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On the Two Methods for Finding 4-Dimensional Duck Solutions

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ABSTRACT

This paper gives the existence of a duck solution in a slow-fast system in R^{2+2} using two ways. One is an indirect way and the other is a direct way. In the indirect way, the original system is once reduced to the slow-fast system in R^{2+1}. In the direct one, it has a 4-dimensional duck solution when having an efficient local model. This is already published in [1,2]. Some sufficient conditions are given to get such a good model.

KEYWORDS

Slow-Fast System; Singular Perturbation; Duck Solutions; Blowing-Up; Nonstandard Analysis

1. Introduction

In the R^{2+2} slow-fast system with an invariant manifold, we first assume that this manifold describing limit cycle has a duck solution in a projected R^2 system in R^{2+2}. We introduce 2-dimensional duck solutions in the Section 2, then introduce 4-dimensional duck solutions in the Section 4. The blowing up method which constructs a local model [3], is published but revised and extended in this paper, and introduced in the Section 6. In general, we do not need the first assumption to get 4-dimensional duck solutions. It is the shortest way to explain these singular solutions. There are some concrete examples in [3-5].

2. Slow-Fast System in R^2

In this section, we shall review some results in Zvonkin and Shubin [6,7]. Let us consider the following system of differential equations

$$\begin{cases} \varepsilon \mathrm{d}x/\mathrm{d}t = w - f(x), \\ \mathrm{d}w/\mathrm{d}t = a - x, \end{cases} \tag{1}$$

where f is defined in R^1 and ε is infinitesimal in the sence of non-standard analysis of Nelson [8]. For the system (1), the graph $w = f(x)$ is called the *slow curve*. We consider the extremum point x_0 that separates the attracting part and the repelling part.

Definition 2.1 A solution $(x(t), w(t))$ of the system (1) is called a *duck solution* if there exist standard numbers t_1, t_0, t_2 $(t_1 < t_0 < t_2)$ such that

1) ${}^*[x(t_0)] = x_0$, where ${}^*[X]$ denotes the standard part of X,

2) for $t \in (t_1, t_0)$ the segment of the trajectory $(x(t), w(t))$ is infinitesimally close to the attracting part of the slow curve,

3) for $t \in (t_0, t_2)$, it is infinitesimally close to the repelling part of the slow curve, and

4) the attracting and repelling parts of the trajectory are not infinitesimal.

We give a necessary condition for the existence of a duck solution close to the extremum point x_0 of $f(x)$.

Proposition 2.2 *If there is a duck solution of the system* (1) *close to the extremum point* x_0, *then* $a \approx x_0$.

We finally obtain the following proposition concerning the existence of duck solutions.

Proposition 2.3 *Suppose that* f *has a nondegenerate extremum point* x_0, *that is,* $f'(x_0) = 0$ *and* $f''(x_0) > 0$. *Then there are the corresponding values of the parameter* a *satisfying Proposition* 2.2 *for which there exist duck solutions in the system* (1).

3. Slow-Fast System in R^3

We shall introduce 3-dimensional duck solutions by E. Benoit to get a concrete image before giving a framework in the 4-dimensional duck solutions. Let us consider the following slow-fast system:

$$\begin{cases} \varepsilon \mathrm{d}x/\mathrm{d}t = h(x, y, \varepsilon), \\ \mathrm{d}y_1/\mathrm{d}t = f_1(x, y, \varepsilon), \\ \mathrm{d}y_2/\mathrm{d}t = f_2(x, y, \varepsilon), \end{cases} \tag{2}$$

where $x \in R^1$, $y = (y_1, y_2) \in R^2$, are variables, and ε is a parameter as the same as in (1). We give the following assumptions in the system (2).

($A1$) $h \in C^2$, $f = (f_1, f_2) \in C^1$ are defined on $R^3 \times R^1$,

($A2$) The set $S_1 = \{(x, y) \in R^3 \,|\, h(x, y, 0) = 0\}$ is a 2-dimensional differentiable manifold and the set S_1 intersects the set $T_1 = \{(x, y) \in R^3 \,|\, \partial h(x, y, 0)/\partial x = 0\}$ transversely so that the pli set $PL = \{(x, y) \in S_1 \cap T_1\}$ is a 1-dimensional differentiable manifold.

($A3$) $f_1(x, y, 0) \neq 0$, or $f_2(x, y, 0) \neq 0$ at any point $(x, y) \in PL$.

