Rational Equiangular Polygons

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ABSTRACT

The main purpose of this note is to investigate equiangular polygons with rational edges. When the number of edges is the power of a prime, we determine simple, necessary and sufficient conditions for the existence of such polygons. As special cases of our investigations, we settle two conjectures involving arithmetic polygons.

Keywords: Equiangular Polygon; Arithmetic Polygon

1. Introduction

A simple way of extending the class of regular polygons is to maintain the congruence of vertex angles while no longer requiring that the edges be congruent. In this generality, the newly obtained equiangular polygons are not all that interesting given one can find plenty of such (nonsimilar) polygons with a given number of edges. Indeed, drawing a parallel line to one of the edges of a regular polygon through an arbitrary point on an adjacent edge yields a trapezoid and a new equiangular polygon with the same number of edges as the initial one (see **Figure 1**). However, if we also require that all edge lengths be rational numbers and that at least two of these numbers be different (thus excluding regular polygons), in general, such equiangular polygons may not even exist. For example, if we start with the regular pentagon $P_1P_2P_3P_4P_5$ and draw the parallel $\overline{Q_1Q_2}$ to $\overline{P_1P_2}$ as in **Figure 1** and if P_1P_2 and P_1Q_1 are rational numbers then, except for Q_1Q_2, all edge lengths of the equiangular pentagon $Q_1Q_2P_3P_4P_5$ are rational. However, $Q_1Q_2 = P_1P_2 + 2P_1Q_1\cos(2\pi/5)$ is irrational. While this, by no means, proves that equiangular pentagons with

Figure 1. New equiangular pentagons from old.

rational edges must be regular, it gives some credibility to the non-existence claim above.

An interesting investigation of equiangular polygons with integer sides is provided in [1], where the author considers the problem of tiling these polygons with either regular polygons or other pattern blocks of integer sides. In particular, he points out that every equiangular hexagon with integer sides can be tiled by a set of congruent equilateral triangles, also of integer sides, and also proposes a general tiling conjecture with an extended tiling set. On the other hand, if one no longer requires integer edges but asks that the vertices be integer lattice points, the only equiangular polygons that will do are squares and octagons (see [2,3]).

Further restricting the class of equiangular polygons with integer sides, in [4], R. Dawson considers the class of arithmetic polygons, *i.e*., equiangular polygons whose edge lengths form an arithmetic sequence (upon a suitable rearrangement) and shows that the existence of arithmetic *n*-gons is equivalent to that of equiangular *n*-gons whose side lengths form a permutation of the set $\{1, 2, \cdots, n\}$. In addition, some interesting existence as well as non-existence results are obtained, but the classification problem for arithmetic polygons with an arbitrary number of edges is left open.

In this note, we address the more general problem of determining all equiangular polygons with rational edges and, as a special case, we settle the classification problem above.

2. Preliminaries

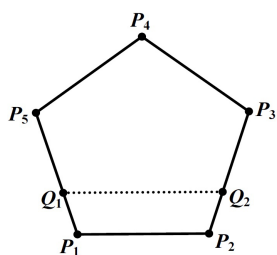First, we derive a necessary and sufficient condition for

the existence of closed polygonal paths in terms of edge lengths and angle measures.

Proposition 1 *Let* $l_1, l_2, \cdots, l_n$ *and* $\theta_1, \theta_2, \cdots, \theta_n$ *be positive real numbers with* $\theta_j < 2\pi, 1 \le j \le n$. *There exists a closed polygonal path* $P_1 P_2 \cdots P_n$ *(with* $P_1 - P_2 - \cdots - P_n - P_1$ *oriented counterclockwise) having edge lengths* $P_1 P_2 = l_1, P_2 P_3 = l_2, \cdots, P_n P_1 = l_n$ *and the measures of the angles* * *formed by* $P_1 P_2$ *with* $P_1 P_n$, $P_2 P_3$ *with* $P_2 P_1$, $P_n P_1$ *with* $P_n P_{n-1}$ *equal to* $\theta_1, \theta_2, \cdots, \theta_n$, *respectively, if and only if*

$$l_1 e^{-i\theta_1} - l_2 e^{-i(\theta_1+\theta_2)} + \cdots + (-1)^{n-1} l_n e^{-i(\theta_1+\cdots+\theta_n)} = 0 \quad (1.1)$$

and

$$\theta_1 + \theta_2 + \cdots + \theta_n = (n - 2k)\pi, \quad (1.2)$$

for some integer k.

