x^2 + ma_3x + a_4\right) \tag{4.3}$$

$$f'''(x)_m := 3!\left(\frac{m}{2}(m+1)x^2 + ma_2x + a_3\right) \tag{4.4}$$

$$f^{(4)}(x)_m := 4!(mx + a_2) \tag{4.5}$$

From Theorem 4.1, we obtain the next theorem.

Theorem 4.2. There exists the rational recurrence formula x_{k+1} obtained from Formula (4.1) so that the denominator equals to (4.2). Similarly, there exists the rational recurrence formula $x_{k+1}^2, x_{k+1}^3, x_{k+1}^4$ so that the denominator equals to $f''(x)_m/2!$ of (4.3), $f'''(x)_m/3!$ of (4.4), $f^{(4)}(x)_m/4!$ of (4.5), respectively.

Proof. We should choose the formulas $x_{k+1}, x_{k+1}^2, x_{k+1}^3, x_{k+1}^4$ as follows.

$$x_{k+1} = \frac{(3m-2)(m-1)x_k^5 + \frac{3m}{2}(m-1)a_2x_k^4 + \left(\frac{m}{2}(m+1)-1\right)a_3x_k^3 + (m-1)a_4x_k^2 - a_6}{\left((3m-2)(m-1)+1\right)x_k^4 + \left(\frac{3m}{2}(m-1)+1\right)a_2x_k^3 + \frac{m}{2}(m+1)a_3x_k^2 + ma_4x_k + a_5} \tag{4.6}$$

$$x_{k+1}^2 = \frac{\frac{3m}{2}(m-1)x_k^5 + \left(\frac{m}{2}(m+1)-1\right)a_2x_k^4 + (m-1)a_3x_k^3 - a_5x_k - a_6}{\left(\frac{3m}{2}(m-1)+1\right)x_k^3 + \frac{m}{2}(m+1)a_2x_k^2 + ma_3x_k + a_4} \tag{4.7}$$

$$x_{k+1}^3 = \frac{\left(\frac{m}{2}(m+1)-1\right)x_k^5 + (m-1)a_2x_k^4 - a_4x_k^2 - a_5x_k - a_6}{\frac{m}{2}(m+1)x_k^2 + ma_2x_k + a_3} \tag{4.8}$$

$$x_{k+1}^4 = \frac{(m-1)x_k^5 - a_3x_k^3 - a_4x_k^2 - a_5x_k - a_6}{mx_k + a_2} \tag{4.9}$$

Furthermore, we obtain the next theorem by a simple calculation.

Theorem 4.3. The recurrence Formula (4.6) obtained from Formula (4.1) is equal to general recurrence Formula (4.10) of the extension of Tsuchikura-Horiguchi's method of x of Formula (4.1).

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)_m} = x_k - \frac{x_k^5 + a_2x_k^4 + a_3x_k^3 + a_4x_k^2 + a_5x_k + a_6}{\left((3m-2)(m-1)+1\right)x_k^4 + \left(\frac{3m}{2}(m-1)+1\right)a_2x_k^3 + \frac{m}{2}(m+1)a_3x_k^2 + ma_4x_k + a_5} \tag{4.10}$$

Similarly recurrence Formulas (4.7)-(4.9) is equal to general recurrence Formulas (4.11)-(4.13) of the extension of Tsuchikura-Horiguchi's method of x^2, x^3, x^4 of Formula (4.1), respectively.

$$x_{k+1}^2 = x_k^2 - 2!\frac{f(x_k)}{f''(x_k)_m} = x_k^2 - \frac{x_k^5 + a_2x_k^4 + a_3x_k^3 + a_4x_k^2 + a_5x_k + a_6}{\left(\frac{3m}{2}(m-1)+1\right)x_k^3 + \frac{m}{2}(m+1)a_2x_k^2 + ma_3x_k + a_4} \tag{4.11}$$

$$x_{k+1}^3 = x_k^3 - 3!\frac{f(x_k)}{f'''(x_k)_m} = x_k^3 - \frac{x_k^5 + a_2x_k^4 + a_3x_k^3 + a_4x_k^2 + a_5x_k + a_6}{\frac{m}{2}(m+1)x_k^2 + ma_2x_k + a_3} \tag{4.12}$$

$$x_{k+1}^4 = x_k^4 - 4!\frac{f(x_k)}{f^{(4)}(x_k)_m} = x_k^4 - \frac{x_k^5 + a_2x_k^4 + a_3x_k^3 + a_4x_k^2 + a_5x_k + a_6}{mx_k + a_2} \tag{4.13}$$

4.3. In the Case of Special Fifth-Degree Equations (4.14) of Murase's Type

$$f(x) = x^5 + a_i x^{6-i} + a_6 = 0 \left(i = 2,3,4,5\right) \tag{4.14}$$

We transform the fifth-degree Equations (4.14), and obtain the next four recurrence formulas.

$$x_{k+1}^{6-i} = \frac{-(m+1)x_k^5 - a_6}{-mx_k^{i-1} + a_i}\left(i = 2,3,4,5\right) \tag{4.15}$$

Because it is a simple matter, we give only theorems without proof in the following.

Theorem 4.4. If $x_k = \alpha, i = 2$ and $m = -1,-2,-3,-4,-5$, then the denominator $a_2 - mx_k$ of the recurrence Formula (4.15) corresponds to the second line of the calculation of Horner's method. Similarly, if $x_k = \alpha$, $\left(i = 3 \text{ and } m = -1,-3,-6,-10\right)$, $\left(i = 4 \text{ and } m = -1,-4,-10\right)$, and $\left(i = 5 \text{ and } m = -1,-5\right)$, then the denominator of recurrence Formula (4.15) corresponds to the third, forth, and fifth line of the calculation of Horner's method, respectively.

Theorem 4.5. The recurrence Formula (4.15) $\left(i = 2,3,4,5\right)$ is equal to the next general recurrence formula x_{k+1}^{6-i} of (4.16) of the extension of Tsuchikura-Horiguchi's method of (4.14).

$$x_{k+1}^{6-i} = x_k^{6-i} - (6-i)x_k^{6-(i+1)}\frac{x_k^5 + a_i x_k^{6-i} + a_6}{f'(x_k)_m} = x_k^{6-i} - (6-i)x_k^{6-(i+1)}\frac{x_k^5 + a_i x_k^{6-i} + a_6}{m'x_k^4 + (6-i)a_i x_k^{5-i}}\left(i = 2,3,4,5\right) \tag{4.16}$$

Corollary 4.6. If $i = 2$, then (4.14) becomes the next formula.

$$f(x) = x^5 + a_2 x^4 + a_6 = 0 \tag{4.14}$$

In this case, Formula (4.13) becomes the next formula.

$$x_{k+1}^4 = x_k^4 - 24\frac{x_k^5 + a_2 x_k^4 + a_6}{f^{(4)}(x_k)_m} = x_k^4 - \frac{x_k^5 + a_2 x_k^4 + a_6}{mx_k + a_2} \tag{4.17}$$

Formula (4.17) is equal to (4.16) if $i = 2$. In the case of $i = 3, 4, 5$, a similar thing holds, respectively.

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