Let $(x(t, \varepsilon), y(t, \varepsilon))$ be a solution of (2). When $\varepsilon = 0$, differentiating $h(x, y, 0)$ with respect to the time t, the following equation holds:

$$h_{y1}(x, y, 0) f_1(x, y, 0) + h_{y2}(x, y, 0) f_2(x, y, 0) + h_x(x, y, 0)\mathrm{d}x/\mathrm{d}t = 0,$$

where $h_i(x, y_1, y_2, 0) = \partial h(x, y_1, y_2, 0)/\partial i$, $i = x, y_1, y_2$. The above system (2) restricted to S_1 on the neighborhood of PL becomes the following system:

$$\begin{cases} \mathrm{d}y_1/\mathrm{d}t = f_1(x, y, 0), \\ \mathrm{d}y_2/\mathrm{d}t = f_2(x, y, 0), \\ \mathrm{d}x/\mathrm{d}t = -\{h_{y1}(x, y, 0) f_1(x, y, 0) + h_{y2}(x, y, 0) f_2(x, y, 0)\}/h_x(x, y, 0), \end{cases} \tag{3}$$

where $(x, y) \in S_1 \setminus PL$. The system (2) coincides with the system (3) at any point $p \in S_1 \setminus PL$. In order to avoid the degeneracy of the system (3), let us consider the following system:

$$\begin{cases} \mathrm{d}y_1/\mathrm{d}t = -h_x(x, y, 0) f_1(x, y, 0), \\ \mathrm{d}y_2/\mathrm{d}t = -h_x(x, y, 0) f_2(x, y, 0), \\ \mathrm{d}x/\mathrm{d}t = h_{y1}(x, y, 0) f_1(x, y, 0) + h_{y2}(x, y, 0) f_2(x, y, 0). \end{cases} \tag{4}$$

As the system (4) is well defined at any point of R^3, it is well defined indeed at any point of PL. The solutions of the system (4) coincide with those of the system (3) on $S_1 \setminus PL$ except the velocity when they start from the same initial points.

($A4$) For any point $(x, y) \in S_1$, either of the following holds; $h_{y1}(x, y, 0) \neq 0$, $h_{y2}(x, y, 0) \neq 0$, that is, the surface S_1 can be expressed as $y_1 = \varphi_1(x, y_2)$ or $y_2 = \varphi_2(x, y_1)$ in the neighborhood of PL. Let

$y_2 = \varphi_2(x, y_1)$ exist, then the projected system (5) is obtained:

$$\begin{cases} \mathrm{d}y_1/\mathrm{d}t = -h_x(x, y_1, \varphi_2(x, y_1), 0) f_1(x, y_1, \varphi_2(x, y_1), 0), \\ \mathrm{d}x/\mathrm{d}t = h_{y1}(x, y_1, \varphi_2(x, y_1), 0) f_1(x, y_1, \varphi_2(x, y_1), 0) \\ \qquad + h_{y2}(x, y_1, \varphi_2(x, y_1), 0) f_2(x, y_1, \varphi_2(x, y_1), 0). \end{cases} \tag{5}$$

If we take $y_1 = \varphi_1(x, y_2)$, it can be analyzed in the same way.

(A5) All the singular points of the system (5) are nondegenerate, that is, the matrix induced from the linearized system of (5) at a singular point has distinct nonzero eigenvalues.

Remark All these points are contained in the set $PS = \{(x, y) \in PL \mid \mathrm{d}x/\mathrm{d}t = 0\}$, which is called the set of *pseudo singular points*. Note that these points are the singular points in the system (4). The above assumptions $(A2)-(A4)$ might be enough to use on the neighborhood of these pseudo singular points, because we aim to analyze only in the neighborhood of the pseudo singular point in the system (2).

Definition 3.1 Let $p \in PS$ and μ_1, μ_2 be two eigenvalues of the matrix associated with the linearized system of (5) at p. The point p is called *pseudo singular saddle* if $\mu_1 < 0 < \mu_2$ and called *pseudo singular node* if $\mu_1 < \mu_2 < 0$ or $\mu_1 > \mu_2 > 0$. When μ_1, μ_2 are complex conjugate, they are called *pseudo singular focus*.

From now on we use *IST* [8]. The "transfer principle" is applied for the approximation to the standard analysis. We take the functions h, f_1, f_2, which are non-standard, that is, they depend on ε. The second derivative of h, and the first derivatives of f_1, f_2 have S-continuity. Let the system (2) have a solution $(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon))$ and let $(x(t,0), y(t,0), z(t,0)) \in S_1$ be a solution of the system (4), then a duck solution is defined as follows.