Proof *Assuming that such a polygon exists, let* z_j *be the complex number associated to* $P_j, 1 \le j \le n$. *As the vector* $P_1 P_2$ *is the* l_1/l_n *- multiple of the rotation of* $P_1 P_n$ *in* P_1 *through* $2\pi - \theta_1$ *(see* **Figure 2**), *we have*

$$z_2 - z_1 = e^{-i\theta_1} (z_n - z_1) \frac{l_1}{l_n}.$$

Based on the same type of argument, regardless of the orientation of triangles $P_j P_{j+1} P_{j+2}, 1 \le j \le n$, we have

$$z_2 - z_1 = -e^{-i\theta_1} (z_1 - z_n) \frac{l_1}{l_n},$$

$$z_3 - z_2 = -e^{-i\theta_2} (z_2 - z_1) \frac{l_2}{l_1} = (-1)^2 e^{-i(\theta_1+\theta_2)} (z_1 - z_n) \frac{l_2}{l_n},$$

$$\vdots$$

$$z_n - z_{n-1} = -e^{-i\theta_{n-1}} (z_{n-1} - z_{n-2}) \frac{l_n-1}{l_{n-2}}$$

$$= \cdots = (-1)^{n-1} e^{-i(\theta_1+\cdots+\theta_{n-1})} (z_1 - z_n) \frac{l_{n-1}}{l_n},$$

$$z_1 - z_n = -e^{-i\theta_n} (z_n - z_{n-1}) \frac{l_n}{l_{n-1}}$$

$$= \cdots = (-1)^n e^{-i(\theta_1+\cdots+\theta_n)} (z_1 - z_n).$$
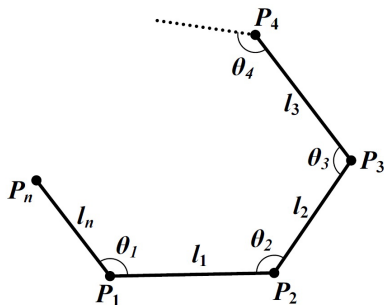
Combining these relations with

Figure 2. The polygon $P_1 P_2 \cdots P_n$.

$$(z_2 - z_1) + (z_3 - z_2) + \cdots + (z_1 - z_n) = 0$$

yields

$$-e^{-i\theta_1} (z_1 - z_n) \frac{l_1}{l_n} + (-1)^2 e^{-i(\theta_1+\theta_2)} (z_1 - z_n) \frac{l_2}{l_n}$$

$$+ \cdots + (-1)^n e^{-i(\theta_1+\cdots+\theta_n)} (z_1 - z_n) \frac{l_n}{l_n} = 0,$$

thus proving relation (1.1) from the conclusion. Relation (1.2) follows easily as a consequence of the last relation in the set of n relations above.

Conversely, to prove the existence of a closed polygonal path with given $l_j, \theta_j, 1 \le j \le n$, satisfying (1.1), observe that, starting with an arbitrary point P_1, we can always consider the points $P_2, \cdots, P_n$ such that, with the exception of $P_1 P_n$ and the measure of the angles formed by $P_1 P_2$ with $P_1 P_n$ and $P_n P_1$ with $P_n P_{n-1}$, all edge lengths and angle measures are as needed. We will prove that the closed polygonal path $P_1 P_2 \cdots P_n$ satisfies the requirements. To do so, if we let $P_1 P_n = l_n'$ and denote the measure of the angle formed by $P_1 P_2$ with $P_1 P_n$ by θ_1' and $P_n P_1$ with $P_n P_{n-1}$ by θ_n' then, we need to show that $l_n = l_n', \theta_n = \theta_n'$, and $\theta_1 = \theta_1'$. By applying the direct implication to our polygonal path (with edge lengths $l_1, l_2, \cdots, l_{n-1}, l_n'$ and angle measures $\theta_1', \theta_2, \cdots, \theta_{n-1}, \theta_n'$), we have

$$l_1 e^{-i\theta_1'} - l_2 e^{-i(\theta_1'+\theta_2)} + \cdots + (-1)^{n-1} l_n' e^{-i(\theta_1'+\cdots+\theta_n')} = 0.$$

