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The Mean Residual Lifetime of ($n-k+1$)-out-of-n Systems in Discrete Setting

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ABSTRACT

In real life, there are situations in which the lifetime of the components of a technical system (and hence the lifetime of the system) is discrete. In this paper, we study the residual life, a ($n-k+1$)-out-of-n system under the assumptions that the components of the system are independent identically distributed with common discrete distribution function F. We define the mean residual lifetime (MRL) of the system and under different scenarios investigate several aging and stochastic properties of MRL.

KEYWORDS

($n-k+1$)-out-of-n System; Discrete Lifetime; Failure Rate; Reliability

1. Introduction

In recent years, researchers in reliability theory have shown intensified interest in the study of stochastic and reliability properties of technical systems. The $(n-k+1)$-out-of-n system structure is a very popular type of redundancy in technical systems. A $(n-k+1)$-out-of-n system is a system consisting of n components (usually the same) and functions if and only if at least $n-k+1$ out of n components are operating $(k \le n)$. Hence, such system fails if k or more of its components fail. Let $T_1, T_2, \cdots, T_n$ denote the component lifetimes of the system and assume that $T_{1:n}, T_{2:n}, \cdots, T_{n:n}$ represent the ordered lifetimes of the components. Then it is easy to argue that the lifetime of the system is $T_{k:n}$ where $T_{k:n}$ denotes the k, the order statistics corresponding to T_i's, $i = 1, 2, \cdots, n$. Under the assumption that T_i's are continuous random variables, several authors have studied the residual lifetime and the mean residual lifetime (MRL) of the system under different conditions. Assuming that at time t at least $n-r+1$ components of the system are working, the residual lifetime of the system can be defined as follows:

$$T_t^{r,k,n} = \left(T_{k:n} - t \middle| T_{r:n} > t\right), r = 1, \cdots, k, k = 1, \cdots, n. \tag{1}$$

Among the researchers who investigated the reliability and aging properties of the conditional random variable $T_t^{r,k,n}$, under various conditions and for different values of k and r, we can refer to Bairamov *et al*. [1], Asadi and Bairamov [2,3], Asadi and Goliforushani [4], Li and Zhao [5] and Zhang and Yang [6]. The extension to coherent systems has also been considered by several authors; see, among others, Li and Zhang [7], Navarro *et al*. [8], Zhang [9,10], Zhang and Li [11], Asadi and Kelkin Nama [12], and references therein.

Recently Mi [13] considered the situation in which the components of the system had discrete lifetimes and investigated some of aging properties of the system. The aim of the present paper is to study the MRL of $(n-k+1)$-out-of-n system under discrete setting. For this purpose, we assume that $T_1, T_2, \cdots, T_n$ are non-negative integer valued random variables denoting the lifetimes of the components of an $(n-k+1)$-out-of-n

system. Furthermore, we assume that T_i, $i=1,\cdots,n$ are independent and have a common probability mass function

$$p(t)=P(T_i=t), t=0,1,2,\cdots$$

and survival function

$$S(t)=P(T_i \geq t)=\sum_{j=t}^{\infty} p(j).$$

The hazard rate of the components, denoted by $h(t)$ and $r(t)$, is defined as follows:

$$h(t)=\frac{P(T_i=t)}{P(T_i \geq t)}=\frac{p(t)}{S(t)}$$

One can easily show that the survival and probability mass functions can be recovered from the hazard rate, respectively, as follows:

$$S(t)=\prod_{j=0}^{t-1}(1-h(j))=\prod_{j=0}^{t-1}(1-H(j)+H(j-1))$$

$$p(t)=h(t)\prod_{j=0}^{t-1}(1-h(j))$$

The MRL function of the components, denoted by $m(t)$, plays an important role in reliability engineering and survival analysis. Assuming each component of the system has survived up to times t, the MRL function $m(t)$ of each component is defined as

$$m(t)=E(T-t|T \geq t)=\frac{\sum_{j=t+1}^{\infty} S(j)}{S(t)}$$

It is not difficult to show that the survival function $S(t)$ can be represented in terms of $L(t)$ as below:

$$S(t)=\prod_{j=0}^{t-1}\frac{m(j)}{1+m(j+1)}.$$

The reset of the paper is organized as follows :

We first assume that at time t all components of the system are working and obtaining the functional form of the mean of $T_t^{1,k,n}$. This is in fact the MRL of the system, denoted by $H_n^k(t)$, under the condition that all components of the system are operating at time t. It is shown that when the components of the system have geometric distribution, $H_n^k(t)$ is free of time. Then, we prove that if the components of the system have increased failure rate, $H_n^k(t)$ is a decreasing function of t. It is also shown that when the components of two independents are ordered in terms of hazard rate ordering, under the condition that all components of the two systems are alive, their corresponding MRLs are also ordered. The results are then extended to the case where at least $(n-r+1)$ components of the system are operating. In this case, we obtain the functional form of the MRL of the system, denoted by $H_n^{r,k}(t)$. It is shown that $H_n^{r,k}(t)$ can be represented as the mixture of $H_n^k(t)$, where the mixing function is

$$P_i(t)=P(T_{i:n}<t<T_{i+1:n}|T_{r:n} \geq t), i=0,\cdots,r-1.$$

We prove that in the case where the components of the system have increased hazard rate, then $H_n^{r,k}(t)$ is decreasing in time. However, it is shown, using a counter example, that when the components of the system have decreased hazard rate, it is not necessarily true in general that $H_n^{r,k}(t)$ is increasing in time.

