Applied Mathematics, **2014, 5, 7-15**
Published Online January 2014 (http://www.scirp.org/journal/am)
http://dx.doi.org/10.4236/am.2014.51002

Scientific Research

Global Attractor of Two-Dimensional Strong Damping KDV Equation and Its Dimension Estimation

Cheng Zhang, Guoguang Lin
Mathematical of Yunnan University, Kunming, China
Email: zhangcheng198910@126.com, gglin@ynu.edu.cn

Received October 10, 2013; revised November 10, 2013; accepted November 17, 2013

ABSTRACT

Firstly, a priori estimates are obtained for the existence and uniqueness of solutions of two dimensional KDV equations, and prove the existence of the global attractor, finally geting the upper bound estimation of the Hausdorff and fractal dimension of attractors.

KEYWORDS

KDV Equation; Strongly Damped; Existence; Global Attractor; Dimension Estimation

1. Introduction

Studies on the infinite dimension system with high dimension have obtained many achievements in recent years, such as [1-5]. In the paper [6,7]. The authors study the estimates of global attractor for one-dimensional KDV equation and its dimension. Based on these work, this paper further studies the global attractor of two-dimensional KDV equations and its upper bound estimation of the Hausdorff and fractal dimension of attractors.

The following form 2D-KDV equation is studied in this paper

$$u_t + u_{xxx} + \alpha u + \beta (uv)_x + \gamma \Delta^2 u = f(x,y),\ (x,y) \in \Omega \tag{1.1}$$

$$u_x(x,y;t) = v_y(x,y;t),\ \ (x,y) \in \Omega \tag{1.2}$$

$$u(x,y;0) = u_0(x,y),\ \ (x,y) \in \Omega \tag{1.3}$$

$$u(x,y;t)\big|_{\partial\Omega} = 0,\ \Delta u(x,y;t)\big|_{\partial\Omega} = 0,\ (x,y) \in \Omega \tag{1.4}$$

where α,β,γ are positive constants. When $\alpha = \beta = \gamma = 0$, the equation is the KDV equation.

The rest of this paper is organized as follows. In Section 2, we introduce basic concepts concerning global attractor. In Section 3, we obtain the existence of the uniqueness global attractor, which has fractal and Hausdorff dimension.

In this paper, C denotes a positive constant whose value may change in different positions of chains of inequalities.

2. Preliminaries

Denoting by $|\cdot|_{L^p}$ the norm in $L^p(\Omega)$, $1 \le p \le \infty$, for simplicity, we denote by $|\cdot|$ and $|\cdot|_\infty$ the norm in the case $p = 2$ and $p = \infty$, respectively. Suppose that $H = L^2(\Omega)$, $H^i(\Omega)$ is a Hilbert space for the scalar product

$$((\cdot,\cdot))_H^i = (\cdot,\cdot) + \sum_{j=1}^{i}\left(D^j\cdot, D^j\cdot\right),\ D = \frac{\partial}{\partial x}.$$

According to the Poincare inequality and (1.2) we can get

$$|v| \le C_1|\nabla u|.$$

In fact,

$$u_x = v_y \Rightarrow u_{xx} = v_{xy} \Rightarrow |u_{xx}| \le |v_{xy}| \le C|\Delta u| \Rightarrow |v_x| \le C|v_{xy}| \le C|\Delta u| \Rightarrow |v| \le C_1|\nabla u|$$

Now, we can do priori estimates for Equation (1.1)

Lemma 1. Assume that $f(x,y)\in L^2(\Omega), u_0(x,y)\in L^2(\Omega)$, $\alpha > \frac{\beta^2C^2}{2\gamma}$ then

$$|u(x,y;t)|^2 \le |u_0(x,y)|^2 \mathrm{e}^{-\left(2\alpha-\frac{\beta^2C^2}{\gamma}\right)t} + \frac{2\gamma}{\beta^2C^2\left(2\alpha-\frac{\beta^2C^2}{\gamma}\right)}|f|^2\left(1-\mathrm{e}^{-\left(2\alpha-\frac{\beta^2C^2}{\gamma}\right)t}\right), \tag{2.1}$$