Equivalently,

$$l_n' = (-1)^n l_1 e^{i(\theta_2+\cdots+\theta_n')} + (-1)^{n-1} l_2 e^{i(\theta_3+\theta_n')} + \cdots + l_{n-1} e^{i\theta_n'}. \quad (1.3)$$

By factoring out $e^{i\theta_n'}$ and applying the modulus on both sides of the equality above, we have

$$l_n' = \left| (-1)^n l_1 e^{i(\theta_2+\cdots+\theta_{n-1})} + (-1)^{n-1} l_2 e^{i(\theta_3+\cdots+\theta_{n-1})} + \cdots + l_{n-1} \right|.$$

However, the same type of operations can also be applied to the relation in our hypothesis (involving $l_1, l_2, \cdots, l_n$, $\theta_1, \theta_2, \cdots, \theta_n$) to obtain

$$l_n = \left| (-1) l_1 e^{i(\theta_2+\cdots+\theta_{n-1})} + (-1)^{n-1} l_2 e^{i(\theta_3+\cdots+\theta_{n-1})} + \cdots + l_{n-1} \right|.$$

But then $l_n = l_n'$ based on the two formulas above. Now, factoring out $e^{i\theta_n'}$ and replacing l_n' by l_n in (1.3), we have

$$l_n = e^{-i\theta_n'} \left((-1)^n l_1 e^{i(\theta_2+\cdots+\theta_{n-1})} \cdot \right.$$

$$\left. + (-1)^{n-1} l_2 e^{i(\theta_3+\cdots+\theta_{n-1})} + \cdots + l_{n-1} \right) \quad (1.4)$$

But we also have

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*All angles are measured counterclockwise from the first to the second referenced ray.

$$l_n = e^{-i\theta_n}\left((-1)^n l_1 e^{i(\theta_2+\cdots+\theta_{n-1})}\right.$$

$$\left.+(-1)^{n-1}l_2 e^{i(\theta_3+\cdots+\theta_{n-1})}+\cdots+l_{n-1}\right) \quad (1.5)$$

Comparing relations (1.4) and (1.5), it follows that $\theta_n = \theta'_n$. To show that $\theta_1 = \theta'_1$, let's note that relation (1.2) applied to $P_1 P_2 \cdots P_n$ implies

$$\theta'_1 + \theta_2 + \cdots + \theta_n = (n-2k')\pi,$$

for some integer k'. By hypothesis,

$$\theta_1 + \theta_2 + \cdots + \theta_n = (n-2k)\pi.$$

Combining the two relations above finishes the proof.

3. Equiangular Polygons

If we consider a convex equiangular $n-$gon, then, with notations as in the previous section, we have

$$\theta_1 = \theta_2 = \theta_n = \frac{(n-2)\pi}{n}. \text{ In addition, if we let}$$

$\xi_n = e^{\frac{2\pi i}{n}}$, then, based on Proposition 1, we obtain

Theorem 1 *Given* $l_1, l_2, \cdots, l_n \in \mathbb{R}^+$, *there exists a convex equiangular n-gon with side lengths* $l_1, l_2, \cdots, l_n$ *(listed counterclockwise) iff*

$$l_1 + l_2\xi_n + l_3\xi_n^2 + \cdots + l_n\xi_n^{n-1} = 0.$$

Definition 1 *A rational polygon is a polygon all of whose edge lengths are rational number.*

Observation 1 *The edges of a non-convex equiangular polygon can be rearranged to form a convex equiangular polygon, so we will only concentrate on the latter.*

As a consequence of Theorem 1, we obtain

Proposition 2 *Let* $\xi_n = e^{\frac{2\pi i}{n}}$ *and let* $N = \phi(n)$ *be the degree of the cyclotomic polynomial* $\Phi_n(X)$. *There exists a convex, rational, equiangular n-gon with edge lengths* $l_1, l_2, \cdots, l_n$ *(ordered counterclockwise) iff the following equalities are satisfied*:

$$l_1 + \sum_{j=N+1}^{n} l_j a_{1,j-1} = 0,$$

$$l_2 + \sum_{j=N+1}^{n} l_j a_{2,j-1} = 0,$$

$$l_N + \sum_{j=N+1}^{n} l_j a_{N,j-1} = 0,$$

where $a_{i,j}$ are defined by

$$\xi_n^j = \sum_{i=0}^{N-1} a_{i+1,j}\xi_n^i.$$

for all $N \leq j \leq n-1, 0 \leq i \leq N-1$.

