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Hybrid Adaptive Synchronization of Hyperchaotic Systems with Fully Unknown Parameters

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ABSTRACT

In this paper, an adaptive control scheme is developed to study the hybrid synchronization behavior between two identical and different hyperchaotic systems with unknown parameters. This adaptive hybrid synchronization controller is designed based on Lyapunov stability theory and an analytic expression of the controller with its adaptive laws of parameters is shown. The adaptive hybrid synchronization between two identical systems (hyperchaotic Chen system) and different systems (hyperchaotic Lorenz and hyperchaotic Lű systems) are taken as two illustrative examples to show the effectiveness of the proposed method. Theoretical analysis and numerical simulations are shown to verify the results.

Keywords: Hybrid Synchronization; Adaptive Control; Unknown Parameters; Hyperchaotic System

1. Introduction

Chaos is an omnipresent phenomenon. Scientists who understand its existence have been struggling to control chaos to our benefit. There is a great need to control the chaotic systems as chaos theory plays an important role in industrial applications particularly in chemical reactions, biological systems, information processing and secure communications [1-3]. Many scientists who are interested in this field have struggled to achieve the synchronization or anti-synchronization of different hyperchaotic systems. Therefore due to its complexity and applications, a wide variety of approaches have been proposed for the synchronization or anti-synchronization of hyperchaotic systems. The types of synchronization used so far include generalized active control [4-8], nonlinear control [9,10], and adaptive control [11-19].

The co-existence of synchronization and anti-synchronization, known as hybrid synchronization, has good application prospects in digital communications. Therefore it attracted a lot of attention in recent years. In hybrid synchronization scheme, one part of the system is anti-synchronized and the others are completely synchronized so that complete synchronization and anti-synchronization co-exist in the system. The co-existence of CS and AS may enhance security in communication and chaotic encryption schemes. Li [20] studied full state hybrid projective synchronization behavior in multi-scroll chaotic systems in symmetrical coordinate subspace. Xie, Chen and Bolt [21], through numerical studies show that an arbitrary signal can be synchronized by hybrid chaotic system and then that particular signal can be stored for password and message identification. They further identify potential applications in information storage, message identification and certain types of secure signal and image communications.

Zhang and Lű [22] introduce a new type of hybrid synchronization called full state hybrid log projective synchronization and apply it to the Rossler systems and the hyperchaotic Lorenz system to numerically verify their results. Similarly Chen, Chen and Lin [23] achieve hybrid synchronization in Chin-Lee system using both linear and non-linear control schemes. Recently Sun et al. [24] analyze the hybrid synchronization of two coupled complex networks using linear feedback and adaptive feedback control methods. They derive a criterion for the hybrid synchronization of the two complex networks and show that under suitable conditions two complex networks can realize hybrid synchronization. More recently, Vaidyanathan and Rasappan [25] investigate the hybrid chaos synchronization of hyperchaotic Qi and Jia systems using active nonlinear control. The idea of the aforementioned type of hybrid synchronization of chaotic systems deals with systems with known parameters. However in practical engineering situations, parameters are probably unknown and may change from time to time. Therefore, there is a vital need to effectively hybrid-synchronize two chaotic systems (identical and different) with unknown parameters. This is typically important in theoretical research as well as practical applications. Among the aforementioned methods, adaptive control [11-19] is an effective option for achieving the synchronization of chaotic systems with fully unknown parameters. Therefore motivated by this, we study the hybrid synchronization of two identical and two different hyperchaotic systems with fully unknown parameters. The rest of the paper is organized as follows. In Section 2, we present a novel adaptive hybrid synchronization scheme with a parameter update law and give a brief description of the systems. In Sections 3 and 4, we present the hyperchaos hybrid synchronization between two identical and different hyper chaotic systems via adaptive control. Conclusions are given in Section 5.

2. Problem Formulation and Systems Description

In the first part of this section, we set up the problem and present novel adaptive hybrid synchronization scheme with parameter update law. By using Lyapunov stability theory we show the co-existence of hybrid synchronization between two systems described below. In the second part of this section we briefly describe the two systems used for further analysis.

2.1. Hybrid Synchronization of Chaotic Systems

Consider the master chaotic system in the form of

$$\dot{x} = f(x) + F(x)\alpha \tag{1}$$

where $x \in \Omega_1 \subset R^n$ is the state vector, $\alpha \in R^m$ is the unknown constant parameter vector of the system, $f(x)$ is an $n \times 1$ matrix, $F(x)$ is an $n \times m$ matrix whose elements $F_{ab}(x) \in L_\infty$. The slave system is assumed by:

$$\dot{y} = g(y) + G(y)\beta + u \tag{2}$$

where $y \in \Omega_2 \subset R^n$ is the state vector, $\beta \in R^q$ is the unknown constant parameter vector of the system $g(y)$ is an $n \times 1$ matrix, $G(y)$ is an $n \times q$ matrix whose elements $G_{ab}(x) \in L_\infty$, and $u \in R^n$ is control input vector. If we divide the master and the slave systems into two parts, then system (1) can be written as:

$$\dot{x}_i = f_i(x) + F_i(x)\alpha_i \tag{3}$$

$$\dot{x}_j = f_j(x) + F_j(x)\alpha_j \tag{4}$$

and the slave system (2) can be written as

$$\dot{y}_i = g_i(y) + G_i(y)\beta_i + u \tag{5}$$

$$\dot{y}_j = g_j(y) + G_j(y)\beta_j + u \tag{6}$$

Let $e_i = y_i - x_i$ and $e_j = y_j + x_j$ be the synchronization and the anti–synchronization error vector's respectively. Our goal is to design a controller u such that the trajectory of the response system (5)-(6) with initial conditions $y_0 = (y_i(0), y_j(0))$ can asymptotically approach the drive system, (3)-(4), with initial condition $x_0 = (x_i(0), x_j(0))$. And finally implement the hybrid synchronization such that, $\lim_{t\to\infty}\|e_i\| = \lim_{t\to\infty}\|y_i(t, y_0) - x_i(t, x_0)\| = 0$ and the anti-synchronization such that

$$\lim_{t\to\infty}\|e_j\| = \lim_{t\to\infty}\|y_j(t, y_0) - x_j(t, x_0)\| = 0$$

where $\|\cdot\|$ is the Euclidean norm.

2.2. Adaptive Hybrid Synchronization Controller Design

Theorem: If the nonlinear control $u(t, x, y)$ is selected as:

$$u(t,x,y) = \begin{cases} f_i(t,x) + F_i(t,x)\hat{\alpha}_i - g_i(t,y) \\ \quad - G_i(t,y)\hat{\beta}_i - e_i \\ -f_j(t,x) - F_j(t,x)\hat{\alpha}_j - g_j(t,y) \\ \quad + G_j(t,y)\hat{\beta}_j - e_j \end{cases}$$

and adaptive laws of parameters are taken as:

$$\begin{cases} \dot{\hat{\alpha}}_i = -[F_i(t,x)]^{\mathrm{T}} e_i \\ \dot{\hat{\alpha}}_j = -[F_j(t,x)]^{\mathrm{T}} e_j \\ \dot{\hat{\beta}}_i = [G_i(t,y)]^{\mathrm{T}} e_i \\ \dot{\hat{\beta}}_j = [G_j(t,y)]^{\mathrm{T}} e_j \end{cases}$$

then the response system (5)-(6) can synchronize and anti-synchronize the drive system (3)-(4) globally and asymptotically, where $\hat{\alpha}_i, \hat{\alpha}_j, \hat{\beta}_i$ and $\hat{\beta}_j$ are respecttively, estimations of the unknown parameters $\alpha_i, \alpha_j, \beta_i$ and β_j.

Proof: From Equations (3)-(6), we get the error dynamical systems as follows:

$$\dot{e}_i = G_i(y)(\beta_i - \hat{\beta}_i) + F_i(x)(\alpha_i - \hat{\alpha}_i) - e_i \tag{7}$$

$$\dot{e}_j = G_j(y)(\beta_j - \hat{\beta}_j) + F_j(x)(\alpha_j - \hat{\alpha}_j) - e_j \tag{8}$$

$$\begin{aligned} \dot{e} &= F_i(x)(\alpha_i - \hat{\alpha}_i) - G_i(y)(\beta_i - \hat{\beta}_i) \\ &+ F_j(x)(\alpha_j - \hat{\alpha}_j) + G_j(y)(\beta_j - \hat{\beta}_j) - e \end{aligned} \tag{9}$$

where $e = e_i + e_j$.

Let $\tilde{\alpha}_i = \alpha_i - \hat{\alpha}_i, \tilde{\alpha}_j = \alpha_j - \hat{\alpha}, \tilde{\beta}_i = \beta_i - \hat{\beta}_i$ and

$\tilde{\beta}_j = \beta_j - \hat{\beta}_j$.

If a Lyapunov function candidate is chosen as

$$V\left(e_i, e_j, \tilde{\alpha}_i, \tilde{\alpha}_j, \tilde{\beta}_i, \tilde{\beta}_j\right) = \frac{1}{2}\left(e_i^{\mathrm{T}} e_i + e_j^{\mathrm{T}} e_j + \tilde{\alpha}_i^{\mathrm{T}} \tilde{\alpha}_i + \tilde{\alpha}_j^{\mathrm{T}} \tilde{\alpha}_j + \tilde{\beta}_i^{\mathrm{T}} \tilde{\beta}_i + \tilde{\beta}_j^{\mathrm{T}} \tilde{\beta}_j\right) \quad (10)$$

The time derivative of V along the error dynamical system is given by:

$$\begin{aligned}\dot{V} &= \dot{e}_i^{\mathrm{T}} e_i + \dot{e}_j^{\mathrm{T}} e_j + \tilde{\alpha}_i^{\mathrm{T}} \dot{\tilde{\alpha}}_i + \tilde{\alpha}_j^{\mathrm{T}} \dot{\tilde{\alpha}}_j + \tilde{\beta}_i^{\mathrm{T}} \dot{\tilde{\beta}}_i + \tilde{\beta}_j^{\mathrm{T}} \dot{\tilde{\beta}}_j \\ &= \left[-G_i(y)\tilde{\beta}_i + F_i(x)\tilde{\alpha}_i - e_i\right]^{\mathrm{T}} e_i - \tilde{\alpha}_i^{\mathrm{T}}\left[F_i(x)\right]^{\mathrm{T}} e_i \\ &\quad + \tilde{\beta}_i^{\mathrm{T}}\left[G_i(y)\right]^{\mathrm{T}} e_i + \left[G_j(y)\tilde{\beta}_j + F(x)\tilde{\alpha}_j - e_j\right]^{\mathrm{T}} e_j \\ &\quad - \tilde{\alpha}_j^{\mathrm{T}}\left[F_j(x)\right]^{\mathrm{T}} e_j - \tilde{\beta}_j^{\mathrm{T}}\left[G_j(y)\right]^{\mathrm{T}} e_j \end{aligned} \quad (11)$$

$$= -\left(e_i^{\mathrm{T}} e_i - e_j^{\mathrm{T}} e_j\right) \le 0) \quad (12)$$

