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Representation of Functions in L_μ^1 Weighted Spaces by Series with Monotone Coefficients in the Walsh Genrealized System*

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ABSTRACT

Let $\{\varphi_n(x)\}$ be the Walsh generalized system. In the paper constructed a weighted space L_μ^1, and series $\sum a_n\varphi_n$ in the Walsh generalized system with monotonically decreasing coefficient $|a_n| \downarrow 0$ such that for each function $f(x) \in L_\mu^1$ in the space one can find a subseries $\sum a_{n_k}\varphi_{n_k}(x)$ that converges to $f(x)$ in the weighted L_μ^1 and almost everywhere on $[0,1]$.

Keywords: Orthonormal System; Convergence; Functional Series

1. Introduction

In the present paper we study the following natural question: does there exist a weighted space $L_\mu^1(0,1)$, with $0 < \mu(x) \le 1$, such that for every function in the space

$$L_\mu^1(0,1) = \left\{f; \int_0^1 |f(x)|\mu(x)\mathrm{d}x < \infty\right\}$$

one can find a series in the Walsh generalized system $\{\varphi_n\}$ of the form

$$\sum_{n=1}^{\infty} a_n\varphi_n, \text{ with } |a_n| \searrow 0,$$

that possess the following property: *for any function $f \in L_\mu^1(0,1)$ there exists a growing sequence of natural numbers n_k such that the subseries $\sum_{k=1}^{\infty} a_{n_k}\varphi_{n_k}$ converges to f in the $L_\mu^1(0,1)$ – norm and a.e.*

Note that the problem of representing a function f by a series in classical and general orthonormal systems has a long history. Of course the problem of the representation of functions was studied before Luzin's work. It goes back to D. Bernoulli, L. Euler and many others.

A question posed by Lusin in 1915 asks whether it is possible to find for every measurable function $[0,2\pi]$ a trigonometric series, with coefficient sequence converging to zero, that converges to the function almost everywhere. For real-valued functions, this question was given an affirmative answer by Men'shov [1] in 1941.

There are many other works (see [2-11]) devoted to representations of functions by series in classical and general orthonormal systems and the existence of different types of universal series in the sense of convergence almost is everywhere and by measure.

Since the trigonometric and Walsh systems have many properties in common, one would think that there should be a corresponding result for the Walsh system. This is, indeed, the case, and, in fact, the same sort of result holding for a multitude of Walsh subsystems, many of them are quite sparse and far from complete.

In this paper we prove the following theorem:

Theorem 1. *For any $0 < \delta < 1$ there exists a measurable function $\mu(x), 0 < \mu(x) \le 1$, with*

$$|\{x \in [0,1]; \mu(x) = 1\}| > 1 - \delta,$$

such that for any $p \ge 1$ and any function $f \in L_\mu^1(0,1)$ there exists a series in the Walsh generalized system $\{\varphi_n\}$ of the following form

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$$\sum_{i=1}^{\infty} a_i \varphi_{k_i}, \text{ where } |a_i| \searrow 0 \text{ and } k_1 < k_2 < \cdots, \quad (1)$$

which converges to f in the $L^1_\mu(0,1)$—metric and almost everywhere.

Note that there exist functions in the space $L^1(0,1)$ that can not be represented by series in the Walsh system $\{\varphi_k\}$ (see [8], pp. 124-125).

Theorem 1 *is a consequence of the more general Theorem* 2, *which is stated as follows*:

Theorem 2. *For any* $0 < \delta < 1$ *there exists a measurable function* $\mu(x), 0 < \mu(x) \le 1$, *with*

$$\left|\{x \in [0,1]; \mu(x) = 1\}\right| > 1 - \delta,$$

and a series in the Walsh generalized system $\{\varphi_n\}$ of the form

$$\sum_{n=0}^{\infty} a_n \varphi_n, \text{ with } |a_n| \searrow 0,$$

that possess the following property: *for any function* $f \in L^1_\mu(0,1)$ *there exists a growing sequence of natural numbers* n_k *such that the subseries*

$$\sum_{k=1}^{\infty} a_{n_k} \varphi_{n_k}$$

converges to f in the $L^1_\mu(0,1)$-norm and a.e.