Proof *Let us first note that the definition of* a_{ij} *makes*

sense. Indeed, since $\left\{1, \xi_n, \xi_n^2, \cdots, \xi_n^{N-1}\right\}$ *forms a basis of* $\mathbb{Q}(\xi_n)$, *for a fixed* $j, N \leq j \leq n$, *we can define* $a_{i,j}$ *to be the coefficients of* ξ_n^j *in this basis.*

For each $N \leq j \leq n-1$, if we replace ξ_n^j in the equality from Theorem 1 by $\sum_{i=0}^{N-1}a_{i+1,j}\xi_n^i$, we obtain

$$l_1 + l_2\xi_n + \cdots + l_N\xi_n^{N-1} + l_{N+1}\left(\sum_{i=0}^{N-1}a_{i+1,N}\xi_n^i\right)$$

$$+\cdots+l_n\left(\sum_{i=0}^{N-1}a_{i+1,n}\xi_n^i\right)=0.$$

By reorganizing the terms, the formula above becomes

$$\left(l_1 + \sum_{j=N+1}^{n} l_j a_{1,j-1}\right) + \left(l_2 + \sum_{j=N+1}^{n} l_j a_{2,j-1}\right)\xi_n$$

$$+\cdots+\left(l_N + \sum_{j=N+1}^{n} l_j a_{N,j-1}\right)\xi_n^{N-1} = 0.$$

But then we get a polynomial of degree $N-1$ with rational coefficients having ξ_n as a root. This is only possible iff all the coefficients are zero, thus proving the proposition.

Observation 2 *By fixing* $l_1, l_2, \cdots, l_{N-1} \in \mathbb{Q}^+$, *the conditions in the proposition above generate a system of equations*

$$\sum_{j=N+1}^{n} a_{s,j-1}l_j = -l_s, 1 \leq s \leq N$$

with N equations and $n-N$ variables $l_{N+1}, \cdots, l_n$. Comparing the number of equations and the number of variables, we obtain three cases depending on whether $N = \phi(n) < n-N, \phi(n) = n-\phi(n)$, or $\phi(n) > n-\phi(n)$.

To better understand the three cases above, we have

Lemma 1 *For any positive integer* n, *we have the following*

1) $n < 2\phi(n)$ *iff* $n = p^k$ *for some odd prime* p *and some positive integer* k.

2) $n = 2\phi(n)$ *iff* $n = 2^k$ *for some positive integer* k.

3) $n > 2\phi(n)$ *iff* $n = 2^k m$, *where the nonnegative integer* k *and the odd integer* $m, m \geq 3$, *are such that either* $k \geq 1$ *or, if* $k = 0$, *then* m *is the product of the powers of at least two distinct primes.*

Proof *Since the third case is the complement of the first two, it is enough to prove the first two cases. So let* $n = p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r}$, *where* $p_1, p_2, \cdots, p_r$ *are distinct primes and* k_i *are nonnegative integers.*

To prove (1), observe that the inequality $n < 2\phi(n)$ is equivalent to

$$p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r} < 2p_1^{k_1-1}(p_1-1)p_2^{k_2-1}(p_2-1)\cdots p_r^{k_r-1},$$

or $p_1p_2\cdots p_r < 2(p_1-1)(p_2-1)\cdots(p_r-1)$. To show that n has the desired form, let us assume by contradiction that $r \geq 2$. But then, since $p_1 \geq 2$ and $p_2 \geq 3$,

we have $(p_1 + 2)(p_2 + 2) > 6$, or equivalently $p_1 p_2 + 2p_2 + 2p_1 - 2 > 0$. This implies $p_1 p_2 > 2p_1 p_2 - 2p_1 - 2p_2 + 2 = 2(p_1 - 1)(p_2 - 1)$. Together with $p_i > p_i - 1, 3 \le i \le r$, the second inequality above yields $n > 2\phi(n)$, which contradicts the hypothesis. Thus $r = 1$. But then we must also have $p_1 > 2$ since otherwise $n = 2^{k_1} = 2 \cdot 2^{k_1 - 1} = 2\phi(n)$. Conversely, it is easy to see that if $n = p^k, p > 2$, then $n < 2\phi(n)$.