Certainly there exist $t_1 = t_1(\Omega) > 0$, such that

$$|u(x,y;t)| \le C_2, \tag{2.2}$$

Proof. We multiply u for both sides of Equation (1.1), we obtain

$$(u_t,u)+(u_{xxx},u)+\alpha(u,u)+\beta((uv)_x,u)+\gamma(\Delta^2u,u)=(f,u), \tag{2.3}$$

where $(u,u_{xxx}) = -(u_{xxx},u)$, we have

$$(u,u_{xxx}) = 0, \tag{2.4}$$

$$\beta|((uv)_x,u)| = \beta|(uv,u_x)| \le \beta|u|_\infty|\nabla u||v| \le \beta|u|_\infty|\nabla u|^2 \le \beta C|\Delta u||u| \le \gamma|\Delta u|^2 + \frac{\beta^2C^2}{4\gamma}|u|^2, \tag{2.5}$$

$$|(u,f)| \le |u||f| \le \frac{\beta^2C^2}{4\gamma}|u|^2 + \frac{\gamma}{\beta^2C^2}|f|^2, \tag{2.6}$$

Substituting (2.4)-(2.6) into (2.3) gets

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + \left(\alpha-\frac{\beta^2C^2}{2\gamma}\right)|u|^2 \le \frac{\gamma}{\beta^2C^2}|f|^2$$

Using the Growall inequality, we can get

$$|u(x,y;t)|^2 \le |u_0(x,y)|^2 \mathrm{e}^{-\left(2\alpha-\frac{\beta^2C^2}{\gamma}\right)t} + \frac{2\gamma}{\beta^2C^2\left(2\alpha-\frac{\beta^2C^2}{\gamma}\right)}|f|^2\left(1-\mathrm{e}^{-\left(2\alpha-\frac{\beta^2C^2}{\gamma}\right)t}\right)$$

■

Lemma 2. Assume that $f(x,y)\in H_0^1(\Omega), u_0(x,y)\in H_0^1(\Omega)$, $\alpha > \frac{\beta^2C^2}{2\gamma}$ then

$$|\nabla u(x,y;t)|^2 \le |\nabla u_0(x,y)|\mathrm{e}^{-2\alpha t} + \frac{|\nabla f(x,y)|^2 + 2\alpha C}{\alpha^2}\left(1-\mathrm{e}^{-2\alpha t}\right), \tag{2.7}$$

certainly, there also exist $t_2 = t_2(\Omega) > 0$, such that

$$\overline{\lim_{t\to\infty}}\left|\nabla u(x,y;t)\right|^2 \le C_3, \tag{2.8}$$

Proof. We take parts of the scalar product in L^2 of (1.1) with $-\Delta u$:

$$\left(u_t,-\Delta u\right)+\left(u_{xxx},-\Delta u\right)+\alpha\left(u,-\Delta u\right)+\beta\left(\left(uv\right)_x,-\Delta u\right)+\gamma\left(\Delta^2 u,-\Delta u\right)=\left(f,-\Delta u\right), \tag{2.9}$$

where $\left(u_{xxx},-\Delta u\right)=\left(\Delta u,u_{xxx}\right)$, thus

$$\left(u_{xxx},-\Delta u\right)=0, \tag{2.10}$$

$$\beta\left|\left(\left(uv\right)_x,-\Delta u\right)\right|=\beta\left|\left(uv,\Delta u_x\right)\right|\le\beta\left|u\right|_\infty\left|\nabla u\right|\left|\nabla\Delta u\right|, \tag{2.11}$$

Noticing

$$\left|u\right|_\infty\le C\left|\nabla\Delta u\right|^{\frac{1}{3}}\left|u\right|^{\frac{2}{3}}, \tag{2.12}$$

$$\left|\nabla u\right|\le C\left|\nabla\Delta u\right|^{\frac{1}{3}}\left|u\right|^{\frac{2}{3}}, \tag{2.13}$$

According to (12) and (13), Lemma 1 and Young inequality, we can obtain that

$$\beta\left|\left(\left(uv\right)_x,-\Delta u\right)\right|\le C\left|\nabla\Delta u\right|^{\frac{5}{3}}\left|u\right|^{\frac{4}{3}}\le\gamma\left|\nabla\Delta u\right|^2+C, \tag{2.14}$$

$$\left|\left(f,-\Delta u\right)\right|\le\left|\nabla f\right|\left|\nabla u\right|\le\frac{1}{2\alpha}\left|\nabla f\right|^2+\frac{\alpha}{2}\left|\nabla u\right|^2, \tag{2.15}$$

