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The Quantum $\mathfrak{sl}_2$-Invariant of a Family of Knots

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ABSTRACT

We give a general formula of the quantum $\mathfrak{sl}_2$-invariant of a family of braid knots. To compute the quantum invariant of the links we use the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$ in its standard two-dimensional representation. We also recover the Jones polynomial of these knots as a special case of this quantum invariant.

KEYWORDS

Quantum Invariant; Jones Polynomial; Braid Knot

1. Introduction

The discovery of the Jones polynomial inspired many people to search for other skein relations compatible with Reidemeister moves and thus defined knot polynomials. This led to the introduction of the HOMFLY and Kauffmans polynomials. It soon became clear that all these polynomials are the first members of a vast family of knot invariants called *quantum invariants*.

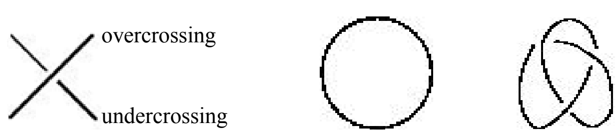
The original idea of quantum invariants was proposed by E. Witten in [1]. Witten's approach coming from physics was not completely justified from the mathematical viewpoint. The first mathematically definition of quantum invariants of links and 3-manifolds was given by Reshetikhin and Turaev [2,3], who used in their construction the notion of *quantum groups* introduced shortly before that by V. Drinfeld in [4] (see also [5]) and M. Jimbo in [6]. In fact, a quantum group is not a group at all. Instead, it is a family of algebras, more precisely, of Hopf algebras, depending on a complex parameter q and satisfying certain axioms. The quantum group $U_q\mathfrak{g}$ of a semisimple Lie algebra $\mathfrak{g}$ is a remarkable deformation of the universal enveloping algebra of $\mathfrak{g}$ (corresponding to the value $q=1$) in the class of Hopf algebras.

This paper is organized as follows: In Section 2, we give the basic ideas about knots, tangles, the Jones polynomial, Lie algebra representations, and construction of quantum invariants. In Section 3, we present the main result along with its specialization to the Jones polynomial.

2. Preliminary Notions

2.1. Basic Concepts of Knots

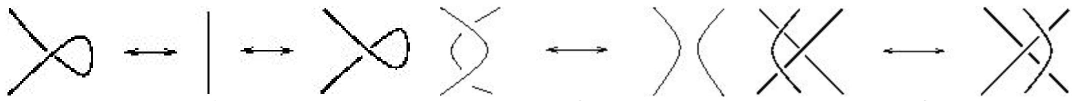
A *knot* is a circle embedded in $\mathbb{R}^3$. Knots are usually studied via projecting them on a plan; a projection with extra information of *overcrossing* and *undercrossing* is called the *knot diagram*.



A crossing　　trivial knot　　trefoil knot

Two knots are called *isotopic* if one of them can be transformed to the other by a diffeomorphism of the ambient space $\mathbb{R}^3$ onto itself. A fundamental result about the isotopic knot diagrams is:

Two unoriented knots K_1 and K_2 are equivalent if and only if a diagram of K_1 can be transformed into a diagram of K_2 by a finite sequence of ambient isotopies of the plane and three local (*Reidemeister*) *moves*:



R1 R2 R3

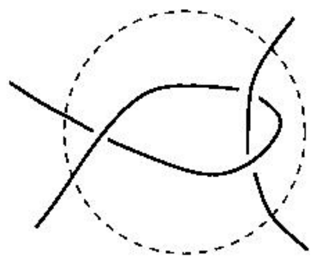
The set of all knots that are equivalent to a knot K is called a *class* of K. By a knot K we shall always mean a class of the knot K.

The main question of knot theory is *Which two links are equivalent and which are not*? To address this question one needs a *knot invariant*, a function that gives one value on all knots that belong to a single class and gives different values (but not always) on knots that belong to different classes. The present work is concerned with this question.