Definition 3.2 The solution $(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon))$ of the systems (2) is called a *duck*, if there exist standard $t_1 < t_0 < t_2$ such that

1) ${}^*[x(t_0,\varepsilon), y(t_0,\varepsilon), z(t_0,\varepsilon)] \in S_1$,

2) for $t \in (t_1, t_0)$ the segment of the trajectory $(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon))$ is infinitesimally close to the attracting part of the slow curves (the constrained surface),

3) for $t \in (t_0, t_2)$, it is infinitesimally close to the repelling part of the slow curves, and

4) the attracting and repelling parts of the trajectory are not infinitesimal.

The definitions of *attracting and repelling* are the same in [9,10].

Theorem 3.3 (Benoit) *If the system has a pseudo singular saddle or node point, then it has duck solutions. In the saddle case, the duck solutions are determined uniquely. In the node case, for the distinct eigenvalues they are determined uniquely, if it has no resonance. If the system has a pseudo singular focus point, it has no duck solutions.*

Remark Note that there are some important conditions on the standardness of the functions. At around the pseudo singular point, we blow up the variables in order to get a local model which is described in the Section 6. In this case, it is determined uniquely. Through the local model, we can get an exact solution as is approximation in the original system. Using *transfer principle*, we can confirm the existence of 3-dimensional duck solutions. See [9].

4. Slow-Fast System in R^4

Now, let us consider a slow-fast system (6):

$$\begin{cases} \varepsilon \mathrm{d}x_1/\mathrm{d}t = h_1(x_1, x_2, y_1, y_2, \varepsilon), \\ \varepsilon \mathrm{d}x_2/\mathrm{d}t = h_2(x_1, x_2, y_1, y_2, \varepsilon), \\ \mathrm{d}y_1/\mathrm{d}t = f_1(x_1, x_2, y_1, y_2, \varepsilon), \\ \mathrm{d}y_2/\mathrm{d}t = f_2(x_1, x_2, y_1, y_2, \varepsilon), \end{cases} \tag{6}$$

where $f = (f_1, f_2)$ and $h = (h_1, h_2)$ are standard defined on $R^4 \times R^1$ and ε is infinitesimal.

First, we assume the following condition $(B1)$ to get an explicit solution.

(B1) f is of class C^1 and h is of class C^2.

Furthermore, we assume that the system (6) satisfies the following generic conditions $(B2)-(B5)$:

(B2) The set $S_2 = \{(x,y)\in R^4 \mid h(x,y,0)=0\}$ is a 2-dimensional differentiable manifold and the set S_2 intersects the set $T_2 = \{(x,y)\in R^4 \mid \det[\partial h(x,y,0)/\partial x]=0\}$, which is a 3-dimensional differentiable manifold, transversely so that the generalized pli set $GPL = \{(x,y)\in S_2 \cap T_2\}$ is a 1-dimensional differentiable manifold.

(B3) The value of f is nonzero at any point $p \in GPL$.

(B4) The $rank[\partial h(x,y,0)/\partial x]=2$ for any $(x,y)\in S_2 \setminus GPL$, and the $rank[\partial h(x, rank[\partial h(x,y,0)/\partial y]=2$ for any $(x,y)\in S_2$. Then, the surface S_2 can be expressed as $y=\varphi(x)$ in the neighborhood of GPL. On the set GPL, $\partial h_1(x,y,0)/\partial x_2 \neq 0$ or $\partial h_2(x,y,0)/\partial x_1 \neq 0$, then $x_2 = \psi_2(x_1,y)$ and $x_1=\psi_1(x_2,y)$, where we use the notations $x=(x_1,x_2)$, and $y=(y_1,y_2)$.

Assume $y=\varphi(x)$. On the set S_2, differentiating both sides of $h(x,\varphi(x),0)=0$ with respect to x,

$$[h_x]+[h_y]D\varphi = 0, \tag{7}$$

where $D\varphi$ is a derivative with respect to x, thus the following is established:

$$D\varphi(x) = -[h_y]^{-1}[h_x]. \tag{8}$$

On the other hand,

$$\mathrm{d}y/\mathrm{d}t = D\varphi(x)\,\mathrm{d}x/\mathrm{d}t,$$

because of $y=\varphi(x)$. We can reduce the slow system to the following:

$$D\varphi(x)\,\mathrm{d}x/\mathrm{d}t = f(x,\varphi(x),0). \tag{9}$$

Using (8), the system (9) is described by

$$[h_x]\mathrm{d}x/\mathrm{d}t = -[h_y]\, f(x,\varphi(x),0).$$

Put $A=[h_x]=[h_{ij}]$ simply, then

$$\mathrm{d}x/\mathrm{d}t = -B[h_y]\, f(x,\varphi(x),0), \tag{10}$$

where B is a cofactor matrix of A, that is, $B=[A_{ji}]$. A_{ij} is a *cofactor* of h_{ij}.