Using (2.10), (2.14) and (2.15), we can get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left|\nabla u\right|^2+\frac{\alpha}{2}\left|\nabla u\right|\le\frac{1}{2\alpha}\left|\nabla f\right|^2+C$$

Using Growall inequality, we have

$$\left|\nabla u\right|^2\le\left|\nabla u_0\right|^2\mathrm{e}^{-2\alpha t}+\frac{\left|\nabla f\right|^2+2\alpha C}{\alpha^2}\left(1-\mathrm{e}^{-2\alpha t}\right)$$

■

Lemma 3. Assume that $f(x,y)\in H_0^2(\Omega), u_0(x,y)\in H_0^2(\Omega)$, $\alpha>\frac{\beta^2C^2}{2\gamma}$ then

$$\left|\Delta u(x,y;t)\right|^2\le\left|\Delta u_0(x,y)\right|\mathrm{e}^{-2\alpha t}+\frac{\left|\Delta f\right|^2+2\alpha C}{\alpha^2}\left(1-\mathrm{e}^{-2\alpha t}\right), \tag{2.16}$$

Thus there exists $t_3 = t_3(\Omega) > 0$, such that

$$\left|\Delta u(x,y;t)\right|\le C_4, \tag{2.17}$$

Proof. We multiply $\Delta^2 u$ for both sides of Equation (1.1), we obtain that

$$\left(u_t,\Delta^2 u\right)+\left(u_{xxx},\Delta^2 u\right)+\alpha\left(u,\Delta^2 u\right)+\beta\left(\left(uv\right)_x,\Delta^2 u\right)+\gamma\left(\Delta^2 u,\Delta^2 u\right)=\left(f,\Delta^2 u\right), \tag{2.18}$$

where

$$\left(u_{xxx},\Delta^2 u\right)=0, \tag{2.19}$$

Noticing

$$\left|u\right|_\infty\le C\left|\Delta^2 u\right|^{\frac{1}{4}}\left|u\right|^{\frac{3}{4}}, \tag{2.20}$$

$$|\Delta u| \le C\left|\Delta^2 u\right|^{\frac{1}{3}}|\nabla u|^{\frac{2}{3}}, \tag{2.21}$$

Using (2.20)-(2.21), we obtain that

$$\beta\left|\left((uv)_x, \Delta^2 u\right)\right| \le \beta C|u|_\infty|\Delta u|\left|\Delta^2 u\right| \le C\left|\Delta^2 u\right|^{\frac{19}{12}}\left|u^{\frac{3}{4}}\right||\nabla u|^{\frac{2}{3}}, \tag{2.22}$$

According to Lemma 1, Lemma 2 and Young inequality, we get that

$$\beta\left|\left((uv)_x, \Delta^2 u\right)\right| \le \gamma\left|\Delta^2 u\right|^2 + C, \tag{2.23}$$

$$\left|\left(f, \Delta^2 u\right)\right| \le |\Delta u||\Delta f| \le \frac{\alpha}{2}|\Delta u|^2 + \frac{1}{2\alpha}|\Delta f|^2, \tag{2.24}$$

Substituting (2.19)-(2.24) into (2.18) gets

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\Delta u|^2 + \frac{\alpha}{2}|\Delta u|^2 \le \frac{1}{2\alpha}|\Delta f|^2 + C$$

Using the Growall inequality, we can get

$$|\Delta u|^2 \le |\Delta u_0|^2 \mathrm{e}^{-2\alpha t} + \frac{|\Delta f|^2 + 2\alpha C}{\alpha^2}\left(1 - \mathrm{e}^{-2\alpha t}\right)$$ ■

Lemma 4. Assume that $f(x,y) \in H_0^2(\Omega), u_0(x,y) \in H_0^2(\Omega)$, $\alpha > \frac{\beta^2 C^2}{2\gamma}$ then

$$|\nabla \Delta u(x,y;t)| \le \frac{Q}{t}, \tag{2.25}$$

here Q and $|u_0|_{H_0^2}$, $|f|_{H_0^2}$ have relations.