2.2. Tangles

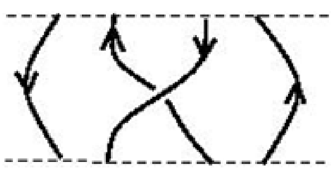
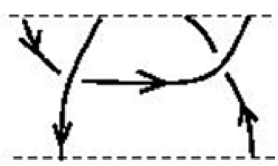
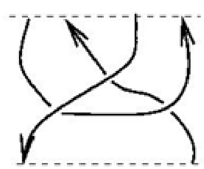
A tangle is a generalization of a knot which at the same time is simpler and more complicated than a knot. On one hand, knots are a particular case of tangles, on the other hand, knots can be represented as combinations of (simple) tangles.

A *tangle* in a knot projection is a region in the projection plane surrounded by a circle such that the knot crosses the circle exactly at four places.

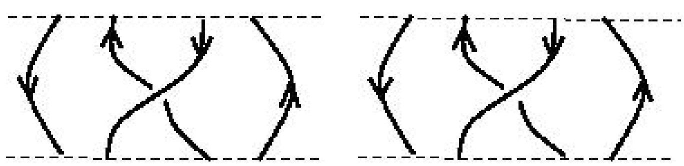


A tangle

The following two operations are defined on tangles: When the bottom of a tangle T_1 coincides with the top of another tangle T_2, the *product* $T_1 \cdot T_2$ is defined by putting T_1 on top of T_2. (For oriented tangles we also require the consistency of orientations.)

$T_1=$  $T_2=$  $T_1 \cdot T_1=$ 

The second operation, *tensor product*, is defined by placing one tangle next to the other tangle (of the same height).

$T_1 \otimes T_2 =$ 

2.3. The Jones Polynomial

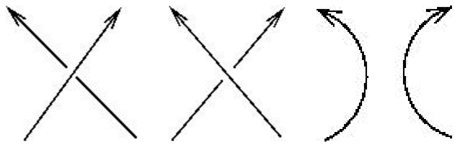
In 1985, V. F. R. Jones revolutionized knot theory by defining the Jones polynomial as a knot invariant via Von

Neumann algebras [7]. However, in 1987 L. H. Kauffman introduced in [8] a state-sum model construction of the Jones polynomial that was purely combinatorial and remarkably simple; we follow this approach.

Definition 1 [7-9] *The Jones polynomial* $V_K(t)$ *of an oriented link* K *is a Laurent polynomial in the variable* $\sqrt{t}$ *satisfying the skein relation*

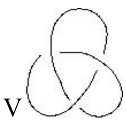
$$t^{-1}V_{K_+}(t) - tV_{K_-}(t) = \left(t^{1/2} - t^{-1/2}\right)V_{K_0}(t),$$

and that the value of the unknot is 1. Here K_+, K_-, and K_0 are three oriented links having diagrams that are isotopic everywhere except at one crossing where they differ as in the figure below:



K_+ K_- K_0

For instance, it is easy to verify that the Jones polynomial of the left-handed trefoil knot (which is denoted by 3_1 in the knot table) is

V  $= -t^{-4} + t^{-3} + t^{-1}$.

2.4. Lie Algebra Representations

Let $\mathfrak{g}$ be a semisimple Lie algebra and let V be its finite-dimensional representation. One can view V as a representation of the universal enveloping algebra $U(\mathfrak{g})$. It is remarkable that this representation can also be deformed with parameter q to a representation of the quantum group $U_q\mathfrak{g}$. The vector space V remains the same, but the action now depends on q. For a generic value of q all irreducible representations of $U_q\mathfrak{g}$ can be obtained in this way. However, when q is a root of unity the representation theory is different and resembles the representation theory of q in finite characteristic. It can be used to derive quantum invariants of 3-manifolds. For the purposes of knot theory it is enough to use generic values of q, that is, those which are not roots of unity.

An important property of quantum groups is that every representation gives rise to a solution R of the quantum Yang-Baxter equation

$$(R\otimes id_V)\circ(id_V\otimes R)\circ(R\otimes id_V) = (id_V\otimes R)\circ(R\otimes id_V)\circ(id_V\otimes R)$$

where R (the R-matrix) is an invertible linear operator $R: V\otimes V \to V\otimes V$, and both sides of the equation are understood as linear transformations $V\otimes V\otimes V \to V\otimes V\otimes V$.