The function $P_i(t)$, mentioned above, has its own interesting interpretation. It shows the probability that there are exactly i failed components in the system, $i=0,\cdots,r-1$, given that at least $(n-r+1)$ components are working at time t. Several properties of $P_i(t)$ are also investigated.

2. The Mean Residual Life Function of System at the Component Level

In this section, we consider a $(n-r+1)$-out-of-n system and assume that the components of the system have independent discrete lifetimes $T_1,T_2,\cdots,T_n$ with common probability mass function $p(t)=P(T_i=t)$ and survival function $S(t)$, where $t=0,1,2,\cdots$. Let also $T_{1:n},\cdots,T_{n:n}$ be the order statistics corresponding to T_i's. In what follows, first, we assume that at time $t>0$, all the components of the system are working, *i.e.* $T_{1:n}\ge t$. The residual lifetime of the system, under the condition that all components of the system are working at time t, is $T_{k:n}-t|T_{1:n}\ge t$ (see Asadi and Bairamoglu [3]).

Using the standard techniques, one can easily show that

$$P\left(T_{k:n}>t+x\middle|T_{1:n}\ge t\right)=\sum_{i=0}^{k-1}\binom{n}{i}\left(\frac{S(t+x+1)}{S(t)}\right)^{n-i}\left(1-\frac{S(t+x+1)}{S(t)}\right)^{i}. \tag{2}$$

Hence the MRL function of the system, denoted by $H_n^k(t)$, can be obtained as follows

$$\begin{aligned}H_n^k(t)&=E\left(T_{k:n}-t\middle|T_{1:n}\ge t\right)=\sum_{x=0}^{\infty}P\left(T_{k:n}>t+x\middle|T_{1:n}\ge t\right)\\&=\sum_{x=0}^{\infty}\sum_{i=0}^{k-1}\binom{n}{i}\left(\frac{S(t+x+1)}{S(t)}\right)^{n-i}\left(1-\frac{S(t+x+1)}{S(t)}\right)^{i}\end{aligned} \tag{3}$$

$$\begin{aligned}&=\sum_{x=0}^{\infty}\sum_{i=0}^{k-1}\binom{n}{i}\left(\frac{S(t+x+1)}{S(t)}\right)^{n-i}\sum_{j=0}^{i}\binom{i}{j}(-1)^j\left(\frac{S(t+x+1)}{S(t)}\right)^{j}\\&=\sum_{i=0}^{k-1}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}(-1)^j\sum_{x=0}^{\infty}\left(\frac{S(t+x+1)}{S(t)}\right)^{n-i+j}\\&=\sum_{i=0}^{k-1}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}(-1)^j M_{n+j-i}(t)\end{aligned} \tag{4}$$

where

$$M_{n+j-i}(t)=\sum_{x=0}^{\infty}\left(\frac{S(t+x+1)}{S(t)}\right)^{n-i+j}$$

denotes the MRL function of a series system consisting of $n+j-i$ components, $j=0,1,2,\cdots,i$, $i=0,1,\cdots,k-1$.

Example 2.1 Let the components of the system have geometric distribution with probability mass function

$$p(t)=P(T=t)=\theta(1-\theta)^{t-1},t=1,2,\cdots$$

and survival function

$$S(t)=P(T\ge t)=\sum_{i=t}^{\infty}\theta(1-\theta)^{i-1}=(1-\theta)^{t-1}.$$

We have

$$M_{n+j-i}(t)=\sum_{x=0}^{\infty}\left(\frac{(1-\theta)^{t+x}}{(1-\theta)^{t-1}}\right)^{n+j-i}=\sum_{x=0}^{\infty}(1-\theta)^{(x+1)(n+j-i)}=\frac{(1-\theta)^{n+j-i}}{1-(1-\theta)^{n+j-i}}$$

$$H_n^k(t)=\sum_{i=0}^{k-1}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}(-1)^j\frac{(1-\theta)^{n+j-i}}{1-(1-\theta)^{n+j-i}}$$

Note that the MRL of a system having independent geometric components does not depend on t.

The distribution function of the order statistics $T_{r:n}$ can be represented in terms of incomplete beta function as follows (see David and Nagaraga [14]):

$$P\left(T_{r:n} \le x\right) = \sum_{j=r}^{n} \binom{n}{j} F^{j}\left(x\right)\left(1-F\left(x\right)\right)^{n-j} = \frac{1}{B\left(r,n-r+1\right)} \int_{0}^{F(x)} u^{r-1}\left(1-u\right)^{n-r}\, \mathrm{d}u$$

where

$$B\left(a,b\right) = \frac{\left(a+b\right)!}{a!b!}$$

Hence the MRL function of the system can be represented as

$$H_{n}^{k}\left(t\right) = \sum_{x=0}^{\infty} \frac{1}{B\left(k,n-k+1\right)} \int_{1-\frac{S(t+x+1)}{S(t)}}^{1} u^{k-1}\left(1-u\right)^{n-k}\, \mathrm{d}u \tag{5}$$

This representation is useful to prove the following two theorems.