Recall the following definition: a series $\sum_{n=1}^{\infty} a_n \varphi_n$ is said to be universal with respect to subseries in the space $L^1_\mu(0,1)$, if for each function $f(x) \in L^1_\mu(0,1)$, one can select a subseries $\sum_{k=1}^{\infty} a_{n_k} \varphi_{n_k}$ which converges to $f(x)$ in $L^1_\mu(0,1)$ norm.

The above-mentioned definitions are given not in the most general form and only in the generality, in which they will be applied in the present paper.

Note that the result of the Theorem 2 is definitive in a certain sense: one can not replace $L^1_\mu(E)$ by $L^1(0,1)$ because no orthonormal system of bounded functions does there exist a series universal in $L^1(0,1)$ with respect to subseries. This is almost obvious.

The following problems remain open.

Question 1. *Are the theorems* 1 *and* 2 *true for the trigonometric system*?

Question 2. *What kind of necessary and sufficient conditions should be imposed on the weight function* $\mu(x)$ *in order to construct a Walsh series* $\sum_{n=1}^{\infty} a_n \varphi_n$ *to be universal in the space with respect to subseries*?

2. Proofs of Main Lemmas

Let $a \ge 2$ be a fixed integer and $\omega_a = e^{\frac{2\pi i}{a}}$. Recall the following definitions.

The Rademacher system of order a is defined inductively as follows. For $n = 0$ let

$$R_0(x) = \omega_a^k \quad \text{if} \quad x \in \left[\frac{k}{a}, \frac{k+1}{a}\right), k = 0, 1, \cdots, a-1,$$

and for $n \ge 1$ let

$$R_n(x+1) = R_n(x) = R_0(a^n x).$$

The Walsh generalized system (see [3] and [13,14]) of order a is defined by

$$\varphi_0(x) = 1,$$

and if $n = \alpha_1 a^{n_1} + \cdots + \alpha_s a^{n_s}$, where $n_1 > \cdots > n_s$, $0 \le \alpha_j < a, j = 1, 2, \cdots, s$ then

$$\varphi_n(x) = R_{n_1}^{\alpha_1}(x) \cdots R_{n_s}^{\alpha_s}(x).$$

We denote the generalized Walsh system of order a by Ψ_a. Note that Ψ_2 is the classical Walsh system. The basic properties of the generalized Walsh system of order a have been obtained by H. E. Chrestenson, J. Fine, C. Vateri, W. Young, N. Vilenkin and others. Next we list some properties of Ψ_a, which will be useful later.

- Each n-th Rademacher function has period a^{-n}.
- $(R_n(x))^k = (R_n(x))^m$, $\forall n, k \in \mathcal{N}$, and $m = k$ (mod a).
- $\varphi_n(x)$ is a finite product of Rademacher functions with values in Ω_a.
- $\varphi_{a^k + j}(x) = R_k(x) \varphi_j(x)$ if $0 \le j \le a^k - 1$.
- φ_a, $a \ge 2$ is a complete orthonormal system in $L^2[0,1)$ and it is basic in $L^p[0,1]$ for $p > 1$.

We put

$$I_k^{(j)}(x) = \begin{cases} 1, & \text{if } x \in [0,1] \setminus \Delta_k^{(j)}, 1 - a^k, \\ 1 - a^k, & \text{if } x \in \Delta_k^{(j)} = \left(\dfrac{j-1}{a^k}, \dfrac{j}{a^k}\right) \end{cases}; \quad (2)$$

$$k = 1, 2, \cdots, 1 \le j \le a^k,$$

and periodically extend these functions on R^1 *with period* 1.

By $\chi_E(x)$ *we denote the characteristic function of the set* E, *i.e.*

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{if } x \notin E. \end{cases} \quad (3)$$

Then, clearly

$$I_k^{(j)}(x) = \varphi_0(x) - a^k \cdot \chi_{\Delta_k^{(j)}}(x), \quad (4)$$

and let for the natural numbers $k \ge 1$, and $j \in [1, a^k]$

$$b_i\left(\chi_{\Delta_k^{(j)}}\right) = \int_0^1 \chi_{\Delta_k^{(j)}}(x) \varphi_i(x) \mathrm{d}x = \pm \frac{1}{a^k}, \quad (5)$$

$$0 \le i < a^k$$

$$a_i\left(I_k^{(j)}\right) = \int_0^1 I_k^{(j)}(x) \varphi_i(x) \mathrm{d}x = \begin{cases} 0, & \text{if } i = 0 \text{ or } i \ge a^k \\ \pm 1, & \text{if } 1 \le i < a^k. \end{cases} \quad (6)$$