In case of Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$ and its standard two-dimensional representation, the R-matrix has the form

$$R = \begin{cases} e_1\otimes e_1 \mapsto q^{1/4}e_1\otimes e_1 \\ e_1\otimes e_2 \mapsto q^{-1/4}e_2\otimes e_1 \\ e_2\otimes e_1 \mapsto q^{-1/4}e_1\otimes e_2 + \left(q^{1/4} - q^{-3/4}\right)e_2\otimes e_1 \\ e_2\otimes e_2 \mapsto q^{1/4}e_2\otimes e_2 \end{cases}$$

for an appropriate basis $\{e_1, e_2\}$ of the space V. The inverse of R is

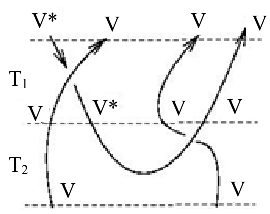
$$R^{-1} = \begin{cases} e_1\otimes e_1 \mapsto q^{-1/4}e_1\otimes e_1 \\ e_1\otimes e_2 \mapsto q^{1/4}e_2\otimes e_1 + \left(-q^{3/4} + q^{-1/4}\right)e_1\otimes e_2 \\ e_2\otimes e_1 \mapsto q^{1/4}e_1\otimes e_2 \\ e_2\otimes e_2 \mapsto q^{-1/4}e_2\otimes e_2 \end{cases}$$

2.5. Construction of Quantum Invariants

The general procedure of constructing quantum invariants is as follows (see details in [10]). Consider a knot diagram in the plane and take a generic horizontal line. To each intersection point of the line with the diagram assign either the representation space V or its dual V^* depending on whether the orientation of the knot at this intersection is directed upwards or downwards. Then take the tensor product of all such spaces over the whole horizontal line. If the knot diagram does not intersect the line, then the corresponding vector space is the ground field $\mathbb{C}$.

A portion of a knot diagram between two such horizontal lines represents a tangle T. We assume that this tangle is framed by the black board framing. With T we associate a linear transformation $\theta^{fr}(T)$ from the vector space corresponding to the bottom of T to the vector space corresponding to the top of T. The following three properties hold for the linear transformation $\theta^{fr}(T)$:

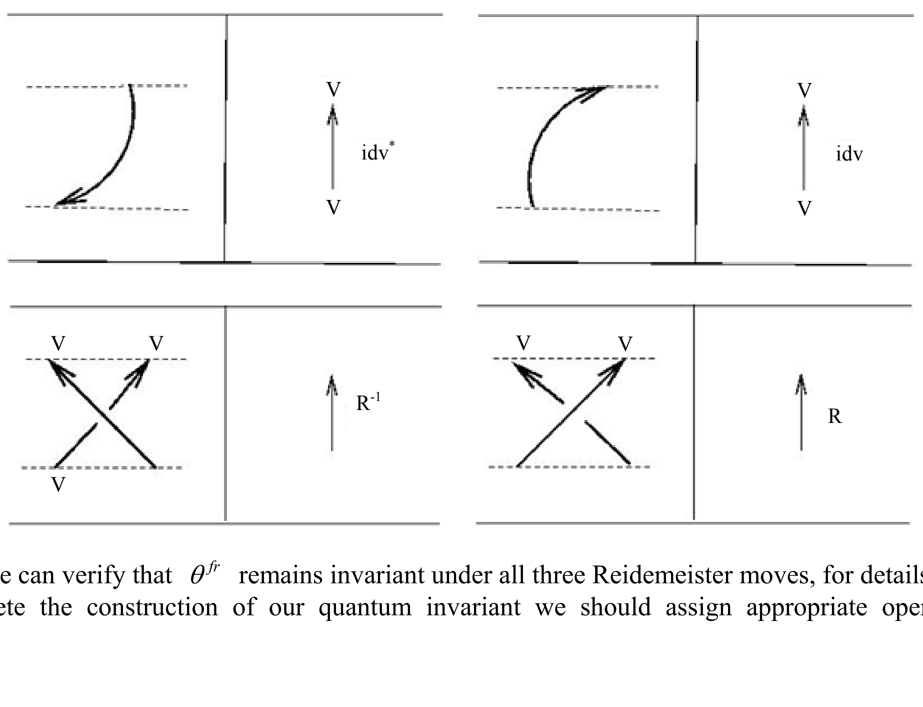
- $\theta^{fr}(T)$ is an invariant of the isotopy class of the framed tangle T;
- $\theta^{fr}(T_1 \cdot T_2) = \theta^{fr}(T_1) \circ \theta^{fr}(T_2)$;
- $\theta^{fr}(T_1 \otimes T_2) = \theta^{fr}(T_1) \otimes \theta^{fr}(T_2)$.