Since V is positive definite, and $\dot{V}$ is negative semi-definite, it follows that from the fact that

$$\int_0^t \|e\|^2 \mathrm{d}t = \frac{1}{2}\left[V(0) - V(t)\right] \le V(0).$$

It can easily be seen that $\dot{e} \in L_\infty$. From Equation (9) have $\lim_{t\to\infty} e = 0$. Thus, by Barbalat's lemma, we have $\lim_{t\to\infty}\left\|y(t, y_0) \pm x(t, x_0)\right\| = 0.$ Thus the response system (2) can be synchronized and anti-synchronized the drive system (1) globally and asymptotically. This completes the proof.

2.3. Systems Description

The hyperchaotic Chen system [26,27] is given by:

$$\begin{aligned}\dot{x} &= a(y - x) + w, \\ \dot{y} &= dx - xz + cy, \\ \dot{z} &= xy - bz, \\ \dot{w} &= yz + rw.\end{aligned} \quad (13)$$

where x, y, z and w are state variables, and a, b, c, d, and r are real constants. When $a = 35, b = 3, c = 12, d = 7, 0 < r < 0.085$, system (13) is chaotic, when $a = 35, b = 3, c = 12, d = 7, 0.085 < r < 0.789$, system (13) is hyperchaotic.

The hyperchaotic Lorenz system [28,29] is described by

$$\begin{aligned}\dot{x} &= a(y - x) + w, \\ \dot{y} &= -xz + rx - y, \\ \dot{z} &= xy - bz, \\ \dot{w} &= -xz + dw.\end{aligned} \quad (14)$$

where x, y, z, and w are state variables, a, b, c and d are real constants. When $a = 36, b = 3, c = 20$ and $d = 1.3$, system (14) has hyperchaotic attractor.

The hyperchaotic Lű system [30] is described by:

$$\begin{aligned}\dot{x} &= a(y - x) + w, \\ \dot{y} &= -xz + cy, \\ \dot{z} &= xy - bz, \\ \dot{w} &= xz + rw.\end{aligned} \quad (15)$$

where x, y, z and w are state variables, a, b, c and r are real constants. When $a = 36, b = 3, c = 20, -0.35 < r < 1.3$, system (15) has hyperchaotic attractor.

3. Adaptive Hybrid Synchronization of Two Identical Hyperchaotic Systems with Unknown Parameters

In order to observe the efficacy of our proposed method, we used two hyperchaotic Chen systems where the master system is denoted with the subscript 1 and the response system having identical equations denoted by the subscript 2. The two systems are defined below.

$$\begin{aligned}\dot{x}_1 &= a(y_1 - x_1) + w_1, \\ \dot{y}_1 &= -x_1 z_1 + c y_1, \\ \dot{z}_1 &= x_1 y_1 - b z_1, \\ \dot{w}_1 &= x_1 z_1 + d w_1.\end{aligned} \quad (16)$$

and

$$\begin{aligned}\dot{x}_2 &= a(y_2 - x_2) + w_2 + u_1, \\ \dot{y}_2 &= -x_2 z_2 + c y_2 + u_2, \\ \dot{z}_2 &= x_2 y_2 - b z_2 + u_3, \\ \dot{w}_2 &= x_2 z_2 + d w_2 + u_4.\end{aligned} \quad (17)$$

where u_1, u_2, u_3, u_4 are four control functions to be designed. For the hybrid synchronization, we define the state errors between the response system that is to be controlled and the controlling drive system as $e_1 = x_2 - x_1, e_2 = y_2 + y_1$, $e_3 = z_2 - z_1$, $e_4 = w_2 + w_1$. The error system is given by

$$\begin{aligned}e_1 &= a(e_2 - e_1) + e_4 - 2ay_1 - 2w_1 + u_1, \\ e_2 &= de_1 - x_2 z_2 - x_1 z_1 + ce_2 + 2dx_1 + u_2, \\ e_3 &= x_2 y_2 - x_1 y_1 - be_3 + u_3, \\ e_4 &= y_2 z_2 + y_1 z_1 + re_4 + u_4\end{aligned} \quad (18)$$

Now our goal is to find proper control functions $u_i\ (i = 1,2,3,4)$ and parameter update rule, such that system (17) globally hybrid synchronizes system (16) asymptotically. *i.e.*, $\lim_{t\to\infty}\|e\| = 0$, where $e = [e_1, e_2, e_3, e_4]^{\mathrm{T}}$. If the two systems are without controls $u_i\ (i = 1,2,3,4)$ and the initial condition is:

$$\left(x_1(0), y_1(0), z_1(0), w_1(0)\right) \ne \left(x_2(0), y_2(0), z_2(0), w_2(0)\right)$$

then the trajectories of the two systems will quickly separate each other and become irrelevant. However, when appropriate controls are applied the two systems will approach hybrid synchronization for any initial conditions. We shall propose the following adaptive control law for system (17).