Hence

$$\chi_{\Delta_k^{(j)}}(x)=\sum_{i=0}^{a^k-1} b_i\left(\chi_{\Delta_k^{(j)}}\right)\varphi_i(x) \qquad (7)$$

$$I_k^{(j)}(x)=\sum_{i=1}^{a^k-1} a_i\left(I_k^{(j)}\right)\varphi_i(x). \qquad (8)$$

Lemma 1. *Let dyadic interval*

$$\Delta=\Delta_m^{(k)}=\left((k-1)/a^m;k/a^m\right),\ k\in\left[1,a^m\right]$$

and numbers $N_0\in\mathbb{N},\ \gamma\neq 0,\ \varepsilon\in(0,1)$ *be given. Then there exists a measurable set* $E\subset[0,1]$ *and a polynomial* Q *in the Walsh generalized system* $\{\varphi_k\}$ *of the following form*

$$Q=\sum_{k=N_0}^{N} c_k\varphi_k$$

which satisfy the following conditions:

1) *the coefficients* $\{c_k\}_{k=N_0}^{N}$ *are* 0 *or* $\pm\gamma|\Delta|$,

2) $|E|>(1-\varepsilon)|\Delta|$,

3) $Q(x)=\begin{cases}\gamma: & \text{if } x\in E\\ 0: & \text{if } x\notin\Delta\end{cases}$,

4) $\max_{N_0\le m\le N}\left(\int_0^1\left|\sum_{k=N_0}^{m} c_k\varphi_k(x)\right|\mathrm{d}x\right)<A|\gamma|\varepsilon^{-1/2}|\Delta|^{1/2}$,

where A *is a constant*,

5) $\max_{N_0\le m\le N}\left(\left|\sum_{k=N_0}^{m} c_k\varphi_k(x)\right|\right)<\frac{|\gamma|}{\varepsilon}, x\in[0;1]$.

Proof. *Let*

$$\nu_0=\left[\log_a\frac{1}{\varepsilon}\right]+1;\ s=\left[\log_a N_0\right]+m \qquad (9)$$

We define the polynomial $Q(x)$ *and the numbers* c_n, a_i *and* b_j *in the following form*:

$$Q(x)=\gamma\cdot\chi_{\Delta_m^{(k)}}(x)\cdot I_{\nu_0}^{(1)}\left(a^s x\right),\ x\in[0;1]. \qquad (10)$$

$$c_n=c_n(Q)=\int_0^1 Q(x)\varphi_n(x)\mathrm{d}x,\ \forall n\ge 0. \qquad (11)$$

$$b_i=b_i\left(\chi_{\Delta_m^{(k)}}\right),\ 0\le i<a^m,$$
$$a_j=a_j\left(I_{\nu_0}^{(1)}\right),\ 0<j<a^{\nu_0}. \qquad (12)$$

Taking into consideration the following equation

$$\varphi_i(x)\cdot\varphi_j\left(a^s x\right)=\varphi_{j\cdot a^s+i}(x),$$
$$\text{if } 0\le i,j<a^s\ (\text{see}(1)),$$

and having the following relations (5)-(8) and (10)-(12), we obtain that the polynomial $Q(x)$ has the following form:

$$\begin{aligned}Q(x)&=\gamma\cdot\sum_{i=0}^{a^m-1} b_i\varphi_i(x)\cdot\sum_{j=1}^{a^{\nu_0}-1} a_j\varphi_j\left(a^s x\right)\\&=\gamma\cdot\sum_{j=1}^{a^{\nu_0}-1} a_j\cdot\sum_{i=0}^{a^m-1} b_i\varphi_{j\cdot a^s+i}(x)\\&=\sum_{k=N_0}^{\bar{N}} c_k\varphi_k(x),\end{aligned} \qquad (13)$$

where

$$c_k=c_k(Q)=\begin{cases}\pm\frac{\gamma}{a^m}\text{ or } 0 & \text{if } k\in\left[N_0,\bar{N}\right]\\ 0, & \text{if } k\notin\left[N_0,\bar{N}\right]\end{cases}, \qquad (14)$$