The system (10) is the time scaled reduced system projected into R^2. Again, we assume the set $T_2 = \{(x,y)\in R^4 \mid \det A = 0\} \neq \phi$.

(B5) All the singular points of the system (10) are nondegenerate, that is, the matrix induced from the corresponding linearized system at the singular point has distinct nonzero eigenvalues.

Remark All these points are contained in the set $GPS = \{(x,y)\in GPL \mid B[h_y]\, f(x,\varphi(x))=0\}$, which is called the set of *generalized pseudo singular points*.

As this approach transforms the original system to the time scaled reduced system directly, it is called a *direct method*.

Definition 4.1 Let $p\in GPS$ and μ_1, μ_2 be two eigenvalues of the matrix associated with the linearized system of (4.7) at $p \in R^4$. The point p is called *generalized pseudo singular saddle* if $\mu_1 < 0 < \mu_2$ and called *generalized pseudo singular node* if $\mu_1 < \mu_2 < 0$ or $\mu_1 > \mu_2 > 0$. It is called *generalized pseudo singular focus* if they are compex conjugate.

Now, we have to give a description on the definition of the duck solution in R^4 along the direct method. The method induces a 2-dimensional projected space directly. Note that we can also once project the original system into a 3-dimensinal space. It is called the indirect method.

Definition 4.2 Let a point p be in GPS. If a trajectory follows first the attractive surface before this point and the saddle point, and then it goes along the slow manifold, which is not infinitesimal, it is called a *duck solution in* R^4.

Furthermore, we assume that the following.

(B6) We assume that there exists the set co-GPL, which may contain GPS and then the transversality condition is also established on co-GPL. In the situation, we assume that the invariant manifold through GPS intersects GPL and co-GPL transversely.

Definition 4.3 If the trajectory near the point of GPS passes through along the slow manifold with not infinitesimal and after that it jumps away, it is called *a single duck solution*. If there exists a co-GPL in $(B6)$ within the interval, it is called *a double duck solution*.

Remark The first part of Definition 4.3 ensures that only one of the eigenvalues of the matrix $\left[h_x\left(x,\varphi(x),0\right)\right]$ on the slow manifold takes zero on GPS, because the fast vector field has saddle after GPS. On another GPL, however, the other eigenvalue takes zero. Note that these two eigenvalues of $\left[h_x\left(x,\varphi(x),0\right)\right]$ are negative when the fast vector field is attractive, and are positive when it is repulsive. It occurs such a state satisfying the assumption $(B6)$. When they have different sign, it is saddle.

5. Lemmas

In this section, we give two Lemmas to make it clear the structure of the 4-dimensional system and the 3-dimensional projected system.

Let the latter of $(B4)$ be satisfied, then the following two projected systems (11), (12) in R^3 are induced. We assume that $\mathrm{d}x_1/\mathrm{d}t, \mathrm{d}x_2/\mathrm{d}t$ are limited, that is, $\varepsilon\left|\mathrm{d}x_1/\mathrm{d}t - \mathrm{d}x_2/\mathrm{d}t\right|$ tends to zero as ε tends to zero.

$$\begin{cases} \varepsilon\,\mathrm{d}x_1/\mathrm{d}t = h_2\left(x_1,\psi_2\left(x_1,y\right),y,\varepsilon\right), \\ \mathrm{d}y_1/\mathrm{d}t = f_1\left(x_1,\psi_2\left(x_1,y\right),y,\varepsilon\right), \\ \mathrm{d}y_2/\mathrm{d}t = f_2\left(x_1,\psi_2\left(x_1,y\right),y,\varepsilon\right), \end{cases} \tag{11}$$

since the relation $x_2 = \psi_2\left(x_1,y\right)$ is established from the above assumption. First, we can analyze the vector field of the system (11) on the constrained surface. Then, we use $h_2\left(x_1,x_2,y_1,y_2,\varepsilon\right)$ instead of $h_1\left(x_1,\psi_2\left(x_1,y_1,y_2\right),y_1,y_2,\varepsilon\right)$ as an approximation. Because we have to avoid redundancy for the system as is using h_1. Actually, we need the above condition: $\mathrm{d}x_1/\mathrm{d}t, \mathrm{d}x_2/\mathrm{d}t$ are limited, in such a case. Therefore, this approach is called *an indirect method*. Using the other relation $x_1 = \psi_1\left(x_2,y\right)$, we can get the following:

$$\begin{cases} \varepsilon\,\mathrm{d}x_2/\mathrm{d}t = h_1\left(\psi_1\left(x_2,y\right),x_2,y,\varepsilon\right). \\ \mathrm{d}y_1/\mathrm{d}t = f_1\left(\psi_1\left(x_2,y\right),x_2,y,\varepsilon\right), \\ \mathrm{d}y_2/\mathrm{d}t = f_2\left(\psi_1\left(x_2,y\right),x_2,y,\varepsilon\right). \end{cases} \tag{12}$$