For (2), by considerations similar to the ones above we must have $r = 1$. Since, by (1), we cannot have $p_1 > 2$, it must be that $p_1 = 2$. Also, it is clear that if $n = 2^k$ then $n = 2\phi(n)$.

Next, we consider convex, rational, equiangular polygons in each of the three cases given by the lemma. For the overdetermined case $(n < 2\phi(n))$ we have the following:

Proposition 3 *If $l_1, l_2, \cdots, l_{p^k}$ are the lengths of the edges of a convex, rational, p^k-gon with $p > 2$ prime, then the polygon is equiangular iff*

$$l_1 = l_{p^{k-1}+1} = \cdots = l_{(p-1)p^{k-1}+1},$$

$$l_2 = l_{p^{k-1}+2} = \cdots = l_{(p-1)p^{k-1}+2},$$

$$\vdots$$

$$l_{p^{k-1}-1} = l_{2p^{k-1}-1} = \cdots = l_{p^k-1},$$

$$l_{p^{k-1}} = l_{2p^{k-1}} = \cdots = l_{p \cdot p^{k-1}} = l_{p^k}.$$

Proof *Let*

$$\Phi_n(X) = 1 + X^{p^{k-1}} + X^{2p^{k-1}} + \cdots + X^{(p-1)p^{k-1}}$$

be the minimal polynomial of ξ_n over $\mathbb{Q}[X]$ (see [5], page 31). In order to apply Proposition 2, we need to write ξ_n^j, for all $N = \phi(p^k) = (p-1)p^{k-1} \le j \le p^k - 1$, as a linear combination with integer coefficients of $1, \xi_n, \xi_n^2, \cdots, \xi_n^{N-1}$. Starting with the equation in ξ_n given by its minimal polynomial and multiplying by $\xi_n, \xi_n^2, \cdots, \xi_n^{n-N-1}$, we have

$$\xi_n^N = -1 - \xi_n^{p^{k-1}} - \xi_n^{2p^{k-1}} - \cdots - \xi_n^{(p-2)p^{k-1}},$$

$$\xi_n^{N+1} = -\xi_n - \xi_n^{p^{k-1}+1} - \xi_n^{2p^{k-1}+1} - \cdots - \xi_n^{(p-2)p^{k-1}+1},$$

$$\vdots$$

$$\xi_n^{p^k-1} = -\xi_n^{2p^{k-1}-1} - \xi_n^{3p^{k-1}-1} - \cdots - \xi_n^{(p-1)p^{k-1}-1}.$$

Thus, $a_{i+1,j} = -1$ if $j - (i+1) \in \{p^{k-1}, 2p^{k-1}, \cdots, (p-1)p^{k-1}\}$ and $a_{i+1,j} = 0$ otherwise. With these values of $a_{i+1,j}$, the conclusion follows.

Consequence 1 *Any rational equiangular polygon with a prime number of edges is regular.*

Proof *This follows based on Observation 1 and the $k = 1$ case in Proposition 3.*

Observation 3 *The consequence above proves conjecture 6 from [4].*

For the fully determined case $(n = 2\phi(n))$ we have the following characterization:

Proposition 4 *Given a convex, rational polygon whose number of edges is a power of two, the polygon is equiangular iff opposite edges are congruent.*

Proof *Let $n = 2^k$ be the number of edges of the polygon. Since $\Phi_{2^k}(X) = X^{2^{k-1}} + 1$, it follows that $\xi_n^{2^{k-1}} = -1, \xi_n^{2^{k-1}+1} = -\xi_n, \cdots, \xi_n^{2^k-1} = -\xi_n^{2^{k-1}-1}$. Thus, the relation from Theorem 1 becomes*

$$l_1 + l_2 \xi_{2^n} + \cdots + l_{2^{k-1}-1} \xi_n^{2^{k-1}-1} - l_{2^{k-1}} - l_{2^{k-1}+1} \xi_n - \cdots - l_{2^k} \xi_n^{2^{k-1}-1},$$

or

$$(l_1 - l_{2^{k-1}}) + (l_2 - l_{2^{k-1}+1})\xi_n + \cdots + (l_{2^{k-1}-1} - l_{2^k})\xi_n^{2^{k-1}-1} = 0.$$

But then ξ_n is a root of a rational polynomial of degree less than that of Φ_n. This is only possible if the polynomial is identically zero, which implies the conclusion.