Theorem 2.2 *If the components of the* $\left(n-k+1\right)$*-out-of-*n *system have an increasing* (*decreasing*) *hazard rate*, *then* $H_{n}^{k}\left(t\right)$ *is decreasing* (*increasing*) *in* t.

Proof:

If $h\left(t\right) = \frac{p\left(t\right)}{S\left(t\right)}$ denotes the hazard rate of the components, then $h\left(t\right)$ is increasing (decreasing) if and only if for non-negative integer valued x,t $\frac{S\left(t+x\right)}{S\left(t\right)}$ is decreasing (increasing) in t. Now the result follows easily by representation (5).

The following example gives an application of this theorem.

Example 2.3 Let the components of the system have discrete Weibull distribution with survival function

$$S\left(t\right) = \left(1-\beta\right)^{t^{\alpha}}, t = 0,1,\cdots$$

Then the MRL $H_{n}^{k}\left(t\right)$ of the system is decreasing for $\alpha > 1$ and increasing for $\alpha < 1$.

Theorem 2.4 *Let* $\mathcal{S}_{\infty}$ *and* $\mathcal{S}_{\in}$ *be two* $\left(n-k+1\right)$*-out-of-*n *systems with independent components. Let the components of* $\mathcal{S}_{\infty}$ *and* $\mathcal{S}_{\in}$ *have the probability mass function* $p\left(t\right)$, *and* $q\left(t\right)$, *survival functions* $S_{1}\left(t\right)$, *and* $S_{2}\left(t\right)$; *and hazard rates* $h_{1}\left(t\right)$ *and* $h_{2}\left(t\right)$, *respectively. If, for* $t = 0,1,2,\cdots$, $h_{1}\left(t\right) \le h_{2}\left(t\right)$, *then* $H_{n}^{1k}\left(t\right) \ge H_{n}^{2k}\left(t\right)$, *where* $H_{n}^{1k}\left(t\right)$ *and* $H_{n}^{2k}\left(t\right)$ *denote the mean residual life of* S_{1} *and* S_{2}, *respectively.*

Proof: Note that, for $t = 0,1,2,\cdots$, $h_{1}\left(t\right) \le h_{2}\left(t\right)$ if and only if

$$\frac{S_{1}\left(t+x+1\right)}{S_{1}\left(t\right)} \ge \frac{S_{2}\left(t+x+1\right)}{S_{2}\left(t\right)}, x = 0,1,\cdots.$$

The required result is immediate now from (5).

Khorashadizadeh *et al*. [15] studied discrete variance residual life function for one component.

Using the fact that

$$P\left(T_{k:n} = j \middle| T_{1:n} \ge t\right) = P\left(T_{k:n} \ge j \middle| T_{1:n} \ge t\right) - P\left(T_{k:n} \ge j+1 \middle| T_{1:n} \ge t\right),$$

one can easily prove the following lemma.

Lemma 2.5

$$E\left(T_{k:n} \middle| T_{1:n} \ge t\right) = t + \sum_{j=t+1}^{\infty} \frac{P\left(T_{k:n} \ge j, T_{1:n} \ge t\right)}{P\left(T_{1:n} \ge t\right)} \tag{6}$$

$$E\left(T_{k:n}^{2} - t \middle| T_{1:n} \ge t\right) = t^{2} + \sum_{j=t+1}^{\infty} \left(2j-1\right) \frac{P\left(T_{k:n} \ge j, T_{1:n} \ge t\right)}{P\left(T_{1:n} \ge t\right)} \tag{7}$$

Using this, the variance of the residual life function of $(n-k+1)$-out-of-n system under the condition that all components are working can be derived in terms of $H_n^k(t)$.

Theorem 2.6 *If* $E\left(T_{k:n}^2\right)<\infty$, *the variance residual life function* $\sigma_{k,n}^2(t)$ *and mean residual life function* $H_n^k(t)$ *are related as*

$$\sigma^2(t)=\frac{2}{P\left(T_{1:n}\geq t\right)}\sum_{j=t+1}^{\infty} jP\left(T_{k:n}\geq j, T_{1:n}\geq t\right)-(2t+1)H_n^k(t)-\left(H_n^k(t)\right)^2.$$

Proof:

We have

$$\sigma_{k,n}^2(t)=\operatorname{Var}\left(T_{k:n}-t\middle|T_{1:n}\geq t\right)=E\left(\left(T_{k:n}-t\right)^2\middle|T_{1:n}\geq t\right)-E^2\left(T_{k:n}-t\middle|T_1\geq t\right)$$

$$=E\left(T_{k:n}^2\middle|T_{1:n}\geq t\right)-tE\left(T_k\middle|T_{1:n}\geq t\right)-tE\left(T_{k:n}-t\middle|T_{1:n}\geq t\right)-\left(H_n^k(t)\right)^2.$$

Using Lemma 2.5, we get the required result.