Proof. We multiply $t^2\Delta^3 u$ for both sides of Equation (1.1), we obtain that

$$\left(u_t, t^2\Delta^3 u\right) + \left(u_{xxx}, t^2\Delta^3 u\right) + \alpha\left(u, t^2\Delta^3 u\right) + \beta\left((uv)_x, t^2\Delta^3 u\right) + \gamma\left(\Delta^2 u, t^2\Delta^3 u\right) = \left(f, t^2\Delta^3 u\right), \tag{2.26}$$

we have

$$\left(u_t, t^2\Delta^3 u\right) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|t\nabla\Delta u|^2 + \left|t^{\frac{1}{2}}\nabla\Delta u\right|^2, \tag{2.27}$$

$$\gamma\left(\Delta^2 u, t^2\Delta^3 u\right) = -\gamma\left|t\nabla\Delta^2 u\right|^2, \tag{2.28}$$

$$\left(u_{xxx}, t^2\Delta^3 u\right) = 0, \tag{2.29}$$

$$\left|\left(f, \Delta^3 u\right)\right| = |\nabla f|\left|\nabla\Delta^2 u\right| \le \frac{\gamma}{6}\left|\nabla\Delta^2 u\right|^2 + \frac{3}{2\gamma}|\nabla f|^2, \tag{2.30}$$

$$\alpha\left|\left(u, \Delta^3 u\right)\right| \le \alpha|\nabla u|\left|\nabla\Delta^2 u\right|^2 \le \frac{\gamma}{6}\left|\nabla\Delta^2 u\right|^2 + C, \tag{2.31}$$

$$\beta\left|\left((uv)_x\right), \Delta^3 u\right| = \left|\left(\nabla\left(u_x v + u v_x\right), \nabla\Delta^2 u\right)\right| \le C\left(3|\nabla u|_\infty|\Delta u| + |u|_\infty|\nabla\Delta u|\right)\left|\nabla\Delta^2 u\right|, \tag{2.32}$$

Noticing

$$|u|_\infty \le C|\Delta u|^{\frac{1}{2}}|u|^{\frac{1}{2}}, \tag{2.33}$$

$$|\nabla u|_\infty \le C|\Delta u|^{\frac{3}{4}}\left|\nabla\Delta^2 u\right|^{\frac{1}{4}}, \tag{2.34}$$

$$|\nabla\Delta u| \le C|u|^{\frac{2}{5}}\left|\nabla\Delta^2 u\right|^{\frac{3}{5}}, \tag{2.35}$$

Taking (2.33)-(2.35) into (2.32) and using Young inequality, we have

$$\beta\left|\left((uv)_x,\Delta^3 u\right)\right| \le \frac{\gamma}{6}\left|\nabla\Delta^2 u\right|^2 + C, \tag{2.36}$$

namely,

$$\beta\left|\left((uv)_x,t^2\Delta^3 u\right)\right| \le \frac{\gamma}{6}\left|t\nabla\Delta^2 u\right|^2 + C, \tag{2.37}$$

Taking (2.27)-(2.37) into (2.26), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|t\nabla\Delta u|^2 + \gamma|t\nabla\Delta u|^2 \le C|\nabla f|^2$$

So, we get

$$|\nabla\Delta u| \le \frac{Q}{t}. \blacksquare$$

From [8], we have

Theorem 2.1 *Let E be a Banach space, $\{S(t)\}$ are the semigroup operators. $S(t):E\to E$, $S(t)S(\tau)$ $=S(t+\tau)$, $S(0)=I$, here I is unit operator.Set $S(t)$ satisfy the following conditions*:

1) $S(t)$ is bounded. namely $\forall R>0, |u|_E \le R$, there exist a constant $C(R)$, such that $|S(t)u|_E$ $\le C(R)\,(t\in[0,+\infty))$.

2) There exist a bounded absorbing set $B_0 \subset E$, namely $\forall B\subset E$, there exist a constant t_0, such that $S(t)B\subset B_0\ (t>t_0)$.

3) When $t>0$, $S(t)$ is a completely continuous operator.

Then, the semigroup operators $S(t)$ exist a compact global attractor A.