$$\tilde{a} = a - \hat{a}, \tilde{b} = b - \hat{b}, \tilde{c} = c - \hat{c}, \tilde{r} = r - \hat{r}, \tilde{d} = d - \hat{d}$$

where $\hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{r}$ are the estimates of a, b, c, d, r respectively. Now, let us choose a controller U and parameters update law $\dot{\hat{a}}, \dot{\hat{b}}, \dot{\hat{c}}, \dot{\hat{d}}, \dot{\hat{r}}$ as follows:

$$\begin{aligned} u_1 &= -\hat{a}(e_2 - e_1) - e_4 + 2ay_1 + 2w_1 - e_1, \\ u_2 &= -\hat{d}e_1 + x_2 z_2 + x_1 z_1 - \hat{c}e_2 - 2dx_1 - e_2, \\ u_3 &= -x_2 y_2 + x_1 y_1 + \hat{b}e_3 - e_3, \\ u_4 &= -y_2 z_2 - y_1 z_1 + \hat{r}e_4 - e_4. \end{aligned} \tag{19}$$

and the parameter update rule.

Consider the following Lyapunov function

$$\begin{aligned} &\dot{\hat{a}} = e_1 e_2 - e_1^2, \dot{\hat{b}} = -e_3^2, \dot{\hat{c}} = e_2^2, \\ &\dot{\hat{d}} = e_1 e_2, \dot{\hat{r}} = e_4^2 \end{aligned} \tag{20}$$

Consider the following Lyapunov function

$$V = \frac{1}{2}\left(e^{\mathrm{T}} e + \tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2 + \tilde{d}^2 + \tilde{r}^2\right).$$

Then the time derivative of V along the trajectories of Equation (18) is:

$$\begin{aligned} \dot{V} &= \left(e^T \dot{e} + \tilde{a}\dot{\tilde{a}} + \tilde{b}\dot{\tilde{b}} + \tilde{c}\dot{\tilde{c}} + \tilde{d}\dot{\tilde{d}} + \tilde{r}\dot{\tilde{r}}\right) \\ &= \dot{e}_1\left[\tilde{a}(e_2 - e_1) - e_1\right] + e_2\left[\tilde{d}e_{11} + \tilde{c}e_2 - e_2\right] \\ &\quad + e_3\left[-\tilde{b}e_3 - e_3\right] + e_4\left[\tilde{r}e_4 - e_4\right] + \tilde{a}\left(-(e_2 - e_1)\right) \\ &\quad + \tilde{b}\left(e_3^2\right) + \tilde{c}\left(-e_2^2\right) + \tilde{d}\left(-e_1 e_2\right) + \tilde{r}\left(-e_4^2\right) \\ &= -e_1^2 - e_2^2 - e_3^2 - e_4^2 \end{aligned} \tag{21}$$

Since V is positive definite function and $\dot{V}$ is negative definite function, it translates to $\lim_{t \to \infty} \|e\| = 0$ based on the Lyapunov stability theorem [31]. Therefore, the hyperchaotic Chen response system (17) is hybrid synchronized with hyperchaotic Chen drive system (16) with fully uncertain parameters under the adaptive controller (19) and the parameters update law (20).

Numerical Simulations

To verify and demonstrate the effectiveness of the proposed method, we discuss the simulation results for hyperchaotic Chen system. In the numerical simulations, the fourth-order Runge-Kutta method is used to solve the systems with time step size 0.001. For these numerical simulations, we used the initial conditions, $(x_1(0), y_1(0), z_1(0), w_1(0)) = (5, 8, -1, -3)$ and $(x_2(0), y_2(0), z_2(0), w_2(0)) = (3, 4, 5, 5)$. Hence, the error system has the initial values $e_1(0) = -2, e_2(0) = 12, e_3(0) = 6$ and $e_4(0) = 2$. The unknown parameters are chosen as $a = 36, b = 3, c = 12, r = 0.5$ and $d = 7$ such that the hyperchaotic Chen system exhibits chaotic behavior. Hybrid synchronization of systems (16) and (17) via adaptive control laws (Equations (19) and (20)) with the initial estimated parameters $\hat{a}(0) = 5, \hat{b}(0) = 11, \hat{c}(0) = 2, \hat{r}(0) = 8$ and $\hat{d}(0) = 6$ are shown in **Figures 1** and **2**. **Figures 1(a)** and **(d)** display state trajectories of drive system (16) and the response system (17). **Figure 2(a)** displays the hybrid synchronization errors between system (16) and (17). **Figure 2(b)** Shows that the estimates $\hat{a}(t), \hat{b}(t), \hat{c}(t), \hat{r}(t)$ and $\hat{d}(t)$ of the unknown parameters converges to $a = 36, b = 3, c = 12, r = 0.5$ and $d = 7$ as $t \to \infty$.

