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Analysis of a Delayed SIR Model with Exponential Birth and Saturated Incidence Rate

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ABSTRACT

In this paper, a delayed SIR model with exponential demographic structure and the saturated incidence rate is formulated. The stability of the equilibria is analyzed with delay: the endemic equilibrium is locally stable without delay; and the endemic equilibrium is stable if the delay is under some condition. Moreover the dynamical behaviors from stability to instability will change with an appropriate critical value. At last, some numerical simulations of the model are given to illustrate the main theoretical results.

1. Introduction

Epidemic models described by ordinary differential equations have become important tools in analyzing the spread and control of infectious diseases. In recent years, more and more delayed models have been investigated during the study of epidemic models [1-8].

When the diseases spread quickly, the population remains constant. Furthermore the population remains constant if the birth is nearly equal to the natural death when the disease we consider it over many years. In fact, in many diseases the birth of the population can not be balanced by the natural death, and then we need to assume that the population is a function of time.

Varying total population has become one of the most important areas in the mathematical theory of epidemicology [3-8]. Anderson and May [1,2] have done a lot of work about varying total population, and Michael Y. Li *et al.* considered a SEIR model with varying total population in [9]. In this paper the authors incorporated exponential natural birth and death, as well as disease-caused death into the model, so that the total population size may vary in time, and they analyzed the stability of the model with normalization method. They also present a new method for proving the local stability of the unique endemic equilibrium.

During the study of the dynamical behaviors of the disease, the standard incidence rate $\beta S(t)I(t)/N(t)$

and the bilinear incidence rate $\beta S(t)I(t)$ are frequently used [3,5,7-11]. In recent years, more and more researchers are interested in the nonlinear incidence rate; especially the saturate incidence rate has been investigated by many authors in [12-22], in which the recruitment rate of the population is considered as a constant.

A class of delayed SIR models has been investigated with nonlinear incidence rate. Capasso and Serio [13] introduced a saturated incidence rate $g(I)S$ into epidemic models, where $g(I)$ tends to a saturation level when $I(t)$ reach the maximum number of effective contacts between infective individuals and susceptible individuals may saturate at high infective levels due to crowding of infective individuals or due to the protection measures by the susceptible individuals[14].

In [15], Rui Xu *et al.* considered the effect of time delay and nonlinear incidence rate $\dfrac{\beta S(t)I(t)}{1+\alpha I(t)}$ on the dynamics of a SIR epidemic model as follows:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = B - \dfrac{\beta S(t)I(t-\tau)}{1+\alpha I(t-\tau)} - \mu_1 S(t), \\[3mm] \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = \dfrac{\beta S(t)I(t-\tau)}{1+\alpha I(t-\tau)} - (\mu_2+\gamma)I(t), \\[3mm] \dfrac{\mathrm{d}R(t)}{\mathrm{d}t} = \gamma I(t) - \mu_3 R(t). \end{cases} \quad (1)$$

Here parameters $\alpha_1, \alpha_2, \alpha_3$ are positive constants

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representing the death rates of susceptibles, infectives, and recovered, respectively. The parameters B and γ are positive constants representing the birth rate of the population and the recovery rate of infectives, respectively. By analyzing the corresponding characteristic equations, they discussed the local stability of an endemic equilibrium and a disease free equilibrium. By comparison arguments, they analyzed the globally asymptotically stable of the disease free equilibrium, and by means of an iteration technique and Lyapunov functional technique, respectively, sufficient conditions are obtained for the global asymptotic stability of the endemic equilibrium.

In [16], Kaddar considered a delayed SIR model with a nonlinear incidence rate $\dfrac{\beta S(t)I(t)}{1+\alpha_1 S(t)+\alpha_2 I(t)}$ as follows:

$$\begin{cases} \dfrac{dS(t)}{dt} = A - \dfrac{\beta S(t-\tau)I(t-\tau)e^{-\mu\tau}}{1+\alpha_1 S(t-\tau)+\alpha_2 I(t-\tau)} - \mu S(t), \\ \dfrac{dI(t)}{dt} = \dfrac{\beta S(t)I(t)e^{-\mu\tau}}{1+\alpha_1 S(t)+\alpha_2 I(t)} - (\mu+\alpha+\gamma)I(t), \\ \dfrac{dR(t)}{dt} = \gamma I(t) - \mu R(t). \end{cases} \quad (2)$$

The characteristic of this model is: the saturated incidence rate $\dfrac{\beta S(t)I(t)}{1+\alpha_1 S(t)+\alpha_2 I(t)}$, which includes the three forms: $\beta S(t)I(t)$ (if $a_1=\alpha_2=0$), $\dfrac{\beta S(t)I(t)}{1+\alpha_1 S(t)}$ (if $\alpha_2=0$) and $\dfrac{\beta S(t)I(t)}{1+\alpha_2 I(t)}$ (if $\alpha_1=0$) saturated with the susceptible and the infective individuals. The inclusion of time delay into susceptible and infective individuals in incidence rate, only on the first equation, because susceptible individuals infected at time $t-\tau$ is able to spread the disease at time t.