Lemma 5.1 *The transversality condition* $(B2)$ *is established if and only if the transversality condition* $(A2)$ *in* Section 3 *is satisfied in the systems* (12) *and* (11) *at the common pseudo singular point.*

Lemma 5.2 *The system* (11) *or* (12) *have a pseudo singular saddle* (*or pseudo singular node*) *point, if the system* (6) *has a generalized pseudo singular saddle or node point and if the trajectory follows first the attractive surface before this point and saddle or repulsive one after the point having* $\partial h_1(p)/\partial x_2 > 0$, *or* $\partial h_2(p)/\partial x_1 > 0$ *on* GPL.

5.1. Proof of Lemma 5.1

Let $\nabla h_i\left(x,y,0\right)$ denote a gradient vector of $h_i\left(x,y,0\right)$. The transversality between S_2 and T_2 at the generalized pseudo singular point $p = \left(x1_0, x2_0, y1_0, y2_0\right) \in R^4$ is checked as follows:

$$rank\begin{pmatrix} \nabla h_1\left(p,0\right) \\ \nabla h_2\left(p,0\right) \\ \nabla \det\left[\partial h\left(p,0\right)/\partial x\right] \end{pmatrix} = 3. \tag{13}$$

The transversality between S_1 and T_1 in the system (11) and (12) are checked as follows. Put

$$g_1\left(x_1,y_1,y_2\right) = h_2\left(x_1,\psi_2\left(x_1,y\right),y_1,y_2,0\right),$$
$$g_2\left(x_2,y_1,y_2\right) = h_1\left(\psi_1\left(x_2,y\right),x_2,y_1,y_2,0\right),$$

and then put

$$\begin{pmatrix} \nabla g_1(p1) \\ \nabla \partial g_1(p1)/\partial x1 \end{pmatrix} = M_{p1}, \tag{14}$$

where $p1 = (x1_0, y1_0, y2_0)$,

$$\begin{pmatrix} \nabla g_2(p2) \\ \nabla \partial g_2(p2)/\partial x2 \end{pmatrix} = N_{p2}, \tag{15}$$

As the gradient vectors satisfy the relation (13), $rankM_{p1} = rankN_{p2} = 2$ holds. In fact, the gradient vectors in (14) and (15) are independent, since the assumption $(B4)$ ensures that only the coordinates are changed. Conversely, pulling back the equations (14), (15) to R^4, that is, embedding the corresponding 2-dimensional manifold into the original R^4, we can confirm that the relation (13) holds. In fact, the second equation in (14), (15) is equivalent to the third one in (13). The proof is complete.

5.2. Proof of Lemma 5.2

Let the original system have a generalized pseudo singular saddle point $p = (x1_0, x2_0, y1_0, y2_0) \in R^4$, that is, the point p is a singular point of the system (10) satisfying

$$-B\left[h_y(p)\right] f\left(x1_0, x2_0, \varphi(x1_0, x2_0)\right) = 0$$

Note that this system is described on the constrained surface.

Now, let us pull it back to the system in R^3. In the case of $\partial h_1(p)/\partial x_2 > 0$, or $\partial h_2(p)/\partial x_1 > 0$, using the assumption $\det\left[\partial h(p)/\partial y\right] \neq 0$, the following slow-fast system describes the current state.

$$\begin{cases} \varepsilon \mathrm{d}x_1/\mathrm{d}t = h_2\left(x_1, \psi_2\left(x_1, y_1, \phi_2(x)\right), y_1, \phi_2(x), \varepsilon\right), \\ \varepsilon \mathrm{d}x_2/\mathrm{d}t = h_1\left(\psi_1\left(x_2, y_1, \phi_2(x)\right), x_2, y_1, \phi_2(x), \varepsilon\right), \\ \mathrm{d}y_1/\mathrm{d}t = f_1\left(x_1, x_2, y_1, \phi_2(x), \varepsilon\right), \end{cases} \tag{16}$$