Now, we study the MRL of $(n-k+1)$-out-of-n system under the condition that at least $(n-r+1)$ components of the system are working. That is, we concentrate on $H_n^{r,k}(t)=E\left(T_t^{r,k,n}\right)$, $r=1,2,\cdots,k, k=1,\cdots,n.$

First note that

$$P\left(T_{k:n}-t>x\middle|T_{r:n}\geq t\right)=\frac{P\left(T_{k:n}>t+x, T_{r:n}\geq t\right)}{P\left(T_{r:n}\geq t\right)}$$

$$=\frac{\sum_{i=0}^{r-1}\binom{n}{i}P^i(T<t)P^{n-i}(T\geq t)\sum_{u=0}^{k-i-1}\binom{n-i}{u}\left(\frac{P(T>t+x)}{P(T\geq t)}\right)^{n-i-u}\left(1-\frac{P(T>t+x)}{P(T\geq t)}\right)^u}{\sum_{i=0}^{r-1}\binom{n}{i}P^{n-i}(T\geq t)P^i(T<t)}$$

$$=\frac{\sum_{i=0}^{r-1}\binom{n}{i}\left(\frac{1-S(t)}{S(t)}\right)^i\sum_{u=0}^{k-i-1}\binom{n-i}{u}\left(\frac{S(t+x+1)}{S(t)}\right)^{n-i-u}\left(1-\frac{S(t+x+1)}{S(t)}\right)^u}{\sum_{i=0}^{r-1}\binom{n}{i}\left(\frac{1-S(t)}{S(t)}\right)^i}$$

$$=\sum_{i=0}^{r-1}P_i(t)\sum_{u=0}^{k-i-1}\binom{n-i}{u}\left(\frac{S(t+x+1)}{S(t)}\right)^{n-i-u}\left(1-\frac{S(t+x+1)}{S(t)}\right)^u$$

where

$$P_i(t)=P\left(Z_t=i\middle|Z_t\leq r-1\right)=\frac{\binom{n}{i}\left(1-S(t)\right)^i S^{n-i}(t)}{\sum_{j=0}^{r-1}\binom{n}{j}\left(1-S(t)\right)^j S^{n-j}(t)}$$

and Z_t is a binomial random variable with parameters $\left(n,1-S(t)\right)$.

$$\begin{aligned}H_n^{r,k}(t)&=\sum_{x=0}^{\infty}P\left(T_{k:n}-t>x\middle|T_{r:n}\geq t\right)\\&=\sum_{i=0}^{r-1}P_i(t)\sum_{u=0}^{k-i-1}\sum_{x=0}^{\infty}\binom{n-i}{u}\left(\frac{S(t+x+1)}{S(t)}\right)^{n-i-u}\left(1-\frac{S(t+x+1)}{S(t)}\right)^u\\&=\sum_{i=0}^{r-1}P_i(t)H_{n-i}^{k-i}(t)\end{aligned}\tag{8}$$

Equation (8) shows that $H_n^{r,k}(t)$ is a convex combination of $H_{n-i}^{k-i}(t)$, $i=0,\cdots,r$. Note that $H_{n-i}^{k-i}(t)$ is given by (2).

Example 2.7 Let $T_1,\cdots,T_n$ denote the lifetimes of n independent components which are connected in a $(n-k+1)$-out-of-n system. Let T_i be distributed as discrete Weibull (α,β) with

$$p(t)=P(T_i=t)=(1-\beta)^{t^\alpha}-(1-\beta)^{(t+1)^\alpha},t=0,1,2,\cdots,0<\beta<1,\alpha>0,$$

and

$$S(t)=P(T_i\ge t)=(1-\beta)^{t^\alpha}.$$

Then

$$P_i(t)=\frac{\binom{n}{i}\left((1-\beta)^{-t^\alpha}-1\right)^i}{\sum_{j=0}^{r-1}\binom{n}{j}\left((1-\beta)^{-t^\alpha}-1\right)^j},$$

and

$$H_{n-i}^{k-i}(t)=\sum_{u=0}^{k-i-1}\binom{n-i}{u}\sum_{x=0}^{\infty}\left((1-\beta)^{(t+x+1)^\alpha-t^\alpha}\right)^{n-i-u}\left(1-(1-\beta)^{(t+x+1)^\alpha-t^\alpha}\right)^i.$$

Hence, the MRL $H_n^{r,k}(t)$ is given by (8). **Figures 1** and **2** show the graphs of $H_n^{r,k}(t)$ in example 2.7 when $\alpha=\frac{3}{2}$, $\beta=\frac{1}{2}$, $n=7$ for different values of r and k.

