Scientific Research

Numerical Solutions for the Time-Dependent Emden-Fowler-Type Equations by B-Spline Method

Mohamed El-Gamel, W. El-bashbashy, Atallah El-Shenawy

Department of Mathematical Sciences, Faculty of Engineering, Mansoura University, Mansoura, Egypt
Email: gamel_eg@yahoo.com

Abstract

A numerical method based on B-spline is developed to solve the time-dependent Emden-Fowler-type equations. We also present a reliable new algorithm based on B-spline to overcome the difficulty of the singular point at $x = 0$. The error analysis of the method is described. Numerical results are given to illustrate the efficiency of the proposed method.

Keywords

B-Spline; Time-Dependent; Numerical Solutions; Emden-Fowler Type

1. Introduction

In recent years, a lot of attentions have been devoted to the study of B-spline method to investigate various scientific models. The efficiency of the method has been formally proved by many researchers [1]-[7].

Spline functions have some attractive properties. Due to the being piecewise polynomial, they can be integrated and differentiated easily. Since they have compact support, numerical methods in which spline functions are used as a basis function lead to matrix systems including band matrices. Such systems have solution algorithms with low computational cost.

In this paper, we employ the B-spline method to solve the time dependent partial differential equations. For clarity, the method is presented for the heat equation

$$u_t = u_{xx} + \frac{r}{x}u_x + a\eta(x,t,u) + \mu(x,t), \quad 0 < x < 1, t > 0 \tag{1}$$

subject to the boundary conditions

$$u_x(0,t) = 0, \quad u(1,t) = \xi(t), \tag{2}$$

and the initial condition

$$u(x,0) = v(x). \tag{3}$$

Some forms of the above equations model several phenomena in mathematical physics and astrophysics such as the diffusion of heat perpendicular to the surface of parallel planes, theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas sphere and theory of thermionic currents [8]-[10].

The solution of the time-dependent Emden-Fowler equation as well as a variety of linear and nonlinear singular IVPs in quantum mechanics and astrophysics is numerically challenging because of singularity behavior at the origin. The singularity behavior that occurs at the point $x = 0$ is the main difficulty in the analysis of Equations (1)-(3).

The approximate analytical solutions to the Lane-Emden equations were presented by Shawagfeh [11] and Wazwaz [12]-[14] using the Adomian decomposition method (ADM). Very recently, Chowdhury and Hashim [15] applied the homotopy-perturbation method (HPM) and the variational iteration method (VIM) [16] to obtain approximate analytical solutions of the time-dependent Emden-Fowler-type equations.

The paper is organized as follows. In Section 2, we review some basic facts about the B-spline that are necessary for the formulation of the discrete linear and nonlinear system. In Section 3, the error analysis of the method is described. In Section 4, we formulate our B-spline collocation method to the solution of (1)-(3). In Section 5, numerical experiments are tested to demonstrate the viability of the proposed method.

2. Some Properties of B-Spline

B-splines are mathematically more sophisticated than other types of splines, so we start with a gentle introduction. We first use basic assumptions to derive the expressions for the cubic uniform B-splines directly and without mentioning knots. We then show how to extend the derivations to uniform B-splines of any order. Following this, we discuss a different, recursive formulation of the weight functions of the uniform B-splines.

The third-degree B-spline is used to construct numerical solutions to a given problem (1). A detailed description of third-degree B-spline functions can be found in [13] [17]. The third-degree B-splines are defined as

$$B_{i-1}(x) = B_0(x - (i-1)h), \quad i = 2,3,\cdots \tag{4}$$

$$B_0(x) = \frac{1}{6h^3} \begin{cases} x^3 & \text{if}; 0 \le x < h \\ -3x^3 + 12hx^2 - 12h^2x + 4h^3, & \text{if}; h \le x < 2h \\ 3x^3 - 24hx^2 + 60h^2x - 44h^3, & \text{if}; 2h \le x < 3h \\ -x^3 + 12hx^2 - 48h^2x + 64h^3 & \text{if}; 3h \le x < 4h \end{cases} \tag{5}$$