and using the assumption $\det\left[\partial h(p)/\partial y\right] \neq 0$,

$$\begin{cases} \varepsilon \mathrm{d}x_1/\mathrm{d}t = h_2\left(x_1, \psi_2\left(x_1, \phi_1(x), y_2\right), \phi_1(x), y_2, \varepsilon\right), \\ \varepsilon \mathrm{d}x_2/\mathrm{d}t = h_1\left(\psi_1\left(x_2, \phi_1(x), y_2\right), x_2, \phi_1(x), y_2, \varepsilon\right), \\ \mathrm{d}y_2/\mathrm{d}t = f_2\left(x_1, x_2, \phi_1(x), y_2, \varepsilon\right). \end{cases} \tag{17}$$

The above systems look like having a 1-dimensional slow manifold in R^3, however, they are tangent each other, because they have a still 2-dimensional differentiable manifold in R^3. Therefore, the orbits of the linearized systems (16), (17) are equivalent to the eigenvectors of the time scaled reduced system in the system (10).

The condition $\partial h_1/\partial x_2 > 0$ on the set GPL ensures that the sign of $\partial h_2\left(x_1, \psi_2\left(x_1, y_1, y_2\right), y_1, y_2, 0\right)/\partial x_1$ changes $+$ to $-$ when one of the eigenvalues in $\left[h_x\left(x, \phi(x)\right)\right]$ on the slow manifold changes as the same sign. If $\partial h_2/\partial x_1 > 0$ on the set GPL changes the sign, then the value of $\partial h_1\left(\psi_1\left(x_2, y_1, y_2\right), x_2, y_1, y_2, 0\right)/\partial x_2$ changes in the same way. Therefore, the system (11) has a pseudo singular saddle.

In fact, the system (16) is equivalent to the system (11) and the system (17) is also equal to the system (12). In the case of the node point, the proof is similar. The proof is complete.

6. Local Models

In this section, we shall give the following two theorems through a local model in R^{2+2}. See [1].

Theorem 6.1 *Let* $0 \in GPS$ *be saddle or node. If the matrix* $\left[h_x\left(0, \varphi(x), 0\right)\right]$ *has one zero eigenvalue and the other one has negative with a local model satisfying the conditions:* (1) $\mathrm{trace}[\mathrm{h}(0, \varphi(0), 0)] < 0$, (2) $f_1(0) \neq 0,\ f_2(0) \neq 0$, *there exists a duck solution in* R^4.

(Proof) As only one of the eigenvalues of the matrix $\left[h_x\left(x, \varphi(x), 0\right)\right]$ on the slow manifold takes zero on GPS, the assumptions $(A2)$, $(A4)$ ensure that two eigenvalues of $\left[h_x\left(x, \varphi(x), 0\right)\right]$ are negative in the fast vector field before GPS. They are maybe negative, respectively positive after GPS. When each coefficient on GPS is limitted, a local model shows a precise structure as an approximation of the original system. Then, the property on GPS reflects directly the whole system. It can be shown that the time scaled reduced system $(\varepsilon = 0)$ is an apprximated one with a singular solution of the whole system $(\varepsilon \neq 0)$, because the corresponding solutions are very close to each other under the only two conditions. Therefore, we can conclude that there exists a duck solution.

Let $0 \in GPS$ be saddle or node. When changing the variables correspond to microscopes $(\alpha \simeq 0)$: $x_1 = \alpha^p u_1, x_2 = \alpha^q u_2, y_1 = \alpha^r v_1, y_2 = \alpha^s v_2$, $p,q,r,s \in N$, the original system is reduced to the system with variables u_1, u_2, v_1, v_2. Then there exist local models which describe the 4 -dimensional duck solutions.

Theorem 6.2 *If the system has a square-linear solution in a local model, for any* $p,q,r,s \in N$ *, there exist essentially two local models describing the explicit duck solutions.*

(Proof)

In the case $p = 2$, $q = 1$, $r = 2$, $s = 2$, changing variables:

$$x_1 = \alpha^2 u_1, x_2 = \alpha u_2, y_1 = \alpha^2 v_1, y_2 = \alpha^2 v_2, \tag{18}$$

we reduce the system as well in (19) as well in (20).