Remark 2.8 Let us consider again the condition random variable $T_t^{r,k,n}=T_{k:n}-t\,|\,T_{r:n}\ge t$ for which the survival function is given by (2). The representation (2) shows that $T_t^{1,k,n}$ is in fact the k, the order statistics

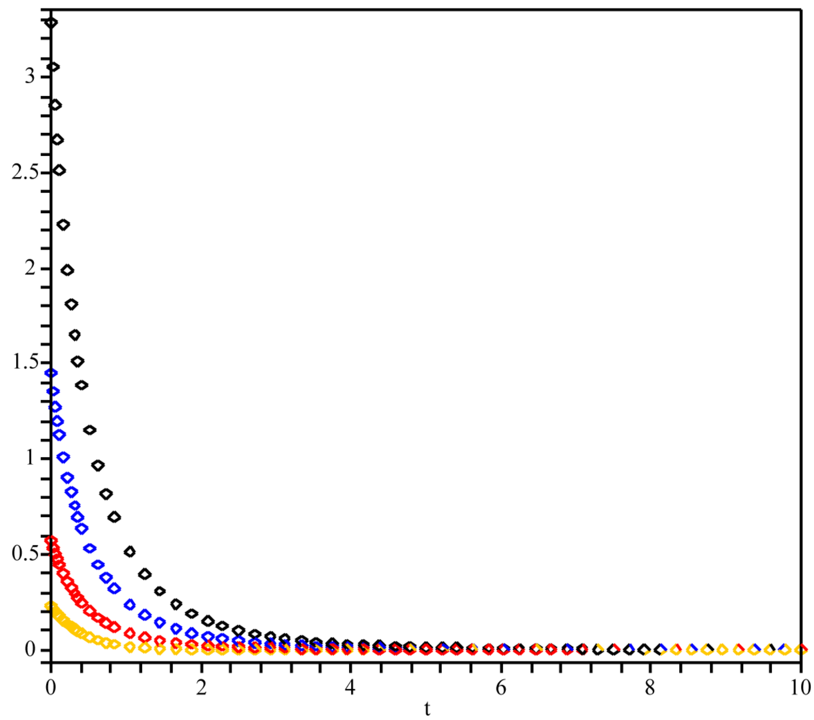

Figure 1. **The MRL $H_n^{r,k}(t)$ of the system for the discrete weibull $\left(\frac{3}{2},\frac{1}{2}\right)$ distribution with $n=7$, $k=5$, and $r=2,3,4,5$ from the top respectively.**

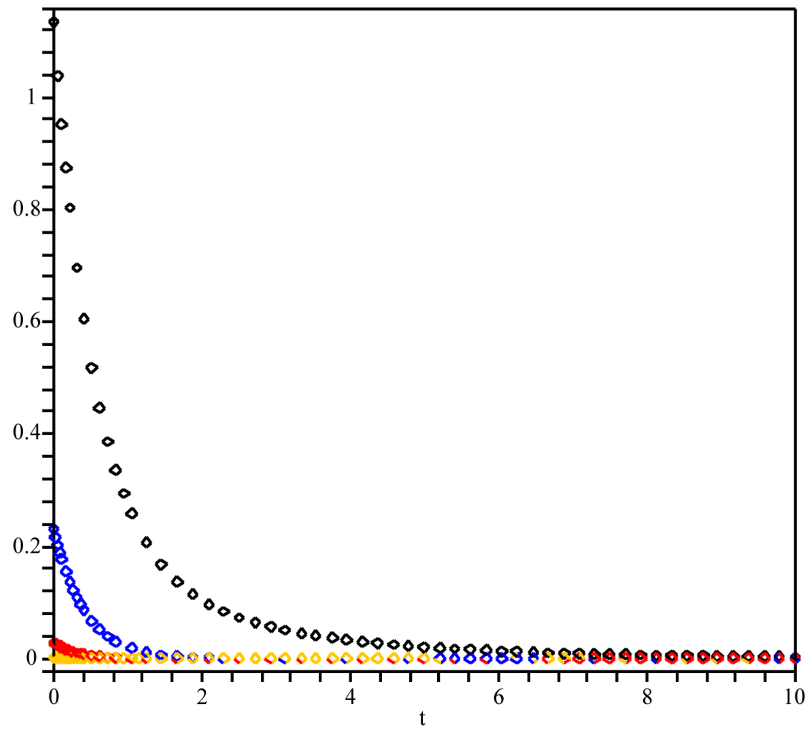

Figure 2. The MRL $H_n^{r,k}(t)$ of the system for the discrete weibull $\left(\frac{3}{2},\frac{1}{2}\right)$ distribution with $n=7$, $r=3$, and $k=3,4,5,6$ from the top respectively.

form of a distribution with survival function $\frac{S(t+x+1)}{S(t)}$. Hence using the result of David and Nagaraje [8], one can write

$$P\left(T_t^{1,k,n} > x\right) = \sum_{j=n-k+1}^{n} (-1)^{j-n+k+1} \binom{j-1}{n-k} \binom{n}{j} \left(\frac{S(t+x+1)}{S(t)}\right)^j .$$

Hence

$$H_n^k(t) = \sum_{j=n-k+1}^{n} (-1)^{j-n+k-1} \binom{j-1}{n-k} \binom{n}{j} H_j^1(t)$$

and

$$H_n^{r,k}(t) = \sum_{i=0}^{r-1} P_i(t) \sum_{j=n-k+1}^{n-i} (-1)^{j-n+k-1} \binom{j-1}{n-k} \binom{n}{j} H_j^1(t). \tag{9}$$

This indicates the MRL $H_n^{r,k}(t)$ can be expressed in terms of simpler MRL $H_j^1(t)$ which is in fact the MRL of series systems.

The following theorem gives bounds for $H_n^{r,k}(t)$.