4. Adaptive Hybrid Synchronization between Two Different Hyperchaotic Systems

In order to observe the hybrid synchronization behavior between hyperchaotic Lorenz system (15) and hyperchaotic Lű system (14), we assume that hyperchaotic Lorenz system with four unknown parameters is the drive system and hyperchaotic Lű system with four unknown parameters is the response system. The drive and response systems are defined as follows:

$$\begin{aligned} \dot{x}_1 &= a_1(y_1 - x_1) + w_1, \\ \dot{y}_1 &= -x_1 z_1 + r_1 x_1 - y_1, \\ \dot{z}_1 &= x_1 y_1 - b_1 z_1, \\ \dot{w}_1 &= -x_1 z_1 + d_1 w_1. \end{aligned}$$

and

$$\begin{aligned} \dot{x}_2 &= a_2(y_2 - x_2) + w_2 + u_1, \\ \dot{y}_2 &= -x_2 z_2 + c_2 y_2 + u_2, \\ \dot{z}_2 &= x_2 y_2 - b_2 z_2 + u_3, \\ \dot{w}_2 &= x_2 z_2 + r_2 w_2 + u_4. \end{aligned} \tag{23}$$

where u_1, u_2, u_3, u_4 are four control functions to be designed. For the hybrid synchronization, we define the state errors between the response system that is to be controlled and the controlling drive system as $e_1 = x_2 - x_1, e_2 = y_2 + y_1, e_3 = z_2 - z_1, e_4 = w_2 + w_1$.

The error system is given by

$$\begin{aligned} e_1 &= a_2(y_2 - x_2) + w_2 - a_1(y_1 - x_1) - w_1 + u_1, \\ e_2 &= -x_2 z_2 + c_2 y_2 - x_1 z_1 + r_1 x_1 - y_1 + u_2, \\ e_3 &= x_2 y_2 - b_2 z_2 - x_1 y_1 + b_1 z_1 + u_3, \\ e_4 &= x_2 z_2 + r_2 w_2 - x_1 z_1 + d_1 w_1 + u_4. \end{aligned} \tag{24}$$

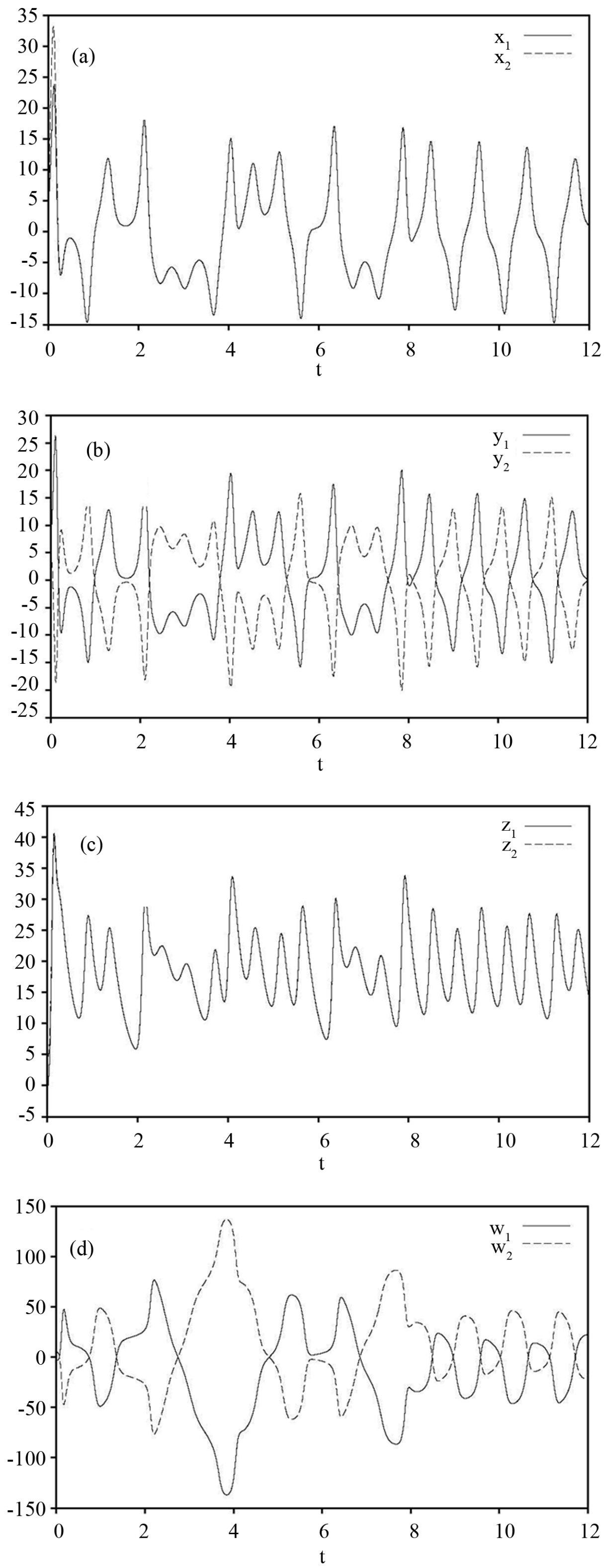

Figure 1. State trajectories of the drive system (16) and the response system (17). (a) Signals x_1 and x_2; (b) Signals y_1 and y_2; (c) Signals z_1 and z_2; and (d) Signals w_1 and w_2.