$$\begin{cases} \varepsilon \mathrm{d}u_1/\mathrm{d}t = h_1\left(\alpha^2 u_1, \alpha u_2, \alpha^2 v_1, \alpha^2 v_2, \varepsilon\right)/\alpha^2, \\ \varepsilon \mathrm{d}u_2/\mathrm{d}t = h_2\left(\alpha^2 u_1, \alpha u_2, \alpha^2 v_1, \alpha^2 v_2, \varepsilon\right)/\alpha, \\ \mathrm{d}v_1/\mathrm{d}t = f_1\left(\alpha^2 u_1, \alpha u_2, \alpha^2 v_1, \alpha^2 v_2, \varepsilon\right)/\alpha^2, \\ \mathrm{d}v_2/\mathrm{d}t = f_2\left(\alpha^2 u_1, \alpha u_2, \alpha^2 v_1, \alpha^2 v_2, \varepsilon\right)/\alpha^2. \end{cases} \tag{19}$$

Multiplying the right hand side of the system (19) by α^2,

$$\begin{cases} \left(\varepsilon/\alpha^2\right)\mathrm{d}u_1/\mathrm{d}t = h_1\left(\alpha^2 u_1, \alpha u_2, \alpha^2 v_1, \alpha^2 v_2, \varepsilon\right)/\alpha^2 \\ \left(\varepsilon/\alpha^2\right)\mathrm{d}u_2/\mathrm{d}t = h_2\left(\alpha^2 u_1, \alpha u_2, \alpha^2 v_1, \alpha^2 v_2, \varepsilon\right)/\alpha, \\ \mathrm{d}v_1/\mathrm{d}t = f_1\left(\alpha^2 u_1, \alpha u_2, \alpha^2 v_1, \alpha^2 v_2, \varepsilon\right), \\ \mathrm{d}v_2/\mathrm{d}t = f_2\left(\alpha^2 u_1, \alpha u_2, \alpha^2 v_1, \alpha^2 v_2, \varepsilon\right). \end{cases} \tag{20}$$

In fact, doing time scaling $t = \alpha^2 \tau$, then $\mathrm{d}t = \alpha^2 \mathrm{d}\tau$. It is easy to show that Formula (20) is equivalent to (19).

By using the assumptions $(B1)$ and $(B4)$, we construct a local model under the most simple conditions:

$$\begin{aligned} &(1)\,\mathrm{trace}\left[\mathrm{h}\left(0,\varphi(0),0\right)\right] < 0 \\ &(2)\ f_1(0) \neq 0,\ f_2(0) \neq 0. \end{aligned} \tag{21}$$

Putting ε/α^2 infinitesimal to δ simply, that is $\delta = L(\varepsilon^3)$

$$\begin{cases} \delta\, \mathrm{d}u_1/\mathrm{d}t = Au_1 + Bv_1 + Cv_2 + Du_2^2/2 + L\delta, \\ \delta\, \mathrm{d}u_2/\mathrm{d}t = Eu_2 + L\delta, \\ \mathrm{d}v_1/\mathrm{d}t = f_1(0) + L\delta, \\ \mathrm{d}v_2/\mathrm{d}t = f_2(0) + L\delta, \end{cases} \tag{22}$$

where $A = \partial h_1(0)/\partial x_1$, $B = \partial h_1(0)/\partial x_2$, $C = \partial h_1(0)/\partial y_1$, $D = \partial h_1(0)/\partial y_2$, $E = \partial^2 h_1(0)/\partial x_2$, $F = \partial h_2(0)/\partial x_2$. Note that the conditions $\mathrm{trace}[\mathrm{h}(0,\varphi(0),0)] < 0$ imply that $0 \in GPS$ is saddle. See Definition 4.3. The corresponding solutions in the local model are as follows: when $\delta = 0$,

$$\begin{aligned} &u_1 = -\left(Cf_1(0) + Df_2(0)\right)t/A, u_2 = 0, \\ &v_1 = f_1(0)t, v_2 = f_2(0)t \end{aligned} \tag{23}$$

when $\delta \neq 0$,

$$
\begin{aligned}
&u_1 = -(Cf_1(0) + Df_2(0))t / A + L(\delta), u_2 = -L(\delta),\\
&v_1 = f_1(0)t + L(\delta), v_2 = f_2(0)t + L(\delta)
\end{aligned}
\tag{24}
$$

In the case $p = 2$, $q = 1$, $r = 3$, $s = 2$, changing variables:

$$x_1 = \alpha^2 u_1, x_2 = \alpha u_2, y_1 = \alpha^3 v_1, y_2 = \alpha^2 v_2, \tag{25}$$

we construct a local model under the conditions:

$$
\begin{aligned}
&(1)\, \text{trace}[\mathrm{h}(0, \varphi(0), 0)] < 0\\
&(2)\, f_1(0) = 0,\ f_2(0) \neq 0.
\end{aligned}
\tag{26}
$$