$\bar{N}=a^{s+\nu_0}+a^m-a^s-1.$

Then let

$$E=\{x;Q(x)=\gamma\}.$$

Clearly that (*see* (2) *and* (10)),

$$|E|=a^{-m}\left(1-a^{-\nu_0}\right)>(1-\varepsilon)|\Delta|, \qquad (15)$$

$$Q(x)=\begin{cases}\gamma, & \text{if } x\in E\\ \gamma\left(1-a^{\nu_0}\right), & \text{if } x\in\Delta\setminus E\\ 0, & \text{if } x\notin\Delta\end{cases} \qquad (16)$$

$$c_n=\int_0^1 Q(x)\varphi_n(x)dx,\ \forall n\in\mathbb{N}.$$

Hence

$$\begin{aligned}&\max_{N_0\le m\le N}\int_0^1\left|\sum_{k=N_0}^{m} c_k\varphi_k(x)\right|\mathrm{d}x\\&\le\max_{N_0\le m\le N}\left(\int_0^1\left|\sum_{k=N_0}^{m} c_k\varphi_k(x)\right|^2\mathrm{d}x\right)^{\frac{1}{2}}\\&\le\left(\int_0^1 Q^2(x)\mathrm{d}x\right)^{\frac{1}{2}}\le A|\gamma|\varepsilon^{-1/2}|\Delta|^{1/2},\end{aligned}$$

where $A=4\bar{A}$, Repeating the arguments in the proof of Lemma 1, we get a proof of the last statement of Lemma 1. *Lemma* 1 *is proved.*

Lemma 2. *Let given the numbers* $\tilde{N}\in\mathbb{N}, 0<\varepsilon<1$. *Then for any function* $f\in L^1(0,1)$, $\|f\|_{L_1}>0$, *one can find a set* $E\subset[0,1]$ *and a polynomial in the Walsh generalized system*

$$Q=\sum_{k=\tilde{N}+1}^{M} a_k\varphi_k,$$

satisfying the following conditions:

1) $0\le|a_k|<\varepsilon$ *and the non-zero coefficients in* $\{|a_k|\}_{k=\tilde{N}+1}^{M}$ *are in decreasing order*,

2) $|E|>1-\varepsilon$,

3) $\left(\int_E |Q(x) - f(x)|\,\mathrm{d}x\right) < \varepsilon,$

4) $\max_{\tilde{N}+1 \le m \le M} \left(\int_e \left|\sum_{k=\tilde{N}+1}^{m} a_k \varphi_k(x)\right| \mathrm{d}x\right) < \left(\int_e |f(x)|\,\mathrm{d}x\right) + \varepsilon$

for every measurable subset e of E,

5) $\max_{\tilde{N}+1 \le m \le M} \left(\left|\sum_{k=\tilde{N}+1}^{m} a_k \varphi_k(x)\right|\right) < \frac{|f(x)|}{\varepsilon},\ x \in [0;1].$

Proof. *We choose some non-overlapping binary intervals* $\{\Delta_\nu\}_{\nu=1}^{\nu_0}$ *and a step function*

$$\varphi(x) = \sum_{\nu=1}^{\nu_0} \gamma_\nu \cdot \chi_{\Delta_\nu}(x),\ \sum_{\nu=1}^{\nu_0} |\Delta_\nu| = 1, \tag{17}$$

satisfying the conditions

$$\max_{1 \le \nu \le \nu_0} |\gamma_\nu| \left(A_2 \varepsilon^{-\frac{1}{2}} |\Delta_\nu|^{\frac{1}{2}} + A|\Delta_\nu| \right) < \frac{\varepsilon}{2}, \tag{18}$$

$$0 < |\gamma_{\nu_0}||\Delta_{\nu_0}| < \cdots < |\gamma_\nu||\Delta_\nu| < \cdots < |\gamma_{\nu_1}||\Delta_{\nu_1}| < \frac{\varepsilon}{2}, \tag{19}$$

$$\left(\int_0^1 |f - \varphi|\,\mathrm{d}x\right) < \frac{\varepsilon}{2}. \tag{20}$$