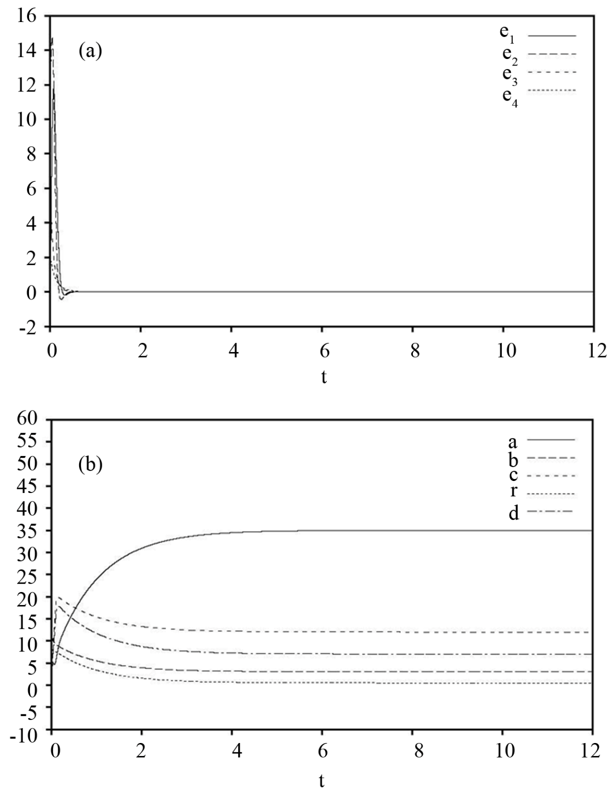

Figure 2. (a) Hybrid synchronization errors e_1, e_2, e_3, e_4 of the drive system (16) and the response system (17) with time t; (b) Changing parameters a,b,c,r,d of the drive system (16) and the response system (17) with time t.

Now our goal is to find proper control functions $u_i\left(i=1,2,3,4\right)$ and parameter update rule, such that system (17) globally hybrid synchronizes system (16) asymptotically. *i.e.* $\lim_{t\to\infty}\|e\|=0$ where $e=\left[e_1,e_2,e_3,e_4\right]^{\mathrm{T}}$. If the two systems are without controls $u_i\left(i=1,2,3,4\right)$ and the initial condition is

$$\left(x_1(0),y_1(0),z_1(0),w_1(0)\right)$$
$$\neq\left(x_2(0),y_2(0),z_2(0),w_2(0)\right)$$

then the trajectories of the two systems will quickly separate each other and become irrelevant. However, when appropriate controls are applied the two systems will approach hybrid synchronization for any initial conditions. We shall propose the following adaptive control law for system (23). We define the parameters error $\tilde{a}_1=a_1-\hat{a}_1,\tilde{b}_1=b_1-\hat{b}_1,\tilde{r}_1=r_1-\hat{r}_1,\tilde{d}_1=d_1-\hat{d}_1$ and $\tilde{a}_2=a_2-\hat{a}_2,\tilde{b}_2=b_2-\hat{b}_2,\tilde{r}_2=r_2-\hat{r}_2,\tilde{c}_2=c_2-\hat{c}_2$ where $\hat{a}_1,\hat{b}_1,\hat{d}_1,\hat{r}_1$ and $\hat{a}_2,\hat{b}_2,\hat{c}_2,\hat{r}_2$ are the estimates of a_1,b_1,d_1,r_1 and a_2,b_2,c_2,r_2 respectively. Now, let us choose a controller U and parameters update law $\dot{\hat{a}}_1,\dot{\hat{b}}_1,\dot{\hat{d}}_1,\dot{\hat{r}}_1,\dot{\hat{a}}_2,\dot{\hat{b}}_2,\dot{\hat{c}}_2,\dot{\hat{r}}_2$ as follows:

$$
\begin{aligned}
u_1 &= -\hat{a}_2\left(y_2 - x_2\right) - w_2 + \hat{a}_1\left(y_1 - x_1\right) + w_1 - e_1,\\
u_2 &= x_2 z_2 - \hat{c}_2 y_2 + x_1 z_1 - \hat{r}_1 x_1 + y_1 - e_2,\\
u_3 &= -x_2 y_2 + \hat{b}_2 z_2 + x_1 y_1 - \hat{b}_1 z_1 - e_3,\\
u_4 &= -x_2 z_2 - \hat{r}_2 w_2 + x_1 z_1 - \hat{d}_1 w_1 - e_4.
\end{aligned} \tag{25}
$$

and the parameter update rule

$$
\begin{aligned}
\dot{\hat{a}}_1 &= -\left(y_1 - x_1\right) e_1,\\
\dot{\hat{b}}_1 &= z_1 e_3,\\
\dot{\hat{r}}_1 &= x_1 e_2,\\
\dot{\hat{d}}_1 &= w_1 e_4\\
\dot{\hat{a}}_2 &= \left(y_2 - x_2\right) e_1,\\
\dot{\hat{b}}_2 &= -z_2 e_3,\\
\dot{\hat{c}}_2 &= y_2 e_2,\\
\dot{\hat{r}}_2 &= w_2 e_4.
\end{aligned} \tag{26}
$$

Consider the following Lyapunov function

$$V = \frac{1}{2}\left(e^{\mathrm{T}} e + \tilde{a}_1^{\,2} + \tilde{b}_1^{\,2} + \tilde{d}_1^{\,2} + \tilde{r}_1^{\,2} + \tilde{a}_2^{\,2} + \tilde{b}_2^{\,2} + \tilde{c}_2^{\,2} + \tilde{r}_2^{\,2}\right).$$