$$\begin{array}{c} V^* \otimes V \otimes V \otimes V \\ \theta^{fr}(T_1)\uparrow \\ V \otimes V^* \otimes V \otimes V \\ \theta^{fr}(T_2)\uparrow \\ V \otimes V \end{array}$$

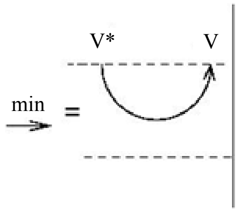
Now we can define a knot invariant $\theta^{fr}(K)$ regarding the knot K as a tangle between the two lines below and above K. In this case $\theta^{fr}(K)$ would be a linear transformation from C to C. Since our linear transformations depend on the parameter q, this number is actually a function of q.

Because of the multiplicity property $\theta^{fr}(T_1 \cdot T_2) = \theta^{fr}(T_1) \circ \theta^{fr}(T_2)$ it is enough to define $\theta^{fr}(T)$ only for elementary tangles T such as a crossing, a minimum or a maximum point. This is precisely where quantum groups come in. Given a quantum group $U_q g$ and its finite-dimensional representation V, one can associate certain linear transformations with elementary tangles in a way consistent with the Turaev oriented moves [11]. The R-matrix appears here as the linear transformation corresponding to a positive crossing. Of course, for a trivial tangle consisting of a single string connecting the top and bottom, the corresponding linear operator should be the identity transformation. So we have the following correspondence valid for all quantum groups:

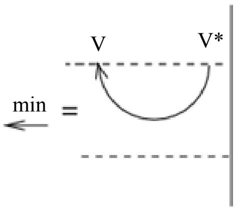


Using this one can verify that θ^{fr} remains invariant under all three Reidemeister moves, for details see [11].

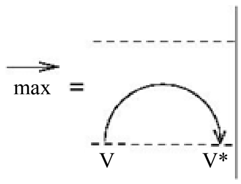
To complete the construction of our quantum invariant we should assign appropriate operators to the minimum and maximum points. These depend on all the data involved: the quantum group, the representation and the R-matrix. For the quantum group $U_q\mathfrak{sl}_2$, its standard two-dimensional representation V and the R-matrix, these operators are:



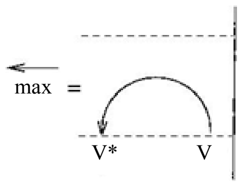
$$\begin{array}{cc} V^*\otimes V & q^{-1/2}e^1\otimes e_1+q^{1/2}e^2\otimes e_2 \\ \uparrow & \uparrow \\ \mathbb{C} & 1 \end{array}$$



$$\begin{array}{cc} V\otimes V^* & e_1\otimes e^1+e_2\otimes e^2 \\ \uparrow & \uparrow \\ \mathbb{C} & 1 \end{array}$$



$$\begin{array}{ccccc} \mathbb{C} & q^{1/2} & 0 & 0 & q^{-1/2} \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ V\otimes V^* & e_1\otimes e^1 & e_1\otimes e^2 & e_2\otimes e^1 & e_2\otimes e^2 \end{array}$$

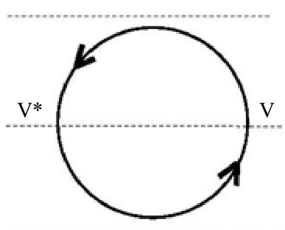


$$\begin{array}{ccccc} \mathbb{C} & 1 & 0 & 0 & 1 \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ V^*\otimes V & e^1\otimes e_1 & e^1\otimes e_2 & e^2\otimes e_1 & e^2\otimes e_2 \end{array}$$

where $\{e^1,e^2\}$ is the basis of V^*, dual to the basis $\{e_1,e_2\}$ of V.