Successively applying Lemma 1, *we determine some sets* $E_\nu \subset [0,1]$ *and polynomials*

$$Q_\nu = \sum_{j=m_{\nu-1}}^{m_\nu - 1} a_j \varphi_j,\ \left(m_0 = \bar{N}+1\right),\ \nu = 1,\cdots,\nu_0, \tag{21}$$

where $a_j = 0$ or $\pm\gamma_j |\Delta_j|$, if $j \in [m_{\nu-1}, m_\nu)$,

$$|E_\nu| > \left(1 - \frac{\varepsilon}{2}\right) \cdot |\Delta_\nu|, \tag{22}$$

$$Q_\nu = \begin{cases} \gamma_\nu : \text{if } x \in E_\nu \\ 0 : \text{ if } x \notin \Delta_\nu \end{cases}, \tag{23}$$

$$\max_{m_{\nu-1} \le m \le m_\nu - 1} \left(\int_0^1 \left| \sum_{k=m_{\nu-1}}^{m} a_k \varphi_k(x) \right| \mathrm{d}x \right) < A|\gamma_\nu| \varepsilon^{-1/2} |\Delta_\nu|^{\frac{1}{2}}, \tag{24}$$

Then let

$$E = \bigcup_{\nu=1}^{\nu_0} E_\nu, \tag{25}$$

$$Q = \sum_{\nu=1}^{\nu_0} Q_\nu = \sum_{k=\tilde{N}+1}^{M} a_k \varphi_k. \tag{26}$$

>*From* (19), (21), (22) *and* (25) *follows*, *that*

$$|E| > 1 - \varepsilon$$

and $0 \le |a_k| < \varepsilon$ and the non-zero coefficients in $\{|a_k|\}_{k=\tilde{N}+1}^{M}$ are in decreasing order, *i.e.* the statements 1) - 3) of Lemma 2 are valid.

To verify the statement 4), *for any* $\tilde{N} < m \le M$ *determine* ν *from the condition* $m_{\nu-1} \le m < m_\nu$. *Then by* (21) *and* (26)

$$\sum_{k=\tilde{N}+1}^{m} a_k \varphi_k = \sum_{n=1}^{\nu-1} Q_n + \sum_{k=m_\nu - 1}^{m} a_k \varphi_k. \tag{27}$$

Since for any point $x \in E$, $Q(x) = \varphi(x)$ *(see* (17), (23) *and* (26)), *then from the conditions* (18), (24), *and* (27) for every measurable subset *e of E*.

We have

$$\left(\int_e |Q(x) - f(x)|\,\mathrm{d}x\right) = \left(\int_e |\varphi(x) - f(x)|\,\mathrm{d}x\right) < \varepsilon.$$

$$\left(\int_e \left| \sum_{k=\tilde{N}+1}^{m} a_k \varphi_k(x) \right| \mathrm{d}x \right) \le \left(\int_e \left| \sum_{n=1}^{\nu-1} \gamma_n \chi_{\Delta_n}(x) \right| \mathrm{d}x \right)$$

$$+ \left(\int_e \left| \sum_{n=m_\nu - 1}^{m} a_n \varphi_n(x) \right| \mathrm{d}x \right)$$

$$\le \left(\int_e |\varphi(x)|\,\mathrm{d}x\right) + \frac{\varepsilon}{2} \le \left(\int_e |f(x)|\,\mathrm{d}x\right) + \varepsilon$$

Repeating the arguments in the proof of Lemma 2, we get a proof of the last statement of Lemma 2. *Lemma* 2 *is proved.*

The main tool in the proof of Theorem 2 *is the following result.*

Lemma 3. Let $\{\varphi_k(x)\}$ the Walsh generalized system, then for any $0 < \delta < 1$ *there exist a weight function* $\mu(x), 0 < \mu(x) \le 1,$ *with* $|\{x \in [0,1]; \mu(x) = 1\}| > 1 - \delta$ *such that for any numbers* $p_0 > 1,\ \tilde{N} \in \mathbb{N}, 0 < \varepsilon < 1$, *and evry function* $f \in L^1_\mu(0,1),\ \|f\|_{L_1} > 0$, *one can find polynomial in the Walsh generalized system*

$$Q = \sum_{k=N}^{M} a_{n_k} \varphi_{n_k},\ N > \tilde{N}$$

satisfying the following conditions:

1) $0 < \left|a_{n_{k+1}}\right| < \left|a_{n_k}\right| < \varepsilon, N < k < M$,

2) $\left(\int_0^1 |Q(x) - f(x)| \mu(x)\,\mathrm{d}x\right) < \varepsilon,$

3) $\max_{\tilde{N}+1 \le m \le M} \left(\int_0^1 \left|\sum_{k=\tilde{N}+1}^{m} a_{n_k} \varphi_{n_k}(x)\right| \mu(x)\,\mathrm{d}x\right)$

$< \left(\int_0^1 |f(x)| \mu(x)\,\mathrm{d}x\right) + \varepsilon$.