3. Global Attractor and Dimension Estimation

3.1. The Existence and Uniqueness of Solution

Theorem 3.1 *Assume that $f(x,y)\in H_0^2(\Omega)$ and $u_0(x,y)\in H_0^2(\Omega)$, $\alpha > \frac{\beta^2 C^2}{2\gamma}$ there exists a unique solution*

$$u(x,y;t)\in L^\infty\left(0,T;H_0^2(\Omega)\right), \tag{3.1.1}$$

Proof. By the Galerkin method, we can easily obtain the existence of solutions. Next, we prove the uniqueness of solutions.

Set $\omega = u_1 - u_2$, where $u_i\,(i=1,2)$ are two solutions of (1.1)-(1.4). then ω satisfies

$$\omega_t + \omega_{xxx} + \alpha\omega + \beta(u_1v_1 - u_2v_2) + \gamma\Delta^2\omega = 0, \tag{3.1.2}$$

$$u_iv_i = u_i\int (u_i)_x\,\mathrm{d}y,\ \ i=1,2, \tag{3.1.3}$$

$$\omega(x,y;0) = 0, \tag{3.1.4}$$

Take the inner product with ω, we gets

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\omega|^2 + \alpha|\omega|^2 + \beta(u_1v_1 - u_2v_2,\omega) + \gamma|\Delta\omega|^2 = 0, \tag{3.1.5}$$

Furthermore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\omega|^2 &\le 2\beta\left|\left(u_1v_1-u_2v_2,\omega\right)\right|+2\alpha|\omega|^2-2\gamma|\Delta\omega|^2\\ &\le 2\beta\left|\left(\omega\int u_{2x}\mathrm{d}y+u_1\int\omega_x\mathrm{d}y,\omega\right)\right|+2\alpha|\omega|^2-2\gamma|\Delta\omega|^2\\ &\le C\left(|\nabla u_2|_\infty|\omega|^2+|u_1|_\infty|\nabla\omega||\omega|\right)+2\alpha|\omega|^2-2\gamma|\Delta\omega|^2,\end{aligned}\tag{3.1.6}$$

Noticing

$$|u|_\infty\le C|\nabla u|^{\frac{1}{4}}|u|^{\frac{3}{4}},\tag{3.1.7}$$

$$|\nabla u|_\infty\le C|\Delta u|^{\frac{1}{4}}|\nabla u|^{\frac{3}{4}},\tag{3.1.8}$$

$$|\nabla\omega|\le C|\Delta\omega|^{\frac{1}{2}}|\omega|^{\frac{1}{2}},\tag{3.1.9}$$

So, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|\omega|^2\le C|\Delta u_2|^{\frac{1}{4}}|\nabla u_2|^{\frac{3}{4}}|\omega|^2+C|\nabla u_1|^{\frac{1}{4}}|u_1|^{\frac{3}{4}}|\nabla\omega||\omega|+2\alpha|\omega|^2-2\gamma|\Delta\omega|^2$$

From Lemmas 1-3, we have

$$|\Delta u_2|\le C,|\nabla u_2|\le C,|\nabla u_1|\le C,|u_1|\le C$$

Using Young inequality, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|\omega|^2\le C|\omega|^2$$

Using Gronwall inequality, we have

$$|\omega|^2\le|\omega(0)|^2\mathrm{e}^{2Ct}=0$$

So, we can get $\omega=0$. ■

3.2. Global Attractor

Theorem 3.2 *Assume that* $f(x,y)\in H_0^2(\Omega)$ *and* $u_0(x,y)\in H_0^2(\Omega)$, $\alpha>\frac{\beta^2C^2}{2\gamma}$ *there exists a compact global attractor* A, *such that*

1) $S(t)A=A,t>0$
2) $\lim_{t\to\infty} dist\left(S(t)B,A\right)=0$

here, B is a bounded set in $H_0^2(\Omega)$.

$$dist(X,Y)=\sup_{x\in X}\inf_{y\in Y}|x-y|_E,$$

$S(t)$ are the semigroup operators.