Theorem 2.9 *It is always true that*

$$H_{n-r+1}^{k-n+1}(t) \le H_n^{r,k}(t) \le H_n^k(t)$$

Proof: The proof is similar to the proof of Theorem 2.3 of [4] and hence is omitted.

The next theorem proves that when the parent distribution has increased hazard rate, $H_n^{r,k}(t)$ increases in terms of time.

Theorem 2.10 *If* $h(t)$ *is increasing in* t, *then* $H_n^{r,k}(t)$ *is decreasing in* t.
Proof: In order to prove the result, we need to show that, for r,k and n fixed,

$$H_n^{r,k}(t) - H_n^{r,k}(t+1) \ge 0.$$

We have, from (8), after some algebra

$$H_n^{r,k}(t) - H_n^{r,k}(t+1) = \sum_{i=0}^{r-1} P_i(t) H_{n-i}^{k-i}(t) - \sum_{i=0}^{r-1} P_i(t+1) H_n^{k-i}(t+1)$$
$$= \sum_{i=0}^{r-1} P_i(t)\left(H_{n-i}^{k-i}(t) - H_{n-i}^{k-i}(t+1)\right) + \sum_{i=0}^{r-1} H_{n-i}^{k-i}(t+1)\left(P_i(t) - P_i(t+1)\right)$$

But the first term in the above equality is positive by Theorem 2.2. Hence we just need to prove that the second term in the above equality is positive. Assume that $\frac{1-S(t)}{S(t)} = \phi(t)$ and note that $\phi(t)$ is an increasing function of t. Then

$$\sum_{i=0}^{r-1} H_{n-i}^{k-i}(t+1)\left(\frac{\binom{n}{j}\phi^i(t)}{\sum_{j=0}^{r-1}\binom{n}{j}\phi^j(t)} - \frac{\binom{n}{i}\phi^i(t+1)}{\sum_{j=0}^{r-1}\binom{n}{j}\phi^j(t+1)}\right) = \frac{\sum_{i=0}^{r-1}\sum_{j=0}^{r-1}\binom{n}{j}\binom{n}{i}\phi^i(t)\phi^j(t+1)\left(H_{n-i}^{k-i}(t+1) - H_{n-j}^{k-j}(t+1)\right)}{\left(\sum_{j=0}^{r-1}\binom{n}{j}\phi^j(t)\right)\left(\sum_{j=0}^{r-1}\binom{n}{j}\phi^j(t+1)\right)}.$$

After some algebraic manipulations, one can show that the numerator of the expression is equal to

$$\sum_{i=0}^{r-2}\sum_{j=i+1}^{r-1}\binom{n}{i}\binom{n}{j}\left(\phi^i(t)\phi^j(t+1) - \phi^j(t)\phi^i(t+1)\right)\left(H_{n-i}^{k-i}(t+1) - H_{n-j}^{k-j}(t+1)\right). \tag{10}$$

It can be easily shown that for $j > i$, $H_{n-i}^{k-i}(t+1) - H_{n-j}^{k-j}(t+1) > 0$ (see, [2,3]). On the other hand, as $\phi(t)$ is an increasing function of t, we have $\left(\phi^i(t)\phi^j(t+1) - \phi^j(t)\phi^i(t+1)\right) \ge 0$. This implies that the expression in (10) is non-negative and hence the proof is complete.

Remark 2.11 As it was already mentioned for a system with decreasing failure rate components, $H_n^k(t)$ is increasing in time. This result, however, is not generally true for MRL $H_n^{r,k}(t)$. **Figures 3** and **4** show the graphs of $h(t)$ and $H_n^{r,k}(t)$ in Example 2.7. As the graphs show that $h(t)$ is a decreasing function of time, however, $H_n^{r,k}(t)$ is an increasing function of t for a period of time and then starts to decrease.

Remark 2.12 In the following, we show that $P_i(t)$ has its own interesting interpretation. In fact, under the condition that the system is working at time t, $P_i(t)$ shows the probability that there is exactly i component failure in the system. The mentioned conditional probability can be written as

$$P\left(T_{i:n} < t < T_{i+1:n} \middle| T_{r:n} \ge t\right) = P\left(T_{i:n} < t \middle| T_{r:n} \ge t\right) - P\left(T_{i+1:n} < t \middle| T_{r:n} \ge t\right)$$
$$= \frac{P\left(T_{i:n} < t \le T_{r:n}\right)}{P\left(T_{r:n} \ge t\right)} - \frac{P\left(T_{i+1:n} < t \le T_{r:n}\right)}{P\left(T_{r:n} \ge t\right)}$$
$$= \frac{\sum_{j=i}^{r-1}\binom{n}{j}(1-S(t))^j S^{n-j}(t)}{\sum_{j=0}^{r-1}\binom{n}{j}(1-S(t))^j S^{n-j}(t)} - \frac{\sum_{j=i+1}^{r-1}\binom{n}{j}(1-S(t))^j S^{n-j}(t)}{\sum_{j=0}^{r-1}\binom{n}{j}(1-S(t))^j S^{n-j}(t)}$$
$$= \frac{\binom{n}{j}(1-S(t))^i S^{n-i}(t)}{\sum_{j=0}^{r-1}\binom{n}{j}(1-S(t))^j S^{n-j}(t)} = \frac{\binom{n}{j}\phi^i(t)}{\sum_{j=0}^{r-1}\binom{n}{j}\phi^j(t)}$$
$$= P_i(t), i = 0, \cdots, r-1$$