***Proof of Lemma* 3**

Let

$$\{f_k(x)\}_{k=1}^{\infty}, x \in [0,1], \tag{28}$$

be the sequence of all algebraic polynomials with rational coefficients. Applying repeatedly Lemma 2, we obtain sequences of $\{E_k\}_{k=1}^{\infty}$ sets and polynomials in the Walsh systems $\{\varphi_n(x)\}$

$$Q_k(x) = \sum_{i=m_{k-1}}^{m_k - 1} a_{n_i} \varphi_{n_i}(x), \tag{29}$$

where

$$m_0 = 1;\ m_k \nearrow$$

which satisfy the following conditions:

$$2^{-k} > \left|a_{n_i}\right| \ge \left|a_{n_{i+1}}\right| > 0,\quad \forall i \in [m_{k-1}, m_k),\quad k = 1,2,\cdots, \tag{30}$$

$$\left(\int_{E_k} \left|Q_k(x) - f_k(x)\right| \mathrm{d}x\right) < 2^{-4(k+1)}, \tag{31}$$

$$\max_{m_{k-1k} \le m < m_k} \left(\int_e \left|\sum_{i=m_{k-1}}^{m} a_{n_i}\varphi_{n_i}(x)\right| \mathrm{d}x\right) < \left(\int_e \left|f_k(x)\right| \mathrm{d}x\right) + 2^{-k-1} \tag{32}$$

for every measurable subset e of E_k

$$\left|E_k\right| > 1 - 2^{-k-1}. \tag{33}$$

Setting

$$\begin{cases} \Omega_n = \bigcap_{s=n}^{\infty} E_s, \quad n = 1,2,\cdots,; \\ E = \Omega_{n_0} = \bigcap_{s=n_0}^{\infty} E_s, \quad n_0 = \left[\log_{1/2}\delta\right] + 1; \\ B = \Omega_{n_0} \bigcup \left(\bigcup_{n=n_0+1}^{\infty} \Omega_n \setminus \Omega_{n-1}\right). \end{cases} \tag{34}$$

It is clear (*see* (33), (34))

$$|B| = 1,\quad |E| > 1 - \varepsilon_0.$$

We define a function $\mu(x)$ in the following way:

$$\mu(x) = \begin{cases} 1, & x \in E \bigcup \left([0,1] \setminus B\right) \\ \mu_n, & x \in \Omega_n \setminus \Omega_{n-1},\ n \ge n_0 + 1 \end{cases} \tag{35}$$

where

$$\mu_n = \left[2^{2n\cdot} \cdot \prod_{s=1}^{n} h_s\right]^{-1}; \tag{36}$$

$$h_k = \sup\left(1 + \int_0^1 \left|f_k(x)\right| \mathrm{d}x + \max_{m_{k-1} < m \le m_k} \int_0^1 \left|\sum_{i=m_{k-1}}^{m} a_{n_i}\varphi_{n_i(x)}\right| \mathrm{d}x\right).$$

It follows from (34)-(36) that for all $k \ge n_0$

$$\int_{[0,1]\setminus\Omega_k} \left|Q_k(x)\right| \mu(x) dx = \sum_{n=k+1}^{\infty} \left(\int_{\Omega_n \setminus \Omega_{n-1}} \left|Q_k(x)\right| \mu_n \mathrm{d}x\right) \le \sum_{n=k+1}^{\infty} 2^{-2n} \left(\int_0^1 \left|Q_k(x)\right| \mathrm{d}x\right) h_k^{-1} < \frac{1}{3} 2^{-2k\cdot}. \tag{37}$$

In a similar way for all $k \ge n_0$ we have

$$\int_{[0,1]\setminus\Omega_k} \left|f_k(x)\right| \mu(x) \mathrm{d}x < \frac{1}{3} 2^{-2k\cdot}. \tag{38}$$

By the conditions (31), (35)-(38) *for all* $k \ge n_0$ *we obtain*

$$\begin{aligned} &\int_0^1 \left|Q_k(x) - f_k(x)\right| \mu(x) \mathrm{d}x \\ &= \int_{\Omega_k} \left|Q_k(x) - f_k(x)\right| \mu(x) \mathrm{d}x \\ &+ \int_{[0,1]\setminus\Omega_k} \left|Q_k(x) - f_k(x)\right| \mu(x) \mathrm{d}x \\ &\le 2 \cdot 2^{-4(k+1)} + 2 \cdot \frac{1}{3} 2^{-2k} \le 2^{-2k}. \end{aligned} \tag{39}$$