We can simply check that each of the function $B_i(x)$ is twice continuously differentiable on the entire real line. Also,

$$B_i(x_j) = \begin{cases} 4, & \text{if}; i = j \\ 1, & \text{if}; i - j = \pm 1 \\ 0, & \text{if}. i - j = \pm 2 \end{cases} \tag{6}$$

Similarly, we can show that:

$$B_i'(x_j) = \begin{cases} 0, & \text{if}; \quad i = j \\ \pm \dfrac{3}{h}, & \text{if}; \quad i - j = \pm 1 \\ 0, & \text{if}. \quad i - j = \pm 2 \end{cases} \tag{7}$$

and

$$B_i''(x_j) = \begin{cases} -\dfrac{12}{h^2}, & \text{if}\,;\, i = j \\[2mm] \dfrac{6}{h^2}, & \text{if}\,;\, i - j = \pm 1 \\[2mm] 0, & \text{if}\,.\, i - j = \pm 2 \end{cases} \tag{8}$$

3. Error Analysis

3.1. Governing Equation

Since $x = 0$ is singular point of Equation (1), we first modify Equation (1) at $x = 0$. By L'Hospital rule, the boundary value problem (1) is transform into

$$u_t = p(x)u_{xx} + q(x)u_x + a\eta(x,t,u) + \mu(x,t), \tag{9}$$

where

$$p(x) = \begin{cases} r+1, & x = 0; \\ 1, & x \neq 0 \end{cases} \qquad q(x) = \begin{cases} 0, & x = 0; \\ \dfrac{r}{x}, & x \neq 0 \end{cases}$$

Let $\eta(x,t,u) = h(x,t)u$, we consider the linear non-homogeneous Emden-Fowler differential Equation (9), difference schemes for this problem considered as following:

$$p(x)\frac{d^2 u_{i+1}}{dx^2} + q(x)\frac{du_{i+1}}{dx} + ah(x,t)u_{i+1} + \mu(x,t) = \frac{u_{i+1} - u_i}{\Delta t} \tag{10}$$

or

$$(\Delta t)\, p(x)u_{i+1}^{(2)} + \Delta t q(x)u_{i+1}^{(1)} + \left[a(\Delta t)h(x,t) - 1 \right] u_{i+1} = -u_i - (\Delta t)\mu(x,t) \tag{11}$$

Suppose $S(x)$ is the cubic B-spline interpolating $u(x_k, t_{i+1})$ at the $(i+1)^{th}$ time level then

$$S(x) = \sum_{k=-3}^{n-1} C_k B(x) \tag{12}$$

Theorem 3.1. Let the collocation approximation $S(x)$ to the solution $u(x)$ of the boundary value problem (1) is (12) then the following relation is hold

$$S'(x) = u_x(x_i) - \frac{h^4}{180} u_{xxxxx}(x_i) + O(h^6)$$

$$S''(x) = u_{xx}(x_i) - \frac{h^2}{12} u_{xxxx}(x_i) + \frac{h^4}{360} u_{xxxxxx} + O(h^6) \tag{13}$$

3.2. Truncation Error of Time Dependent Emden-Fowler Equation

Theorem 3.2. By using the combination of the finite difference approximation and cubic B-spline method, the truncation error is $O(\Delta t, h^2)$.