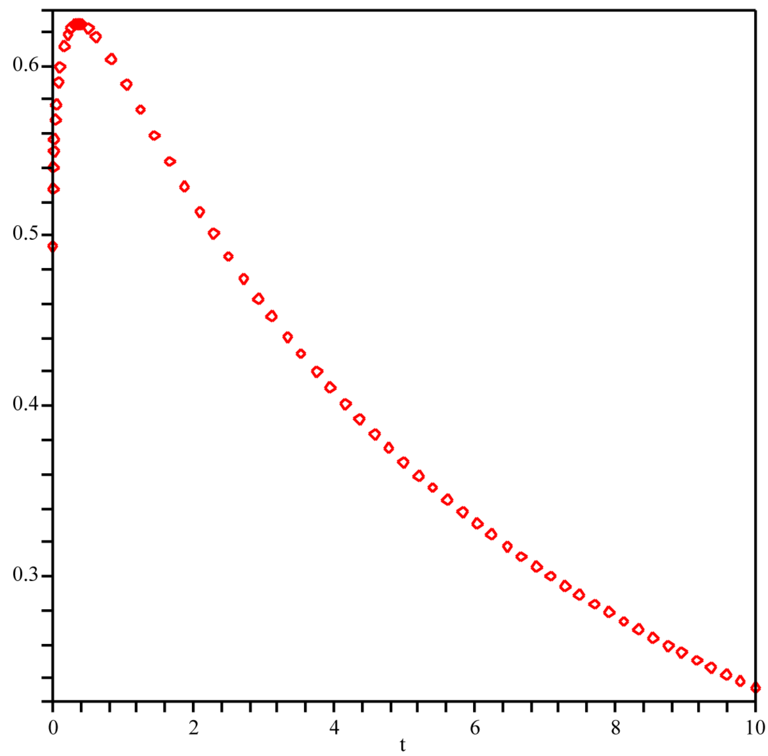

Figure 3. **The MRL $H_n^{r,k}(t)$ of the system for the discrete weibull $\left(\frac{1}{2},\frac{1}{2}\right)$ distribution with $n=7$, $k=5$, and $r=5$.**

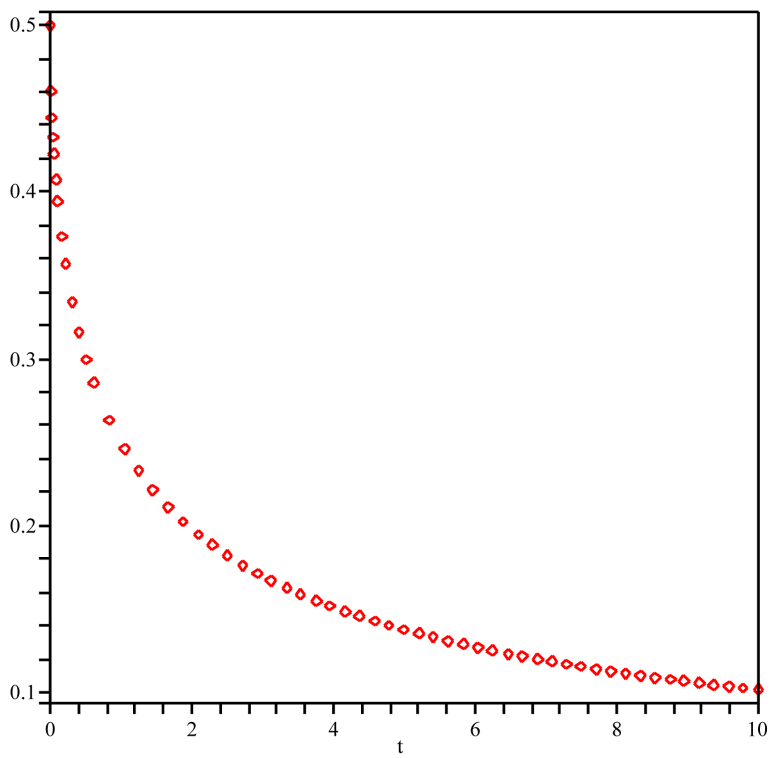

Figure 4. **The failure rate of the system for the discrete weibull $\left(\frac{1}{2},\frac{1}{2}\right)$ distribution.**

where $\phi(t)=\frac{1-S(t)}{S(t)}$ for t such that $S(t)>0$ shows the odds of the event that a component has a lifetime less than t. Also in the following, we study some properties of $P_i(t)$.