In the following example we compute the quantum $\mathfrak{sl}_2$-invariant for the unknot.

Example 1 *Let us compute the sl_2-quantum invariant of the unknot. Represent the unknot as a product of two tangles and compute the composition of the corresponding transformations.*



$$\begin{array}{cc} \mathbb{C} & \underbrace{q^{-1/2}}+\underbrace{q^{1/2}} \\ \uparrow & \uparrow \\ V^*\otimes V & \overbrace{q^{-1/2}e^1\otimes e_1}+\overbrace{q^{1/2}e^2\otimes e_2} \\ \uparrow & \uparrow \\ \mathbb{C} & 1 \end{array}$$

So, $\theta^{fr}(\text{unknot})=q^{1/2}+q^{-1/2}$. Therefore, in order to normalize our invariant so that its value on the unknot is equal to 1, we must divide $\theta^{fr}(\cdot)$ by $q^{1/2}+q^{-1/2}$, and denote this normalized invariant by $\tilde{\theta}^{fr}(\cdot)$. (We shall write the precise formula for $\tilde{\theta}^{fr}(\cdot)$ in the main result.)

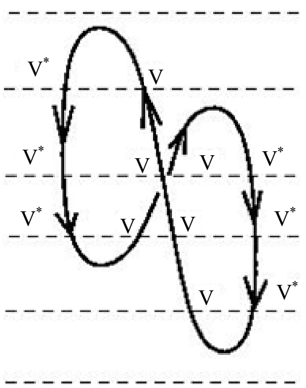
3. Main Result

Here we give the general formulas of the quantum $\mathfrak{sl}_2$-invariants of the braid knot $\widehat{x_1^n}$ for odd n.

Proposition 1 *The quantum invariant of* $\widehat{x_1^n}$, *when* n *is odd*, *is*

$$\tilde{\theta}\left(\widehat{x_1^n}\right) = -q^{\frac{3n-1}{2}} + q^{\frac{3n-3}{2}} - q^{\frac{3n-5}{2}} + \cdots + q^{\frac{n+3}{2}} + q^{\frac{n-1}{2}}. \tag{1.1}$$

Proof 1 *We prove it by induction on* n.
For $n = 1$, we receive the following braid knot along with its tensor product.



$$\begin{array}{c} \mathbb{C} \\ \uparrow \\ V^* \otimes V \\ \uparrow \\ V^* \otimes V \otimes V \otimes V^* \\ \uparrow id_{V^*} \otimes R^{-1} \otimes id_{V^*} \\ V^* \otimes V \otimes V \otimes V^* \\ \uparrow \\ V \otimes V^* \\ \uparrow \\ \mathbb{C} \end{array}$$

Note that the map $\mathbb{C} \to V \otimes V^*$ sends $1 \in \mathbb{C}$ into the tensor $e_1 \otimes e^1 + e_2 \otimes e^2$. Also, the map $V \otimes V^* \to V^* \otimes V \otimes V \otimes V^*$ sends $e_1 \otimes e^1 + e_2 \otimes e^2$ into the tensor

$$q^{-1/2} e^1 \otimes e_1 \otimes e_1 \otimes e^1 + q^{-1/2} e^1 \otimes e_1 \otimes e_2 \otimes e^2 + q^{1/2} e^2 \otimes e_2 \otimes e_1 \otimes e^1 + q^{1/2} e^2 \otimes e_2 \otimes e_2 \otimes e^2.$$