Proof. Let us verify theorem 2.1 conditions (1), (2), (3). In Theorem 3.2 conditions, we know that there exist the solution semigroup $S(t)$, $E=H_0^2(\Omega)$, $S(t):H_0^2(\Omega)\to H_0^2(\Omega)$. form Lemmas 1-3, we can get that $\forall B\subset H_0^2(\Omega)$ is a bounded set and B included in the ball $\left\{|u|_{H_0^2}\le R\right\}$,

$$\left|S(t)u_0\right|_{H_0^2}^2=\left|u(x,y;t)\right|_{H^2}^2\le|u_0|^2+C_1|f|^2+C_2\quad\left(t\ge0,u_0\in B\right).$$

This shows that $S(t)(t\ge0)$ is uniformly bounded in $H_0^2(\Omega)$. Furthermore, when $t\ge\max\{t_1,t_2,t_3\}$, we have

$$\left|S(t)u_0\right|_{H_0^2}^2=\left|u(x,y;t)\right|^2\le2\left(C_2+C_3+C_4\right)$$

so, we can get that $B_0 \ge \left\{u(x,y;t) \in H_0^2(\Omega), |u|_{H_0^2} \le \sqrt{2(C_2+C_3+C_4)}\right\}$ is bounded absorbing set of semigroup $S(t)$.

From Lemma 4, we have $|\nabla \Delta u| \le \frac{Q}{t}, (t>0)$, $|u_0|_{H_0^2} \le R$. Since $H_0^3(\Omega) \to H_0^2(\Omega)$ is tightly embedded.

So the semigroup operator $S(t): H_0^2(\Omega) \to H_0^2(\Omega)$ for $\forall t>0$ is continuous.

3.3. Dimension Estimation

Considering the following first variation equations

$$\omega_t(x,y;t) + L(u(x,y;t))\omega(x,y;t) = 0, \tag{3.3.1}$$

$$v(x,y;t) = \int u_x(x,y;t)\mathrm{d}y, \tag{3.3.2}$$

$$\omega(x,y;0) = 0, \tag{3.3.3}$$

$$\omega(x,y;t)\big|_{\partial\Omega} = 0,\ \Delta\omega(x,y;t)\big|_{\partial\Omega} = 0 \tag{3.3.4}$$

where

$$\omega(x,y;0) \in H_0^1(\Omega)$$

$$L(u(t))\omega(t) = \omega_{xxx}(t) + \alpha\omega(t) + \beta\omega_x(t)v(t) + \beta\int \omega_{xx}(t)\mathrm{d}y + \gamma\Delta^2\omega(t)$$

It's easy to prove that the equation has a unique solution. $\omega(x,y;t) \in L^\infty(0,T;H_0^1(\Omega))$.

Furthermore, Let $u(t)=S(t)u_0$, $(DS(t)u_0)\omega_0 = \omega(t)$, $S(t)(u_0+\omega_0) = u^*(t)$, we can get $\forall R_1$, R_2 and T are constants. There exist a constant $C = C(R_1,R_2,T)$ such that for u_0, ω_0, t with $|u_0|_{H_0^1(\Omega)} \le R_1$, $|\omega_0|_{H_0^1(\Omega)} \le R_2$, $|t| \le T$, we have

$$\left|u^*(t) - u(t) - \omega(t)\right|_{H_0^1(\Omega)} \le C|\omega_0|_{H_0^1(\Omega)}^2, \tag{3.3.5}$$

That suggests that $S(t)$ is Frechet differential at $u_0(x,y)$.

Let $V_1(t), V_2(t), \cdots, V_N(t)$ be the solutions of the linear variational equations corresponding to the initial value $V_1(0)=\xi_1, V_2(0)=\xi_2, \cdots, V_N(0)=\xi_N$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|V_1(t)\Lambda V_2(t)\Lambda\cdots\Lambda V_N(t)\right|^2 - 2tr\left(L(u(t))\cdot Q_N\right)\left|V_1(t)\Lambda V_2(t)\Lambda\cdots\Lambda V_N(t)\right|^2 = 0, \tag{3.3.6}$$

here Λ represents the outer product, tr represents the trace, Q_N means that the $L^2(\Omega)$ to the orthogonal projection on the span $\{V_1(t), V_2(t), \cdots, V_N(t)\}$. So, from (3.3.8) we can obtain

$$\omega_N(t) = \sup_{u_0 \in A}\ \sup_{\xi_n \in L^2, |\xi_n| \le 1} \left|V_1(t)\Lambda V_2(t)\Lambda\cdots\Lambda V_N(t)\right|_{\Lambda_{L^2}^N}^2, \tag{3.3.7}$$

where ω_N is called Secondary index, namely

$$\omega_N(t+t') \le \omega_N(t)\omega_N(t'),\ t,t' \ge 0$$

so

$$\lim_{t\to\infty}\omega_N(t)^{\frac{1}{t}} = \Pi_n,\ 1 \le n \le N$$

$$\Pi_n \le \mathrm{e}^{-q_N}$$

here

$$q_N = \lim_{t\to\infty}\sup\left(\inf_{u_0\in A}\frac{1}{t}\int_0^t \inf\left(tr\left(L(s(\tau)u_0)Q_N(\tau)\right)\mathrm{d}\tau\right)\right).$$