As a consequence of the proposition above, we obtain a different proof of Theorem 3 from [4].

Consequence 2 *There does not exist an equiangular 2^k-gon with integer edge lengths, all distinct.*

For the underdetermined case $(n > 2\phi(n))$, given the lack of a simple formula for $\Phi_n(X)$ in this case, we will only consider the following example.

Lemma 2 *$l_1, l_2, \cdots, l_{15} \in \mathbb{Q}^+$ are the edge lengths of a convex equiangular 15-gon, with the edges ordered counterclockwise, iff*

$$l_3 - l_8 = l_6 - l_{11} = l_9 - l_{14} = l_{12} - l_2 = l_{15} - l_5$$

and

$$l_1 - l_6 = l_4 - l_9 = l_7 - l_{12} = l_{10} - l_{15} = l_{13} - l_3.$$

Proof *In this case, $\phi(15) = 8$, $\Phi_{15}(X) = X^8 - X^7 + X^5 - X^4 + X^3 - X + 1$. By letting $\xi_{15} = \xi$, we have*

$$\xi^8 = \xi^7 - \xi^5 + \xi^4 - \xi^3 + \xi, \xi^9 = -1 + \xi^2 - \xi^3 - \xi^6 + \xi^7,$$

$$\xi^{10} = -1 - \xi^5, \xi^{11} = -\xi - \xi^6, \xi^{12} = -\xi^2 - \xi^7,$$

$$\xi^{13} = 1 - \xi - \xi^4 + \xi^5 - \xi^7.$$

Based on these relations and Proposition 2, we must have

$$l_1 - l_9 - l_{10} - l_{11} + l_{14} + l_{15} = 0, l_2 + l_9 - l_{12} - l_{14} = 0,$$

$$l_3 + l_{10} - l_{13} - l_{15} = 0, l_4 - l_9 - l_{10} + l_{15} = 0,$$

$$l_5 + l_9 - l_{14} - l_{15} = 0, l_6 - l_9 - l_{11} + l_{14} = 0,$$

$$l_7 - l_{10} - l_{12} + l_{15} = 0, l_8 + l_9 + l_{10} - l_{13} - l_{14} - l_{15} = 0.$$

If we let

$$l_3 - l_8 = c_1, l_6 - l_{11} = c_2, l_9 - l_{14} = c_3,$$

$$l_{12} - l_2 = c_4, l_{15} - l_5 = c_5$$

and

$$l_1 - l_6 = d_1, l_4 - l_9 = d_2, l_7 - l_{12} = d_3,$$
$$l_{10} - l_{15} = d_4, l_{13} - l_3 = d_5,$$

the relations above become

$$d_1 + c_2 - c_3 - d_4 = 0, c_3 = c_4, d_4 = d_5, d_2 = d_4,$$
$$c_5 = c_3, c_2 = c_3, d_3 = d_4, -c_1 - d_5 + d_4 + c_3 = 0.$$

Clearly, these relations are equivalent to $c_1 = c_2 = c_3 = c_4 = c_5$ and $d_1 = d_2 = d_3 = d_4 = d_5$, thus proving the lemma.

4. Arithmetic Polygons

Following the terminology from [4], a polygon is said to be *arithmetic* if it is equiangular and its edge lengths (in some order) form a nontrivial arithmetic sequence. As shown in the same paper, an arithmetic n-gon exists iff there exists an equiangular polygon with edge lengths (in some order) $1, 2, \cdots, n$. In this section we find a necessary and sufficient condition for the existence of arithmetic polygons in terms of the number of edges. First, we have the following:

Consequence 3 *There are no arithmetic polygons whose number of edges is the power of a prime.*

Proof *This follows as a consequence of Propositions* 3, 4, *and Observation* 1.

One case when arithmetic polygons do exist is provided by the example below.