Then the time derivative of V along the trajectories of Equation (24) is

$$
\begin{aligned}
\dot{V} &= \Big(e^{\mathrm{T}}\dot{e} + \tilde{a}_1\dot{\tilde{a}}_1 + \tilde{b}_1\dot{\tilde{b}}_1 + \tilde{d}_1\dot{\tilde{d}}_1 + \tilde{r}_1\dot{\tilde{r}}_1 + \tilde{a}_2\dot{\tilde{a}}_2\\
&\qquad + \tilde{b}_2\dot{\tilde{b}}_2 + \tilde{c}_2\dot{\tilde{c}}_2 + \tilde{r}_2\dot{\tilde{r}}_2\Big)\\
&= \dot{e}_1\left[\hat{a}_2\left(y_2 - x_2\right) - \hat{a}_1\left(y_1 - x_1\right) - e_1\right]\\
&\quad + e_2\left[\hat{c}_2 y_2 + \hat{r}_1 x_1 - e_2\right]\\
&\quad + e_3\left[-\hat{b}_2 z_2 + \hat{b}_1 z_1 - e_3\right] + e_4\left[\hat{r}_2 w_2 + \hat{d}_1 w_1 - e_4\right]\\
&\quad + \tilde{a}_1\left(\left(y_1 - x_1\right)e_1\right) + \tilde{b}_1\left(-z_1 e_3\right) + \tilde{d}_1\left(-w_1 e_4\right)\\
&\quad + \tilde{r}_1\left(-x_1 e_2\right) + \tilde{a}_2\left(-\left(y_2 - x_2\right)e_1\right) + \tilde{b}_2\left(z_2 e_3\right)\\
&\quad + \tilde{c}_2\left(y_2 e_2\right) + \tilde{r}_2\left(-w_2 e_4\right)\\
&= -e_1^{\,2} - e_2^{\,2} - e_3^{\,2} - e_4^{\,2}
\end{aligned} \tag{27}
$$

Since V is positive definite function and $\dot{V}$ is negative definite function, it translates to $\lim_{t\to\infty}\|e\| = 0,$ based on the Lyapunov stability theorem [31]. Therefore, the hyperchaotic Lű response system (14) is hybrid synchronized the hyperchaotic Lorenz drive system (15) with fully uncertain parameters under the adaptive controller (25) and the parameters update law (26).

Numerical Simulations

To verify and demonstrate the effectiveness of the proposed method, we discuss the simulation result for the hybrid synchronization between hyperchaotic Lorenz

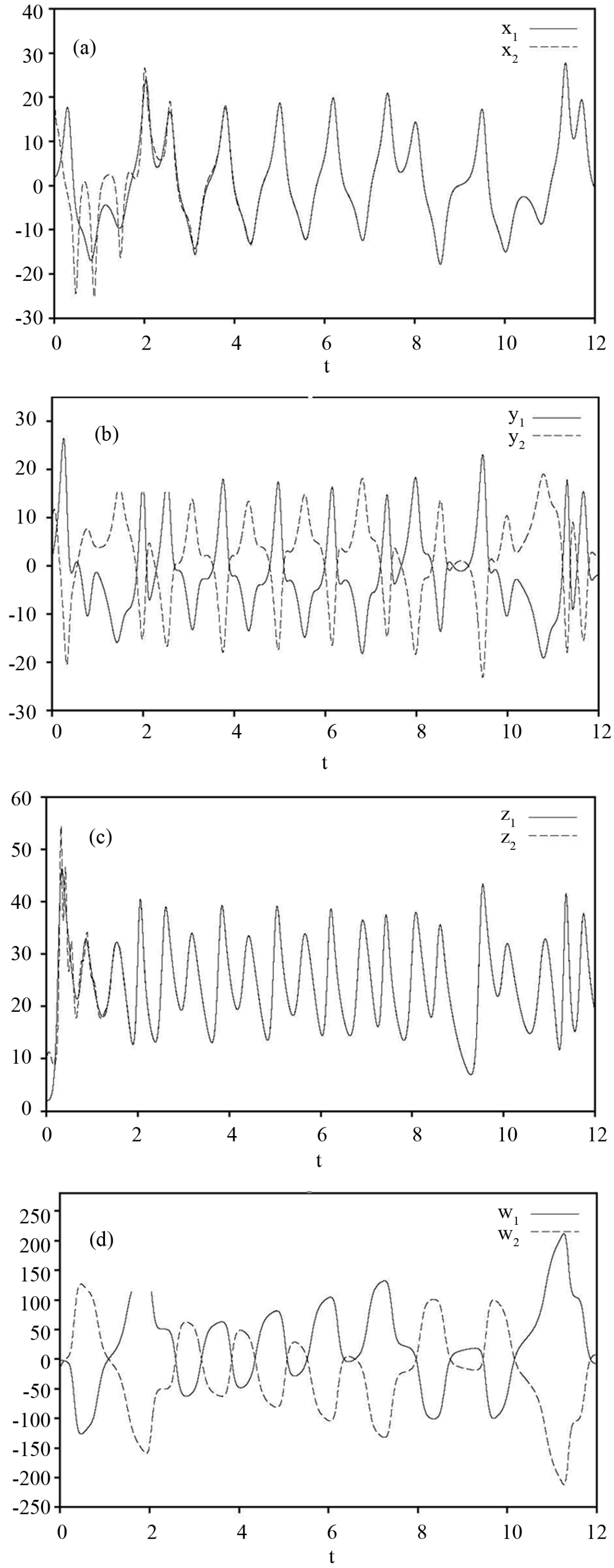

Figure 3. State trajectories of the drive system (22) and the response system (23). (a) Signals x_1 and x_2; (b) Signals y_1 and y_2; (c) Signals z_1 and z_2; and (d) Signals w_1 and w_2.