Proof: Consider the Equation (10) when apply finite difference

$$\frac{1}{\Delta t}(u_{i+1} - u_i) = p(x)u_{xx} + q(x)u_x + ah(x,t)u + \mu(x,t)$$

apply taylor expansion for L.H.S

$$\frac{1}{\Delta t}\left(u_i + \Delta t u_t + \frac{(\Delta t)^2}{2!} u_{tt} + \frac{(\Delta t)^3}{3!} u_{ttt} + \cdots - u_i \right) = p(x)u_{xx} + q(x)u_x + ah(x,t)u + \mu(x,t)$$

$$u_t + \frac{\Delta t}{2!}u_{tt} + \frac{(\Delta t)^2}{3!}u_{ttt} + \cdots = p(x)\left(u_{xx} - \frac{1}{12}h^2 u_{xxxx} + \frac{1}{360}h^4 u_{xxxxxx} + O(h^6)\right)$$

$$+ q(x)\left(u_x - \frac{1}{180}h^4 u_{xxxx} + O(h^6)\right) + ah(x,t)u + \mu(x,t)$$

if

$$p(x)u_{xx} + q(x)u_x + ah(x,t)u + \mu(x,t) = \lambda(x,t)$$

we can calculate the truncation error which is defined as $e_i = u_t - \lambda(x,t)$ as:

$$e = \frac{\Delta t}{2!}u_{tt} + \frac{(\Delta t)^2}{3!}u_{ttt} + \cdots + \left(\frac{h^2}{12} - \frac{h^4}{180}\right)u_{xxxx} - O(h^6)$$

4. B-Spline Solutions for Time-Dependent Emden-Fowler

In this section, we shall introduce a reliable algorithm based on B-spline method to handle singular initial value problems (IVPs) in a realistic and efficient way considering Emden Fowler equation as a model problem.

4.1. Linear Time-Dependent Emden-Fowler

Let

$$u(x) = \sum_{j=-3}^{n-1} C_j B_j(x) \tag{14}$$

be an approximate solution of Equation (1), where C_i are unknown real coefficients and $B_j(x)$ are third-degree B-spline functions. Let $x_0, x_1, \cdots, x_n$ be $n+1$ grid points in the interval $[a,b]$, so that

$$x_i = a + ih, \; i = 0,1,\cdots,n, \quad x_0 = a, \; x_n = b, \; h = \frac{b-a}{n}$$

We can deduce the following

$$u^{(1)}(x) = \sum_{j=-3}^{n-1} C_j B_j'(x)$$

$$u^{(2)}(x) = \sum_{j=-3}^{n-1} C_j B_j''(x) \tag{15}$$

Theorem 4.1. If the assumed approximate solution of the problem (10) is (14), then the discrete collocation system for the determination of the unknown coefficients $\{C_j\}_{j=-3}^{n-1}$ is given by

$$(\Delta t)p(x_i)\sum_{j=-3}^{n-1} C_j B_j''(x_i) + \Delta t q(x_i)\sum_{j=-3}^{n-1} C_j B_j'(x_i)$$

$$+\left[a(\Delta t)h(x_i,t)-1\right]\sum_{j=-3}^{n-1} C_j B_j(x_i) = -u_i - (\Delta t)\mu(x_i,t), \quad 0 \le i \le n. \tag{16}$$

Proof: If we replace each term of (10) with its corresponding approximation given by (14) and (15) and substituting $x = x_k$ and applying the collocation to it.

The boundary conditions (3) can be written as

$$\sum_{j=-3}^{n-1} C_j B_j'(0) = 0$$

$$\sum_{j=-3}^{n-1} C_j B_j(1) = \xi(t) \tag{17}$$

Then use them with the system of Equation (16), which can be written in the matrix form

$$AC = F \tag{18}$$

where

$$C = \begin{pmatrix} C_{n-1} \\ C_{n-2} \\ \vdots \\ C_{-2} \\ C_{-3} \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ \vdots \\ \xi(t) \end{pmatrix}$$