Example 1 *There exists a* (*convex*) *arithmetic* 15-*gon*.
Proof *If we select*

$$l_1 = 15, l_2 = 7, l_3 = 4, l_4 = 11, l_5 = 8, l_6 = 5, l_7 = 12, l_8 = 9,$$
$$l_9 = 1, l_{10} = 13, l_{11} = 10, l_{12} = 2, l_{13} = 14, l_{14} = 6, l_{15} = 3,$$

then the conditions in Example 2 *are satisfied since*

$$l_3 - l_8 = l_6 - l_{11} = l_9 - l_{14} = l_{12} - l_2 = l_{15} - l_5 = -5$$

and

$$l_1 - l_6 = l_4 - l_9 = l_7 - l_{12} = l_{10} - l_{15} = l_{13} - l_3 = 10.$$

Observation 4 *The proposition above provides a counterexample to conjecture* 7 *from* [4] *claiming that no arithmetic n-gons exist if n is odd*.

The example above suggests the following:

Theorem 2 *There exists an arithmetic n-gon if and only if n is not the power of a prime, i.e., n has at least two distinct primes factors.*

Proof *By Consequence* 3, *it is enough to prove the converse. So, let's consider* $n = pq$, *for some positive integers* p, q. *Since* n *is not the power of a prime,* p *and* q *can be chosen to be relatively prime. If* ξ *denotes a primitive* (pq)-*th root of unity, then*

$$1 + \xi^p + \xi^{2p} + \cdots + \xi^{(q-1)p} = 0 \qquad (1.6)$$

and

$$1 + \xi^q + \xi^{2q} + \xi^{(p-1)q} \qquad (1.7)$$

Multiplying relations (1.6) by $a\xi^a, 1 \le a \le p$ and (1.7) by $(b-1)p\xi^b, 1 \le b, \le q$ we have

$$\sum_{k=0}^{q-1} a\xi^{a+kp} = 0 \qquad (1.8)$$

and

$$\sum_{l=0}^{p-1} (b-1) p\xi^{b+lq} = 0. \qquad (1.9)$$

Let us now observe that every integer between 1 and pq appears exactly once as an exponent in both (1.8) and (1.9) due to the fact that p and q are relatively prime. If we add all p equations (1.8) and all q equations (1.9), we obtain

$$\sum_{a=1}^{p}\sum_{k=0}^{q-1} a\xi^{a+kp} + \sum_{b=1}^{q}\sum_{l=0}^{p-1} (b-1) p\xi^{b+lq} = 0.$$

Whenever $a + kp = b + lq$, the sum of the corresponding coefficients $a + (b-1)p$ is an integer between 1 and pq Moreover, different a and b with $1 \le a \le p$, $1 \le b \le q$, generate different values for $a + (b-1)p$ because p and q are relatively prime. Since there are exactly pq pairs (a,b), the values of $a + (b-1)p$ will represent a permutation of the set $\{1, 2, \cdots, pq\}$.

5. Conclusions

In this note we determined all rational equiangular polygons whose number of sides a prime power. Although we also determined all rational equiangular 15-gons, the general problem remains open. In addition, we provided a complete characterization of arithmetic polygons.

As an interesting application, we note that, as mentioned in [6], there is a nice correspondence arising from the Schwarz-Christoffel transformations between equiangular n-gons and certain areas determined by binary forms of degree n with complete factorizations over $\mathbb{R}$. It would be interesting to investigate the consequences of our results in the language of binary forms.

REFERENCES

[1] D. Ball, "Equiangular Polygons," *The Mathematical Gazette*, Vol. 86, No. 507, 2002, pp. 396-407. http://dx.doi.org/10.2307/3621131

[2] P. R. Scott, "Equiangular Lattice Polygons and Semiregular Lattice Polyhedra," *College Mathematics Journal*, Vol. 18, No. 4, 1987, pp. 300-306. http://dx.doi.org/10.2307/2686799

[3] R. Honsberger, "Mathematical Diamonds," The Mathematical Association of America, Washington DC, 2003.

[4] R. Dawson, "Arithmetic Polygons," *American Mathe-*

matical Monthly, Vol. 119, No. 8, 2012, pp. 695-698.
http://dx.doi.org/10.4169/amer.math.monthly.119.08.695

[5]   P. Samuel, "Algebraic Theory of Numbers," Kershaw, 1972.

[6]   M. A. Bean, "Binary Forms, Hypergeometric Functions and the Schwarz-Christoffel Mapping Formula," *Trans-* *actions of the American Mathematical Society*, Vol. 347, No. 12, 1995, pp. 4959-4983.
http://dx.doi.org/10.1090/S0002-9947-1995-1307999-2