In the SIR model (2), they consider the period in the evolution of susceptible class, and not in the evolution of infectious class. They discuss the local stability and the existence of Hopf bifurcation. At last some numerical simulations are given to illustrate the theoretical analysis.

In this paper we consider a delayed SIR model with the saturation incidence rate $\dfrac{\beta S(t-\tau)I(t-\tau)e^{-\mu\tau}}{1+\alpha I(t-\tau)}$ and exponential birth rate. We also analyze the stability and the existence conditions of Hopf bifurcation. The organization of this paper is as follows: In Section 2, we consider a delayed SIR model with saturation incidence rates and exponential birth rate. Then we also consider an

exceptional case. In this case the saturation incidence rate becomes a bilinear incidence rate. Numerical simulations with different values of the delay are given in Section 3.

2. Stability Analysis of the Delayed SIR Model

In Section 2.1, we consider the delayed SIR model with the saturated incidence rate $\dfrac{\beta S(t-\tau)I(t-\tau)e^{-\mu\tau}}{1+\alpha I(t-\tau)}$. Then we consider the parameter measure $\alpha=0$, the saturation incidence rate will become a bilinear incidence rate $\beta S(t-\tau)I(t-\tau)e^{-\mu\tau}$, we consider it in Section 2.2.

2.1. The Delayed SIR Model with the Saturated Incidence Rate

In the section, we consider the following SIR model with the saturation incidence rate $\dfrac{\beta S(t-\tau)I(t-\tau)e^{-\mu\tau}}{1+\alpha I(t-\tau)}$ and a time delay describing a latent period. Let $S(t)$ is the number of susceptible individuals, $I(t)$ is the number of infective individuals, and $R(t)$ is the number of recovered individuals, then we have the following model:

$$\begin{cases} \dfrac{dS(t)}{dt} = b(S(t)+I(t)+R(t)) \\ \qquad - \dfrac{\beta S(t-\tau)I(t-\tau)e^{-\mu\tau}}{1+\alpha I(t-\tau)} - \mu S(t), \\ \dfrac{dI(t)}{dt} = \dfrac{\beta S(t-\tau)I(t-\tau)e^{-\mu\tau}}{1+\alpha I(t-\tau)} - (\mu+\rho+\delta)I(t), \\ \dfrac{dR(t)}{dt} = \rho I(t) - \mu R(t). \end{cases} \quad (3)$$

The parameter $b>0$ is the rate of natural birth, $\mu>0$ is the rate of natural death $(b>\mu)$, $\delta>0$ is the rate of disease-related death, $\rho>0$ is the rate of recovery, $\dfrac{1}{\tau}$ is the incubation period. α is the parameter that measure in infections with the inhibitory effect. Define the basic reproduction number by

$$R_0 = \dfrac{\beta e^{-\mu\tau}\left[\mu(\mu+\rho+\delta)-b(\mu+\rho)\right]}{\alpha\mu(b-\mu)(\mu+\rho+\delta)}$$

and we have the following theorem.

Theorem 1. If $R_0<1$, the solution of system (3) is $(S(t),I(t),R(t))\to(\infty,0,0)$ with $t\to\infty$. If $R_0>1$, system (3) has a unique endemic equilibrium $E^*=(S^*,I^*,R^*)$, where:

$$S^* = \dfrac{(\mu+\rho+\delta)(1+\alpha I^*)}{\beta e^{-\mu\tau}},$$

$$I^* = \frac{\mu(b-\mu)(\mu+\rho+\delta)}{\beta e^{-\mu\tau}\left[\mu(\mu+\rho+\delta)-b(\mu+\rho)\right]-\alpha\mu(b-\mu)(\mu+\rho+\delta)},$$

$$R^* = \frac{\rho}{\mu}I^*.$$

Proof: Considering two case: $I(t)=0$ and $I(t)>0$. If $I(t)=0$, from the third equation in (3), we get $R(t)=0$, then from the first equation in (3), it follows that

$$\frac{dS(t)}{dt} = (b-\mu)S(t).$$

When $t\to\infty$, we have $S(t)\to\infty$. Then we can get that the solution of system (3)

$$I^* = \frac{\mu(b-\mu)(\mu+\rho+\delta)}{\beta e^{-\mu\tau}\left[\mu(\mu+\rho+\delta)-b(\mu+\rho)\right]-\alpha\mu(b-\mu)(\mu+\rho+\delta)}.$$