Taking relations (32), (34)-(36) *into account we obtain that for all* $m \in [m_{k-1}, m_k)$, *and* $k \ge n_0 + 1$

$$\begin{aligned} &\int_0^1 \left|\sum_{i=m_{k-1}}^{m} a_{s_i}\varphi_{s_i}(x)\right| \mu(x) \mathrm{d}x \\ &= \int_{\Omega_k} \left|\sum_{i=m_{k-1}}^{m} a_{n_i}\varphi_{n_i}(x)\right| \mu(x) \mathrm{d}x \\ &+ \int_{[0,1]\setminus\Omega_k} \left|\sum_{i=m_{k-1}}^{m} a_{n_i}\varphi_{ns_i}(x)\right| \mu(x) \mathrm{d}x \\ &\le \frac{1}{3} \cdot 2^{-2k} + \sum_{n=n_0+1}^{k} \left[\int_{\Omega_n \setminus \Omega_{n-1}} \left|\sum_{i=m_{k-1}}^{m} a_{n_i}\varphi_{n_i}(x)\right| \mathrm{d}x\right] \cdot \mu_n \\ &\le \frac{1}{3} \cdot 2^{-2k} + \sum_{n=n_0+1}^{k} \left[2^{-2(k+1)} + \left(\int_{\Omega_n \setminus \Omega_{n-1}} \left|f_k(x)\right| \mathrm{d}x\right)\right] \cdot \mu_n \\ &= \frac{1}{3} \cdot 2^{-2k} + \sum_{n=n_0+1}^{k} \left[\frac{\mu_n}{2^{2(k+1)}} + \left(\int_{\Omega_n \setminus \Omega_{n-1}} \left|f_k(x)\right| \cdot \mu_n \mathrm{d}x\right)\right] \\ &\le \frac{1}{3} \cdot 2^{-2k} + \sum_{n=n_0+1}^{k} 2 \cdot \left[\frac{\mu_n}{2^{2(k+1)}} + \int_{\Omega_n \setminus \Omega_{n-1}} \left|f_k(x)\right| \cdot \mu_n \mathrm{d}x\right] \\ &\le \frac{1}{3} \cdot 2^{-2k} + 2 \cdot 2^{-2(k+1)} \cdot \sum_{n=n_0+1}^{k} \mu_n + 2 \cdot \int_{\Omega_k} \left|f_k(x)\right| \cdot \mu(x) \mathrm{d}x \\ &\le 2\left(2^{-2k} + \int_0^1 \left|f_k(x)\right| \cdot \mu(x) \mathrm{d}x\right). \end{aligned} \tag{40}$$

From the sequence (28) we choose a function $f_{k_0}(x)$ *such that*

$$\int_0^1 \left|f(x) - f_{k_0}(x)\right| \mathrm{d}x < \left(\frac{\varepsilon}{4}\right); \tag{41}$$

$$k_0 > \left[\log_{\frac{1}{2}}(\varepsilon\delta)\right] + 2,;\quad m_{k_0} > \tilde{N}. \tag{42}$$

Then, we set

$$Q(x) = Q_{k_0}(x),\quad \left(N = m_{k_0-1}, M = m_{k_0}\right).$$

Now, it is not difficult to verify (see (30), (39)-(42)) that the function $\mu(x)$ and the polynomials $Q(x)$ satisfy the requirements of Lemma 3.

Remark: In Lemma 3 polynom $Q(x)$ can be chosen such that

$$\max_{\tilde{N}+1 \le m \le M} \left(\left|\sum_{k=\tilde{N}+1}^{m} a_k\varphi_k(x)\right|\right) \le A\frac{|f(x)|}{\varepsilon},\quad x \in [0;1]$$

Lemma 3 is proved.