The corresponding local model is

$$
\begin{aligned}
&\delta du_1 / dt = Au_1 + Bu_2 + Cv_2 + Du_2^2 / 2 + L(\delta),\\
&\delta du_2 / dt = Eu_2 + L(\delta),\\
&dv_1 / dt = Fu_2 + L(\delta),\\
&dv_2 / dt = f_2(0) + L(\delta)
\end{aligned}
\tag{27}
$$

where $A = \partial h_1(0)/\partial x_1$, $B = \partial h_1(0)/\partial x_2$, $C = \partial h_1(0)/\partial y_2$, $D = \partial^2 h_1(0)/\partial x_2^2$, $E = \partial h_2(0)/\partial x_2$, $F = \partial f_1(0)/\partial x_2$.

Notice that we assume again that $\text{trace}[\mathrm{h}(0, \varphi(0), 0)] < 0$, because the fast vector field has one zero eigenvalue and the other one is negative. The corresponding solutions in the local model are as follows: when $\delta = 0$,

$$
\begin{aligned}
&u_1 = -Cf_2(0)t / A, u_2 = 0,\\
&v_1 = k(constant), v_2 = f_2(0)t
\end{aligned}
\tag{28}
$$

when $\delta \neq 0$,

$$
\begin{aligned}
&u_1 = -Cf_2(0)t / A + L(\delta), u_2 = -L(\delta),\\
&v_1 = k + L(\delta), v_2 = f_2(0)t + L(\delta)
\end{aligned}
\tag{29}
$$

In another case, it is impossible to get an explicit solution with a square-linear one but a cubic-linear (or much higher order) one.

In this approach, an invertible affine transformation must be needed for a general point $p \in GPS$, because the conditions (21), (26) are assumed at only $0 \in GPS$. These conditions may not be satisfied at the general pseudo singular point. We have to change the coordinates from the point p to 0. Notice that we do not know if the corresponding affine transformation keeps the conditions (21). In many cases, however, it is feasible.

7. Remark

It is easy to find that any solutions (u_1, u_2, v_1, v_2) at the same time t in (23) and (24) are very near. This fact implies that the time scaled reduced system is an approximated one. As blowing up the coordinates, the microscopes give a freedom on the solutions with respect to the initial values. The corresponding local model has higher possible polynomial solutions (not to be unique generally). We choose the smallest polynomial order.

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REFERENCES

[1] K. Tchizawa, "On a Local Model for Finding 4-Dim Duck Solutions," Selected Topics in Mathematical Methods and Computational Techniques in Electrical Engineering, WSEAS Press, 2010, pp. 177-183.

[2] K. Tchizawa, "On Relative Stability in 4-Dimensional Duck Solutions," *Journal of Mathematics and System Science*, Vol. 2, No. 9, 2012, pp. 558-563.

[3] K. Tchizawa and S. A. Campbell, "On Winding Duck Solutions in R^4," *Proceedings of Neural*, *Parallel*, *and Scientific Computations*, Vol. 2, 2002, pp. 315-318.

[4] S. A. Campbell and M. Waite, "Multistability in Coupled Fitzhugh-Nagumo Oscillators," *Nonlinear Analysis*, Vol. 47, No. 2, 2000, pp. 1093-1104. http://dx.doi.org/10.1016/S0362-546X(01)00249-8

[5] H. Miki, H. Nishino and K. Tchizawa, "On the Possible Occurrence of Duck Solutions in Domestic and Two-Region Business Cycle Models," *Nonlinear Studies*, Vol. 19, No. 1, 2012, pp. 39-55.

[6] F. Diener and M. Diener, "Nonstandard Analysis in Practice," Springer-Verlag, Berlin, 1995.

[7] A. K. Zvonkin and M. A. Shubin, "Non-Standard Analysis and Singular Perturbations of Ordinary Differential Equations," *Russian Mathematical Surveys*, Vol. 39, No. 2, 1984, pp. 69-131. http://dx.doi.org/10.1070/RM1984v039n02ABEH003091

[8] E. Nelson, "Internal Set Theory," *Bulletin of the American Mathematical Society*, Vol. 83, No. 6, 1977, pp. 1165-1198. http://dx.doi.org/10.1090/S0002-9904-1977-14398-X

[9] E. Benoit, "Canards et Enlacements," *Publications Mathématiques de l'IHÉS*, Vol. 72, No. 1, 1990, pp. 63-91. http://dx.doi.org/10.1007/BF02699131

[10] E. Benoit, "Canards en un Point Pseudo-Singulier Noeud," Bulletin de la Société Mathématique de France, 1999, pp. 2-12.