Also the matrix A can be written as

$$A = \begin{pmatrix} \dfrac{-3}{h} & 0 & \dfrac{3}{h} & \cdots & 0 & 0 \\ \alpha_1(x_0) & \alpha_2(x_0) & \alpha_3(x_0) & \cdots & 0 & 0 \\ 0 & \alpha_1(x_1) & \alpha_2(x_1) & & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & & \alpha_3(x_{n-1}) & 0 \\ 0 & 0 & \cdots & & \alpha_2(x_n) & \alpha_3(x_n) \\ 0 & 0 & \cdots & & 4 & 1 \end{pmatrix}$$

where

$$\alpha_1(x_i) = p(x_i)\left(\frac{6}{h^2}\right) - q(x_i)\left(\frac{3}{h}\right) + \left[a(\Delta t)\eta(x_i,t) - 1\right]$$

$$\alpha_2(x_i) = p(x_i)\left(-\frac{12}{h^2}\right) + \left[a(\Delta t)\eta(x_i,t) - 1\right] \cdot 4$$

$$\alpha_3(x_i) = p(x_i)\left(\frac{6}{h^2}\right) + q(x_i)\left(\frac{3}{h}\right) + \left[a(\Delta t)\eta(x_i,t) - 1\right]$$

It is easily seen that the matrix A is strictly diagonally dominant and hence nonsingular. Now we have a linear system of $n+3$ equations for the $n+3$ unknown coefficients, namely, $\{C_j\}_{j=-3}^{n-1}$. We can obtain the coefficient of the approximate solution by solving this linear system by Q-R method.

4.2. Non-Linear Time-Dependent Emden-Fowler

Let $\eta(x,t,u) = h(x,t)g(u)$, we consider the nonlinear non-homogeneous Emden-Fowler differential equation

$$p(x)u_{xx} + q(x)u_x + ah(x,t)g(u) + \mu(x,t) = u_t \tag{19}$$

We can use finite difference method

$$p(x)\frac{\mathrm{d}^2 u_{i+1}}{\mathrm{d}x^2} + q(x)\frac{\mathrm{d}u_{i+1}}{\mathrm{d}x} + ah(x,t)g(u_{i+1}) + \mu(x,t) = \frac{u_{i+1} - u_i}{\Delta t} \tag{20}$$

Theorem 4.2. If the assumed approximate solution of the problem (20) is (14), then the discrete collocation system for the determination of the unknown coefficients is given by

$$p(x)\sum_{j=-3}^{n-1} C_j B_j''(x_k) + q(x)\sum_{j=-3}^{n-1} C_j B_j'(x_k) + ah(x_k,t_{i+1})g\left(\sum_{j=-3}^{n-1} C_j B_j(x_k)\right)$$

$$+ \mu(x_k,t_{i+1}) = \frac{\displaystyle\sum_{j=-3}^{n-1} C_j B_j(x_k) - u_i}{\Delta t}, \quad k = 0,1,\cdots,n. \tag{21}$$

Then the boundary condition (17) with the system of Equation (21). Now we have a nonlinear system of

$n+3$ equations in the $n+3$ unknown coefficients. We can obtain the coefficients of the approximate solution by solving this nonlinear system by *Newton's method*.

5. Examples and Comparisons

In this section, we will present three of our numerical results of time dependent problems using the method outlined in the previous section. The performance of the B-spline method is measured by the maximum absolute error ε_k which is defined as

$$\varepsilon_k = \left| u_{\text{exact}} - u_{\text{B-spline}} \right|$$

Example 1: [18] Consider the problem

$$u_t = u_{xx} + \frac{2}{x} u_x - \left(6 + 4x^2 - \cos t \right) u, \quad 0 < x < 1, t > 0$$

subject to boundary conditions

$$u_x(0,t) = 0, \quad u(1,t) = e^{1+\sin t},$$

and the initial condition

$$u(x,0) = e^{x^2}.$$

whose exact solution is

$$u(x,t) = e^{x^2 + \sin t}.$$

The maximum absolute errors at different n and different time t with $\Delta t = 0.001$ is given in **Table 1**.