Now applying R^{-1} to each middle factor, we get

$$\begin{aligned} & q^{-1/2} e^1 \otimes \left(q^{-1/4} e_1 \otimes e_1\right) \otimes e^1 + q^{-1/2} e^1 \otimes \left(q^{1/4} e_2 \otimes e_1 + \left(-q^{3/4} + q^{-1/4}\right) e_1 \otimes e_2\right) \otimes e^2 \\ & + q^{1/2} e^2 \otimes \left(q^{1/4} e_1 \otimes e_2\right) \otimes e^1 + q^{1/2} e^2 \otimes \left(q^{-1/4} e_2 \otimes e_2\right) \otimes e^2 \\ & = q^{-1/2} e^1 \otimes e_1 \otimes q^{-1/4} e_1 \otimes e^1 + q^{-1/2} e^1 \otimes e_2 \otimes q^{1/4} e_1 \otimes e^2 \\ & + q^{-1/2} e^1 \otimes e_1 \otimes \left(-q^{3/4} + q^{-1/4}\right) e_2 \otimes e^2 + q^{1/2} e^2 \otimes e_1 \otimes q^{1/4} e_2 \otimes e^1 + q^{1/2} e^2 \otimes e_2 \otimes q^{-1/4} e_2 \otimes e^2 \end{aligned}$$

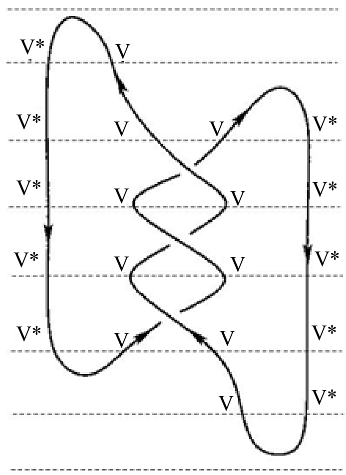
Finally, the two maps at the top contract the whole tensor into the linear transformation

$$\theta^{fr}\left(\widehat{x_1}\right) = q^{-1/2} q^{-1/4} q^{1/2} + q^{-1/2}\left(-q^{3/4} + q^{-1/4}\right) q^{-1/2} + q^{1/2} q^{-1/4} q^{-1/2} = q^{-1/4} + q^{-5/4}.$$

Hence the unframed normalized $\mathfrak{sl}_2$-quantum invariant of $\widehat{x_1}$ is

$$\begin{aligned} \tilde{\theta}\left(\widehat{x_1}\right) &= q^{\frac{-3wr(K)}{4}} \left[\frac{\theta^{fr}\left(\widehat{x_1}\right)}{q^{1/2} + q^{-1/2}}\right] = q^{\frac{-3/2(-1)}{2}} \left[q^{1/4} + q^{-3/4} - q^{1/4}\right] \\ &= q^{3/4}\left[q^{1/4} + q^{-3/4} - q^{1/4}\right] = q + q^0 - q = 1. \end{aligned}$$

To get a clear picture, we also compute the quantum invariant of the knots $\widehat{x_1^3}$ (which is actually the left trefoil) and $\widehat{x_1^5}$. First of all, we proceed for $\widehat{x_1^3}$:



$$\begin{array}{l}
\mathbb{C} \\
\uparrow \\
V^{*}\otimes V \\
\uparrow \\
V^{*}\otimes V\otimes V\otimes V^{*} \\
\uparrow id_{V^{*}}\otimes R^{-1}\otimes id_{V^{*}} \\
V^{*}\otimes V\otimes V\otimes V^{*} \\
\uparrow id_{V^{*}}\otimes R^{-1}\otimes id_{V^{*}} \\
V^{*}\otimes V\otimes V\otimes V^{*} \\
\uparrow \\
V\otimes V^{*} \\
\uparrow \\
\mathbb{C}
\end{array}$$

The map at the bottom sends $1 \in \mathbb{C}$ into the tensor $e_1 \otimes e^1 + e_2 \otimes e^2$.
Now the map $V \otimes V^* \to V^* \otimes V \otimes V \otimes V^*$ sends the above tensor into the tensor

$$q^{-1/2}e^1 \otimes e_1 \otimes e_1 \otimes e^1 + q^{-1/2}e^1 \otimes e_1 \otimes e_2 \otimes e^2 + q^{1/2}e^2 \otimes e_2 \otimes e_1 \otimes e^1 + q^{1/2}e^2 \otimes e_2 \otimes e_2 \otimes e^2.$$