Theorem 3.3 *The global attractor* A *of Theorem 3.2 has finite fractal and Hausdorff dimension in*

$H_0^1(\Omega)$,

$$d_H(A) \le J_0,\ d_F(A) \le 2J_0, \tag{3.3.8}$$

J_0 is a minimal positive integer of the following inequality

$$J_0 = \frac{c-3a+\sqrt{a^2+c^2+8ab+2ac}}{4a}, \tag{3.3.9}$$

here

$$a = \frac{\gamma C'}{6}, b = \alpha + \frac{\beta}{2}C_1|\Delta u|_\infty + \frac{5}{2}C_6|u|_\infty|\Delta u|_\infty, c = \frac{C_6C'}{2}|u|_\infty.$$

Proof. From [9], we need to estimate $tr\big(L(u(t))\cdot Q_N\big)$ of the lower bound. Let $\varphi_1,\varphi_2,\cdots,\varphi_N$ be the orthogonal basis of subspace of $Q_N L^2(\Omega)$,

$$\begin{aligned} tr\big(L(u(t))\cdot Q_N\big) &= \sum_{j=1}^N\Big\{\Big(\varphi_{jxxx}+\beta\varphi_{jx}v+\beta\int\varphi_{jxx}\mathrm{d}y+\gamma\Delta^2\varphi_j+\alpha\varphi_j,\varphi_j\Big)\Big\} \\ &= \sum_{j=1}^N\Big\{\alpha|\varphi_j|^2+\gamma|\Delta\varphi_j|^2+\beta\Big(\varphi_{jx}v+u\int\varphi_{jxx}\mathrm{d}y,\varphi_j\Big)\Big\}, \end{aligned} \tag{3.3.10}$$

where

$$(\varphi_{jx}v,\varphi_j) = -(\varphi_j, v_x\varphi_j+v\varphi_{jx})$$

So, we can obtain

$$(\varphi_{jx}v,\varphi_j) = -\frac{1}{2}(v_x,\varphi_j^2)$$

Furthermore

$$\beta\left|\sum_{j=1}^N(\varphi_{jx}v,\varphi_j)\right| = \frac{\beta}{2}\left|\sum_{j=1}^N(v_x,\varphi_j^2)\right| \le \frac{\beta}{2}C\left|\sum_{j=1}^N\varphi_j^2\right||v_x|_\infty \le \frac{\beta}{2}\left|C_1\sum_{j=1}^N\varphi_j^2\right||\Delta u|_\infty, \tag{3.3.11}$$

$$\begin{aligned} \Big|\Big(u\int\varphi_{jxx}\mathrm{d}y,\varphi_j\Big)\Big| &= \Big|\Big(\int\varphi_j\mathrm{d}y, u_{xx}\varphi_j+2u_x\varphi_{jx}+u\varphi_{jxx}\Big)\Big| \\ &= \big|\big(C_2y\varphi_j, u_{xx}\varphi_j+2u_x\varphi_{jx}+u\varphi_{jxx}\big)\big| \\ &\le C_2|u|_\infty\big|\big(\varphi_j, u_{xx}\varphi_j+2u_x\varphi_{jx}+u\varphi_{jxx}\big)\big| \\ &\le C_3|u|_\infty|\Delta u|_\infty|\varphi_j|^2+C_4|u|_\infty\big|(\varphi_j,2u_x\varphi_{jx})\big|+C_5|u|_\infty\big|(\varphi_j,u\varphi_{jxx})\big|, \end{aligned} \tag{3.3.12}$$

$$(\varphi_j,2u_x\varphi_{jx}) = -2(\varphi_{jx}u_x+\varphi_ju_{xx},\varphi_j) = -2(\varphi_{jx}u_x,\varphi_j)-2(u_{xx},\varphi_j^2)$$

hence

$$(\varphi_j,2u_x\varphi_{jx}) = -(u_{xx},\varphi_j^2), \tag{3.3.13}$$

$$(\varphi_j,u\varphi_{jxx}) = -(\varphi_{jx}u,\varphi_{jx})+\frac{1}{2}(\varphi_j^2,u_{xx}), \tag{3.3.14}$$