***Proof Theorem* 2**

Let $\delta \in (0,1)$ and let

$$\{f_k(x)\}_{k=1}^{\infty}, x \in [0,1], \quad (43)$$

be the sequence of all algebraic polynomials with rational coefficients. Applying repeatedly Lemma 3, we obtain a weight function $\mu(x)$ with $0 < \mu(x) \le 1$ and $|\{x \in [0,1]; \mu(x) = 1\}| > 1 - \delta$, *a sequences of polynomials in the Walsh generalized systems* $\{\varphi_n(x)\}$

$$Q_k(x) = \sum_{i=N_k}^{M_k} a_{n_i} \varphi_{n_i}(x), \quad (44)$$

where

$$N_1 = 1; N_k = M_{k-1} + 1, k \ge 2,$$

which satisfy the following conditions:

$$2^{-k} > |a_{n_i}| \ge |a_{n_{i+1}}| > 0, \quad \forall i \in [N_k, M_k], \quad k = 1, 2, \cdots, \quad (45)$$

$$\left(\int_0^1 |Q_k(x) - f_k(x)| \mu(x) dx\right) < 2^{-4k}, \quad (46)$$

$$\max_{N_k \le m \le M_k} \left(\int_0^1 \left|\sum_{i=N_k}^{m} a_{n_i} \varphi_{n_i}(x)\right| \mu(x) dx\right) < \left(\int_0^1 |f_k(x)| \mu(x) dx\right) + 2^{-k-1}. \quad (47)$$

Consider a series

$$\sum_{s=1}^{\infty} a_s \varphi_s(x), \text{ where } a_s = a_{n_i} \text{ if } s \in [n_i, n_{i+1}) (\text{see}(30)). \quad (48)$$

Clearly (*see* (45), (48))

$$|a_k| \searrow 0 \text{ and}$$

let $p \ge 1$ *and let* $f(x) \in L_\mu^p(0,1)$. *We choose some* $f_{\nu_1}(x)$ *from sequence* (43), *to have*

$$\left(\int_0^1 |f(x) - f_{\nu_1}(x)| \mu(x) dx\right) < 2^{-4}, \quad \nu_1 > k_0.$$

Suppose that the numbers $k_0 < \nu_1 < \cdots < \nu_{q-1}$ and polynomials $Q_{\nu_1}(x), \cdots, Q_{\nu_{q-1}}(x)$ are already determined satisfying to the following conditions:

$$\left(\int_0^1 \left|f(x) - \sum_{n=1}^{s} Q_{\nu_n}(x)\right| \mu(x) dx\right) < 2^{-4s}, \quad s \in [2, q-1], \quad (49)$$

$$\max_{N_{\nu_n} \le m \le M_{\nu_n}} \left(\int_0^1 \left|\sum_{i=N_{\nu_n}}^{m} a_{n_i} \varphi_{n_i}(x)\right| \mu(x) dx\right) < 2^{-n}, \quad n \in [2, q-1]. \quad (50)$$

Let a function $f_{\nu_q}(x)$, $\nu_q > \nu_{q-1}$ *be chosen from the sequence* (43) *such that*

$$\left(\int_0^1 \left|\left[f(x) - \sum_{j=1}^{q-1} Q_j(x)\right] - f_{\nu_q}(x)\right| \mu(x) dx\right) < 2^{-4(q+1)}. \quad (51)$$

Hence by (49) we obtain

$$\left(\int_0^1 |f_{\nu_q}| \mu(x) dx\right) < 2^{-q-1}. \quad (52)$$

From the conditions(46) (47), (52) follows that

$$\left(\int_0^1 \left|f(x) - \sum_{n=1}^{q} Q_{\nu_n}(x)\right| \mu(x) dx\right) < 2^{-4q}, \quad (53)$$

$$\max_{N_{\nu_q} \le m \le M_{\nu_q}} \left(\int_0^1 \left|\sum_{i=N_{\nu_n}}^{m} a_{n_i} \varphi_{n_i}(x)\right| \mu(x) dx\right) < 2^{-q}. \quad (54)$$

Then we obtain that the series

$$\sum_{k=1}^{\infty} \delta_k a_k \varphi_k(x) \quad (\text{see (29) (34) , see (44), (48)})$$

where

$$\delta_k = \begin{cases} 1, & \text{if } k = n_i, \text{ where } i = \bigcup_{q=1}^{\infty} [N_{\nu_q}, M_{\nu_q}], \\ 0, & \text{otherwise} \end{cases}$$

converges to $f(x)$ in the $L_\mu^1(0,1)$-norm. Repeating the arguments in the proof of Theorem 2 and using Lemma 1, Lemma 2 and remark of Lemma 3 we get the proof of the second statement of Theorem 2.

Theorem 2 is proved.

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