Example 2: [18] Consider the linear problem

$$u_t + \left(6 - 5x^2 - 4x^4 \right) = u_{xx} + \frac{2}{x} u_x - \left(5 + 4x^2 \right) u,$$

subject to boundary conditions

$$u_x(0,t) = 0, \quad u(1,t) = 1 + e^{1+t},$$

and the initial condition

$$u(x,0) = x^2 + e^{x^2}.$$

whose exact solution is

Table 1. Maximum absolute errors for Example 1.

n	$T = 0.1$	$T = 0.5$	$T = 1.0$
10	0.002756	0.00493	0.007185
20	6.556E−04	0.00119	0.001819
30	2.693E−04	5.124E−04	8.373E−04
40	1.348E−04	2.742E−04	4.947E−04
50	7.304E−05	1.643E−04	3.364E−04
60	3.999E−05	1.049E−04	2.506E−04
70	2.074E−05	6.951E−05	1.990E−04
80	1.261E−05	4.683E−04	1.657E−04

$$u(x,t) = e^{x^2+t} + x^2.$$

The maximum absolute errors at different n and different time t with $\Delta t = 0.001$ is given in **Table 2**.

Example 3: [18] Now we turn to a singular nonlinear problem

$$u_t = u_{xx} + \frac{5}{x}u_x - \left(24t + 16t^2x^2\right)e^u - 2x^2e^{u/2},$$

subject to boundary conditions

$$u_x(0,t) = 0, \quad u(1,t) = -2\ln(1+t),$$

and the initial condition

$$u(x,0) = 0.$$

whose exact solution is

$$u(x,t) = -2\ln\left(1+tx^2\right).$$

The maximum absolute errors at different n and different time t with $\Delta t = 0.001$ is given in **Table 3**.

6. Conclusions

We have presented a B-spline collocation method for the solution of time-dependent Emden Fowler type Equations (1)-(3). This method produces a spline function which is useful to obtain the solution at any point of the interval, whereas the finite difference method gives the solution only at selected nodal points. The numerical results given in tables show that the present method approximates the exact solution very well.

This present analysis exhibits the reliable applicability of B-spline method to solve linear and nonlinear problems with singular feature. In this work, we demonstrate that this method can be well suited to attain solution to the type of examined equations, linear and nonlinear as well. The difficulty in this type of equations,

Table 2. Maximum absolute errors for Example 2.

n	$T = 0.1$	$T = 0.5$	$T = 1.0$
10	0.002756	0.00498	0.00821
20	6.528E−04	0.00117	0.00192
30	2.664E−04	4.729E−04	7.799E−04
40	1.319E−04	2.309E−04	3.807E−04
50	7.021E−05	1.203E−04	1.984E−04
60	3.727E−05	6.203E−05	1.022E−04
70	1.823E−05	2.906E−05	4.791E−05
80	1.541E−05	2.431E−05	4.008E−05

Table 3. Maximum absolute errors for Example 3.

n	$T = 0.01$	$T = 0.1$	$T = 0.2$
10	6.597E−05	5.805E−04	0.001176
20	1.494E−05	1.340E−04	2.807E−04
30	6.481E−06	5.202E−05	1.148E−04

due to the existence of singular point at $x = 0$, is overcome here. Finally, we conclude that the B-spline method is a promising tool for both linear and nonlinear singular time-dependent Emden-Fowler-type equations.

References

[1] Ali, A.H.A., Gardner, G.A. and Gardner, L.R.T. (1992) A Collocation Solution for Burgers? Equation Using Cubic B-Spline Finite Elements. *Computer Methods in Applied Mechanics and Engineering*, **100**, 325-337. http://dx.doi.org/10.1016/0045-7825(92)90088-2

[2] Aziz, T., Khan, A. and Rashidinia, J. (2005) Spline Methods for the Solution of Fourth-Order Parabolic Partial Differential Equations. *Applied Mathematics and Computation*, **167**, 153-166. http://dx.doi.org/10.1016/j.amc.2004.06.095