Then applying R^{-3} to two tensor factors in the middle we get

$$\begin{aligned}
&q^{-1/2}e^1 \otimes \left(q^{-3/4}e_1 \otimes e_1\right) \otimes e^1 \\
&+q^{-1/2}e^1 \otimes \left[\left(-q^{9/4}+q^{5/4}-q^{1/4}+q^{-3/4}\right)e_1 \otimes e_2 + \left(-q^{7/4}-q^{3/4}-q^{-1/4}\right)e_2 \otimes e_1\right] \otimes e^2 \\
&+q^{1/2}e^2 \otimes \left[\left(q^{7/4}-q^{3/4}+q^{-1/4}\right)e_1 \otimes e_2 + \left(-q^{5/4}+q^{1/4}\right)e_2 \otimes e_1\right] \otimes e^1 \\
&+q^{1/2}e^2 \otimes \left(q^{-3/4}e_2 \otimes e_2\right) \otimes e^2.
\end{aligned}$$

Finally, the two maps at the top contract the whole tensor into a number

$$\begin{aligned}
\theta^{fr}\left(3_1\right) &= q^{-1/2}q^{-3/4}q^{-1/2} + q^{-1/2}\left(-q^{9/4}+q^{5/4}-q^{1/4}+q^{-3/4}\right)q^{-1/2} \\
&\quad + q^{1/2}\left(-q^{5/4}+q^{1/4}\right)q^{1/2} + q^{1/2}q^{-3/4}q^{-1/2} \\
&= 2q^{-3/4} - q^{5/4} + q^{1/4} - q^{-3/4} + q^{-7/4} - q^{9/4} + q^{5/4} \\
&= q^{-7/4} + q^{-3/4} + q^{1/4} - q^{9/4}
\end{aligned}$$

Dividing by the normalizing factor $q^{1/2} + q^{-1/2}$ we get

$$\frac{\theta^{fr}\left(3_1\right)}{q^{1/2}+q^{-1/2}} = q^{-5/4} + q^{3/4} - q^{7/4}.$$

The invariant $\theta^{fr}\left(K\right)$ remains unchanged under the second and third reidemeister moves. However it varies under the first reidemeister move and thus depends on the framing. One can deframe it, that is, manufacture an invariant of unframed knots out of it, according to the formula

$$\theta\left(K\right) = q^{\frac{-c\cdot w(K)}{2}}\theta^{fr}\left(K\right),$$

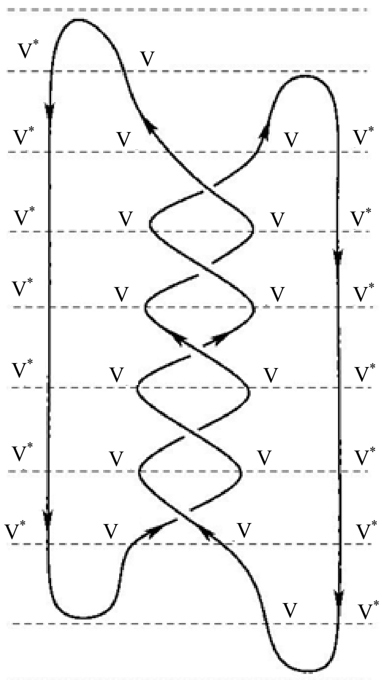
where $w\left(K\right)$ is the *writhe* of the knot diagram and c is the quadratic *Casimir number* defined by the Lie algebra $\mathfrak{g}$ and its representation. For $\mathfrak{sl}_2$ and the standard 2-dimensional representation $c = \frac{3}{2}$. Since the writhe of the left trefoil is -3, the unframed *normalized quantum invariant* is

$$\tilde{\theta}(3_1)=\frac{\theta(3_1)}{q^{1/2}+q^{-1/2}}=q^{9/4}\left(q^{-5/4}+q^{3/4}-q^{7/4}\right)=q+q^3-q^4.$$