Theorem 2.13 *For* $i=0$ $P_i(t)$ *is decreasing function of t and for* $i=r-1$, *it is increasing function of t. Also, for* $0\le i\le r-1$

$$\lim_{t\to 0} P_i(t)=\begin{cases}1 & i=0\\ 0 & i\ne 0\end{cases}$$

$$\lim_{t\to \infty} P_i(t)=\begin{cases}1 & i=r-1\\ 0 & i\ne r-1\end{cases}$$

Proof: We have

$$P_0(t)=\frac{1}{\sum_{j=0}^{r-1}\binom{n}{j}\phi^j(t)} \tag{11}$$

which is obviously a decreasing function of t ($\phi(t)$ is a increasing function) since $\lim_{t\to 0}\phi(t)=0$ and $\lim_{t\to\infty}\phi(t)=1$. From (11), we easily conclude that $\lim_{t\to 0}P_0(t)=1$ and $\lim_{t\to\infty}P_0(t)=0$.

$$P_{r-1}(t)=\frac{\binom{n}{r-1}}{\sum_{j=0}^{r-1}\binom{n}{j}\phi^{j-r+1}(t)}.$$

In this case, it is easily seen that $P_{r-1}(t)$ is an increasing function of t, $\lim_{t\to 0}P_{r-1}(t)=0$ and $\lim_{t\to\infty}P_{r-1}(t)=1$.

Theorem 2.14 *The survival function* $S(t)$ *can be uniquely determined by* $P_i(t)$ *and* $P_{i+1}(t)$, $i=0,1,\cdots,r-1$ *as follows*:

$$S(t)=\frac{(n-i)P_i(t)}{(n-i)P_i(t)+(i+1)P_{i+1}(t)}. \tag{12}$$

Proof:
The result easily follows from the fact that for $i=0,\cdots,r-1$,

$$\frac{P_{i+1}(t)}{P_i(t)}=\frac{n-i}{i+1}\cdot\frac{1-S(t)}{S(t)},$$

which gives (12).

Consider the vector $\boldsymbol{P}(t)=\left(P_0(t),\cdots,P_{r-1}(t)\right)$. Obviously, $\boldsymbol{P}(t)$ is a probability vector. we can then prove the following theorem.

Theorem 2.15 *For all* $0\le t_1\le t_2$,

$$\boldsymbol{P}(t_1)\le_{s_t}\boldsymbol{P}(t_2).$$

Proof: In order to prove the result, we need to show that for $j=0,\cdots,r-1$,

$$\sum_{i=j}^{r-1}P_i(t_1)\le\sum_{i=j}^{r-1}P_i(t_2),\ j=0,\cdots,r-1$$

or equivalently

$$\frac{\sum_{i=j}^{r-1}\binom{n}{i}\phi^{i}(t_1)}{\sum_{k=0}^{r-1}\binom{n}{k}\phi^{k}(t_1)} \le \frac{\sum_{i=j}^{r-1}\binom{n}{i}\phi^{i}(t_2)}{\sum_{k=0}^{r-1}\binom{n}{k}\phi^{k}(t_2)}.$$

This is equivalent to show that

$$\frac{\sum_{k=0}^{j-1}\binom{n}{k}\phi^{k}(t_2)}{\sum_{i=j}^{r-1}\binom{n}{i}\phi^{i}(t_2)} \le \frac{\sum_{k=0}^{j-1}\binom{n}{k}\phi^{k}(t_1)}{\sum_{i=j}^{r-1}\binom{n}{i}\phi^{i}(t_1)}$$

or

$$\sum_{k=0}^{j-1}\sum_{l=j}^{r-1}\binom{n}{k}\binom{n}{l}\left(\phi^{k}(t_2)\phi^{l}(t_1)-\phi^{k}(t_1)\phi^{l}(t_2)\right)\le 0. \tag{13}$$

But, as $\phi(t)$ is increasing in t, the bracket in the summations, for $k<l$, is always negative. Hence the inequality in (13) is valid. This completes the proof of the theorem.

Theorem 2.16 *Consider two* $(n-k+1)$*-out-of-*n *systems. Assume that the components of the systems have independent lifetimes, with survival function* $S_1(t)$ *and* $S_2(t)$*, respectively and odds functions* $\phi_1(t)$ *and* $\phi_2(t)$*, respectively. If for all* t, $S_1(t)\le S_2(t)$*, then* $\boldsymbol{P}_1(t)\ge_{s_t}\boldsymbol{P}_2(t)$.

Proof: Asadi & Berred [16] proved that $\eta_{r,n}^{i}(t)=\frac{\sum_{k=i}^{r-1}\binom{n}{k}t^{k}}{\sum_{j=0}^{r-1}\binom{n}{j}t^{i}}$ for fixed i and n is an increasing function of t.

The assumption that $S_1(t)\le S_2(t)$ implies $\phi_1(t)\ge\phi_2(t)$, then

$$\eta_{r,n}^{i}(\phi_1(t))\ge\eta_{r,n}^{i}(\phi_2(t))$$

which is equivalent to say that for all $i=0,\cdots,r-1$ and all t,

$$\frac{\sum_{k=i}^{r-1}\binom{n}{k}\phi_1^{k}(t)}{\sum_{j=0}^{r-1}\binom{n}{j}\phi_1^{j}(t)} \ge \frac{\sum_{k=i}^{r-1}\binom{n}{k}\phi_2^{k}(t)}{\sum_{j=0}^{r-1}\binom{n}{j}\phi_2^{j}(t)},\left(P_{01}(t)\ge_{st}P_{02}(t)\right).$$

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