[3] Caglar, N. and Caglar, H. (2006) B-Spline Solution of Singular Boundary Value Problems. *Applied Mathematics and Computation*, **182**, 1509-1513. http://dx.doi.org/10.1016/j.amc.2006.05.035

[4] Caglar, H., Ozer, M. and Caglar, N. (2008) The Numerical Solution of the One-Dimensional Heat Equation by Using Third Degree B-Spline Functions. *Chaos*, *Solitons & Fractals*, **38**, 1197-1201. http://dx.doi.org/10.1016/j.chaos.2007.01.056

[5] Fazal-i-Haq, Siraj-ul-Islam and Tirmizi, I.A. (2010) A Numerical Technique for Solution of the MRLW Equation Using Quartic B-Splines. *Applied Mathematical Modelling*, **34**, 4151-4160. http://dx.doi.org/10.1016/j.apm.2010.04.012

[6] Lakestani, M. and Dehghan, M. (2012) Numerical Solutions of the Generalized Kuramoto-Sivashinsky Equation Using B-Spline Functions. *Applied Mathematical Modelling*, **36**, 605-617. http://dx.doi.org/10.1016/j.apm.2011.07.028

[7] Mittal, R.C. and Jain, R.K. (2011) B-Splines Methods with Redefined Basis Functions for Solving Fourth Order Parabolic Partial Differential Equations. *Applied Mathematics and Computation*, **217**, 9741-9755. http://dx.doi.org/10.1016/j.amc.2011.04.061

[8] Chandrasekhar, S. (1967) Introduction to the Study of Stellar Structure. Dover, New York.

[9] Davis, H.T. (1962) Introduction to Nonlinear Differential and Integral Equations. Dover, New York.

[10] Richardson, O.U. (1921) The Emission of Electricity from Hot Bodies. Logmans Green, London.

[11] Shawagfeh, N.T. (1993) Nonperturbative Approximate Solution for Lane-Emden Equation. *Journal of Mathematical Physics*, **34**, 4364-4369. http://dx.doi.org/10.1063/1.530005

[12] Wazwaz, A. (2005) Analytical Solution for the Time-Dependent Emden-Fowler Type of Equations by Adomian Decomposition Method. *Applied Mathematics and Computation*, **166**, 638-665. http://dx.doi.org/10.1016/j.amc.2004.06.058

[13] Wazwaz, A.M. (2005) Adomian Decomposition Method for a Reliable Treatment of the Emden-Fowler Equation. *Applied Mathematics and Computation*, **161**, 543-560. http://dx.doi.org/10.1016/j.amc.2003.12.048

[14] Wazwaz, A. (2001) A New Algorithm for Solving Differential Equations of Lane?—Emden Type. *Applied Mathematics and Computation*, **118**, 287-310. http://dx.doi.org/10.1016/S0096-3003(99)00223-4

[15] Chowdhury, M.S.H. and Hashim, I. (2007) Solutions of Time-Dependent Emden? Fowler Type Equations by Homotopy-Perturbation Method. *Physics Letters A*, **368**, 305-313. http://dx.doi.org/10.1016/j.physleta.2007.04.020

[16] Wazwaz, A. (2011) The Variational Iteration Method for Solving Nonlinear Singular Boundary Value Problems Arising in Various Physical Models. *Communications in Nonlinear Science and Numerical Simulation*, **16**, 3881-3886. http://dx.doi.org/10.1016/j.cnsns.2011.02.026

[17] de Boor, C. (1978) A Practical Guide to Splines. Springer Verlag, New York. http://dx.doi.org/10.1007/978-1-4612-6333-3

[18] Matinfar, M. and Mahdavi, M. (2012) Approximate Analytical Solutions for the Time-Dependent Emden-Fowler-Type Equations by Variational Homotopy Perturbation Method. *WJMS*, **8**, 58-65.