This can be further written as

$$\tilde{\theta}\left(\widehat{x_1^3}\right)=-q^{\frac{3(3)-1}{2}}+q^{\frac{3(3)-3}{2}}+q\left[\tilde{\theta}\left(\widehat{x_1^1}\right)\right]. \tag{1.2}$$

For $n=5$, the knot and the corresponding tensor products are:



$$\begin{array}{c}
\mathbb{C} \\
\uparrow \\
V^*\otimes V \\
\uparrow \\
V^*\otimes V\otimes V\otimes V^* \\
\uparrow id_{V^*}\otimes R^{-1}\otimes id_{V^*} \\
V^*\otimes V\otimes V\otimes V^* \\
\uparrow id_{V^*}\otimes R^{-1}\otimes id_{V^*} \\
V^*\otimes V\otimes V\otimes V^* \\
\uparrow id_{V^*}\otimes R^{-1}\otimes id_{V^*} \\
V^*\otimes V\otimes V\otimes V^* \\
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V^*\otimes V\otimes V\otimes V^* \\
\uparrow id_{V^*}\otimes R^{-1}\otimes id_{V^*} \\
V^*\otimes V\otimes V\otimes V^* \\
\uparrow \\
V\otimes V^* \\
\uparrow \\
\mathbb{C}
\end{array}$$

With some computations, similar to the computations of $\tilde{\theta}\left(\widehat{x_1^3}\right)$, we get

$$\tilde{\theta}\left(\widehat{x_1^5}\right)=-q^7+q^6-q^5+q^4+q^2=-q^{\frac{3(5)-1}{2}}+q^{\frac{3(5)-3}{2}}+q\left[\tilde{\theta}\left(\widehat{x_1^3}\right)\right]. \tag{1.3}$$

Similarly,

$$\tilde{\theta}\left(\widehat{x_1^7}\right)=-q^{\frac{3(7)-1}{2}}+q^{\frac{3(7)-3}{2}}+q\left[\tilde{\theta}\left(\widehat{x_1^5}\right)\right]. \tag{1.4}$$

We now assume the result (1.1) holds for $n=k$, that is

$$\tilde{\theta}\left(\widehat{x_1^k}\right)=-q^{\frac{3k-1}{2}}+q^{\frac{3k-3}{2}}-q^{\frac{3k-5}{2}}+\cdots+q^{\frac{k+3}{2}}+q^{\frac{k-1}{2}} \tag{1.5}$$

Now for $n = k + 2$ we have

$$\begin{aligned}\tilde{\theta}\left(\widehat{x_1^{k+2}}\right) &= -q^{\frac{3(k+2)-1}{2}} + q^{\frac{3(k+2)-3}{2}} + q\left(\tilde{\theta}\left(\widehat{x_1^{k}}\right)\right) \\ &= -q^{\frac{3(k+2)-1}{2}} + q^{\frac{3(k+2)-3}{2}} + q\left[-q^{\frac{3k-1}{2}} + q^{\frac{3k-3}{2}} - q^{\frac{3k-5}{2}} + q^{\frac{3k-7}{2}} + \cdots + q^{\frac{k+3}{2}} + q^{\frac{k-1}{2}}\right] \\ &= -q^{\frac{3(k+2)-1}{2}} + q^{\frac{3(k+2)-3}{2}} - q^{\frac{3(k+2)-5}{2}} + q^{\frac{3(k+2)-7}{2}} + \cdots + q^{\frac{(k+2)+3}{2}} + q^{\frac{(k+2)-1}{2}},\end{aligned}$$

and the proof is finished.

Proposition 2 *The Jones polynomial of the knot* $\widehat{x_1^n}$, *when* n *is odd, is*

$$V_{\widehat{x_1^n}}(t) = -t^{\frac{1-3n}{2}} + t^{\frac{3-3n}{2}} - t^{\frac{5-3n}{2}} + \cdots + t^{\frac{-n-3}{2}} + t^{\frac{1-n}{2}}.$$

Proof 2 *Nothing to prove; just substitute* t^{-1} *in place of* q *in* $\tilde{\theta}\left(\widehat{x_1^n}\right)$.

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