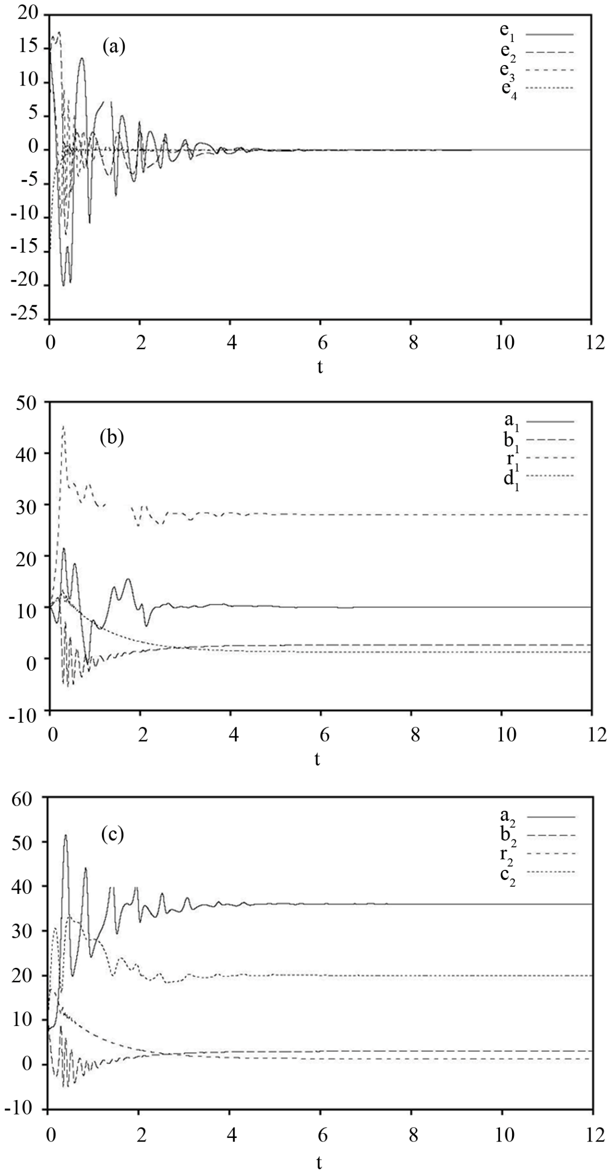

Figure 4. (a) Hybrid synchronization errors, e_1, e_2, e_3, e_4 of the drive system (22) and the response system (23) with time t; (b) and (c) Changing parameters $a_1, b_1, r_1, d_1, a_2, b_2, r_2, c_2$ of the drive system (22) and the response system (23) with time t.

system and hyperchaotic Lű system. In the numerical simulations, the fourth-order Runge-Kutta method is used to solve the systems with time step size 0.001. For this numerical simulation, we assume that the initial condition, $(x_1(0), y_1(0), z_1(0), w_1(0)) = (2,3,2,-2)$, and $(x_2(0), y_2(0), z_2(0), w_2(0)) = (20,10,10,-15)$ is employed. Hence the error system has the initial values $e_1(0)=18, e_2(0)=12, e_3(0)=8$ and $e_4(0)=-17$. The unknown parameters are chosen as

$a_1 = 10, b_1 = \frac{8}{3}, d_1 = 1.3,\ r_1 = 28$ and

$a_2 = 36, b_2 = 3, c_2 = 20, r_2 = 1.3$ in simulations so that both the systems exhibits a hyperchaotic behavior. Hybrid synchronization of and (26) with the initial estimated parameters

$a_1(0) = 10, b_1(0) = 10, d_1(0) = 10, r_1(0) = 10$ and $a_2(0) = 10, b_2(0) = 10, c_2(0) = 10, r_2(0) = 10$ are shown in **Figures 3** and **4**. **Figure 3** displays state trajectories of drive system (22) and the response system (23). **Figure 4(a)** displays the hybrid synchronization errors between system (22) and (23). **Figures 3(b)** and **(c)** show that the estimates $\hat{a}_1(t), \hat{b}_1(t),\ \hat{d}_1(t),\ \hat{r}_1(t)$ and $a_2(t), b_2(t), c_2(t), r_2(t)$ of the unknown parameters converges to $a_1 = 10, b_1 = \frac{8}{3}, d_1 = 1.3, r_1 = 28$ and $a_2 = 36,\ b_2 = 3, c_2 = 20, r_2 = 1.3$ as $t \to \infty$.

5. Conclusion

In this paper, we discussed the problem of adaptive hybrid synchronization of hyperchaotic systems with fully unknown parameters. On the basis of the Lyapunov stability theory and the adaptive control theory, a new adaptive hybrid synchronization control law and a novel parameter estimation update law are proposed to achieve hybrid synchronization between the two identical and different hyperchaotic systems with uncertain parameters. This shows that our proposed method has strong robustness. Finally, the simulation results are presented to show the effectiveness of this approach.

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