Taking (3.3.15)-(3.3.16) into (3.3.14), we can get

$$\Big|\Big(u\int\varphi_{jxx}\mathrm{d}y,\varphi_j\Big)\Big| \le C_6|u|_\infty\left(\frac{5}{2}|\Delta u|_\infty|\varphi_j^2|+|\nabla\varphi_j^2||u|_\infty\right), \tag{3.3.15}$$

Set $\lambda_j, j=(1,2,3,\cdots)$ are eigenvalues of $-\Delta u=\lambda u$ and φ_j are the corresponding eigenfunctions. Satisfying

$$|\nabla\varphi_j|^2=\lambda_j, |\Delta\varphi_j|^2=\lambda_j^2, |\varphi_j|^2=1, \lambda_j \ge \left[\frac{(j-1)^{\frac{1}{2}}}{2}-1\right]^2 \sim C'j, \tag{3.3.16}$$

so, we can get

$$tr\left(L\left(u\left(t\right)\right)\cdot Q_N\right) \geq \gamma\sum_{j=1}^{N}\lambda_j^2 - N\alpha - \frac{\beta}{2}NC_1\left|\Delta u\right|_\infty - \frac{5}{2}C_6N\left|u\right|_\infty\left|\Delta u\right|_\infty - C_6\left|u\right|_\infty^2\sum_{j=1}^{N}\lambda_j, \quad (3.3.17)$$

Let

$$a = \frac{\gamma C'}{6}, b = \alpha + \frac{\beta}{2}C_1\left|\Delta u\right|_\infty + \frac{5}{2}C_6\left|u\right|_\infty\left|\Delta u\right|_\infty, c = \frac{C_6C'}{2}\left|u\right|_\infty, \quad (3.3.18)$$

when

$$N > \frac{c - 3a + \sqrt{a^2 + c^2 + 8ab + 2ac}}{4a}$$

we have

$$tr\left(L\left(u\left(t\right)\right)\cdot Q_N\right) > 0$$

so, we can obtain

$$d_H\left(A\right) \leq J_0, d_F\left(A\right) \leq 2J_0. \blacksquare$$

Funding

This work is supported by the National Natural Sciences Foundation of People's Republic of China under Grant 11161057.

REFERENCES

[1] G. R. Sell, "Global Attractors for the Three-Dimensional Navier-Stokes Equations," Journal of Dynamics and Di_erential Equations, Vol. 1, 1996.

[2] J. K. Hale, "Asymptotic Behaviour of Dissipative Systems," *Mathematical Surveys and Monographs*, Vol. 25, 1988.

[3] G. Sell and M. Taboada, "Local Dissipativity and Attractors for the K-S Equation in Thin 2D Domains," *Journal of Nonlinear Analysis*, Vol. 7, 1992.

[4] R. Temam and S. Wang, "Inertial Forms of Navier-Stokes Equations on the Sphere," *Journal of Functional Analysis*, Vol. 2, 1993.

[5] B. L. Guo and Y. S. Li, "Attractor for Dissipative Klein-Gordon-Schrdinger Equations in R3," *Journal of Differential Equations*, Vol. 2, 1997.

[6] X. Y. Du and Z. D. Dai, "Global Attractor of Dissipative KDV Equation about Cauchy Problem," *Acta Mathematica Scientia*, Vol. 20, 2000.

[7] L. X. Tian, Y. R. Liu and Z. R. Liu, "The Narrow 2D Weak Local Attractor for Damped KDV Equation," *Applied Mathematics and Mechanics*, Vol. 21, 2000.

[8] B. L. Guo, "Method of Vanishing Viscosity and Viscosity Difference Format," Science Press, Beijing, 1993.

[9] G. G. Lin, "Nonlinear Evolution Equation," Yunnan University Press, Yunnan, 2011.