If $I^* > 0$, we must have

$$\beta e^{-\mu\tau}\left[\mu(\mu+\rho+\delta)-b(\mu+\rho)\right]$$
$$-\alpha\mu(b-\mu)(\mu+\rho+\delta) > 0$$

It means that we must make sure $R_0 > 1$. Thus we get if $R_0 > 1$, system (3) has a unique endemic equilibrium.

In the next, we will analyze the stability of the endemic equilibrium E^* with $R_0 > 1$. We analyze the stability by the characteristic equation.

The characteristic equation of system (3) at the endemic equilibrium E^* is of the form

$$f_1(\lambda) = \left(\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3\right) + \left(q_1\lambda^2 + q_2\lambda + q_3\right)e^{-\lambda\tau}, \quad (4)$$

where

$$p_1 = (2\mu-b)+(\mu+\rho+\delta),$$
$$q_1 = \frac{\beta I^* e^{-\mu\tau}}{1+\alpha I^*} - \frac{\beta S^* e^{-\mu\tau}}{(1+\alpha I^*)^2},$$
$$p_2 = (2\mu-b)(\mu+\rho+\delta)-\mu(b-\mu)$$

$$a_2 = p_2 + q_2 = (2\mu-b)\frac{\alpha I^*(\mu+\rho+\delta)}{1+\alpha I^*} + \frac{\mu(b-\mu)(\mu+\rho+\delta)(\mu+\rho+\delta-b)+\mu^2 b(b-\mu)+b\mu\rho(b-\mu)}{\mu(\mu+\rho+\delta)-b(\mu+\rho)} > 0,$$

$$a_3 = p_3 + q_3 = \mu(b-\mu)(\mu+\rho+\delta)\frac{1}{1+\alpha I^*} > 0.$$

According to the Hurwitz criterion, we can know when $\tau = 0$, endemic equilibrium E^* of system (3) is stability.

When $\tau > 0$, we suppose system (3) has a purely

$(S(t),I(t),R(t)) \to (\infty,0,0)$.
If $I(t) > 0$, from the third equation in (3), we get:

$$R^* = \frac{\rho}{\mu}I^*.$$

And from the second equation in (3), we can get

$$S^* = \frac{(\mu+\rho+\delta)(1+\alpha I^*)}{\beta e^{-\mu\tau}}.$$

Then substituting the above equations into (2) we get the unique root

$$q_2 = (2\mu-b)\left(\frac{\beta I^* e^{-\mu\tau}}{1+\alpha I^*} - \frac{\beta S^* e^{-\mu\tau}}{(1+\alpha I^*)^2}\right) + (\rho+\delta)\frac{\beta I^* e^{-\mu\tau}}{1+\alpha I^*},$$

$$p_3 = -\mu(b-\mu)(\mu+\rho+\delta)$$

$$q_3 = \left[\mu(\mu+\rho+\delta)-b(\mu+\rho)\right]\frac{\beta I^* e^{-\mu\tau}}{1+\alpha I^*}$$
$$+ \mu(b-\mu)\frac{\beta S^* e^{-\mu\tau}}{(1+\alpha I^*)^2}$$

Theorem 2. If $R_0 > 1$, suppose $2\mu-b > 0$ and when $\tau = 0$, the endemic equilibrium E^* is stable, and when $\tau > 0$, it is unstable.

Proof. We consider the case without $\tau = 0$, the characteristic Equation (4) reads as:

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0.$$

It is easy to show that

$$a_1 = p_1 + q_1 = (2\mu-b)+\frac{\alpha I^*(\mu+\rho+\delta)}{1+\alpha I^*}+\frac{\beta I^* e^{-\mu\tau}}{1+\alpha I^*} > 0,$$

imaginary root $\omega i (\omega > 0)$, then separating real and imaginary parts, we have

$$\omega^3 - \omega p_2 = \omega q_2 \cos(\omega\tau)+\left(\omega^2 q_1 - q_3\right)\sin(\omega\tau), \quad (5)$$

$$\omega^2 p_1 - p_3 = \omega q_2 \sin(\omega\tau)+\left(-\omega^2 q_1 + q_3\right)\cos(\omega\tau). \quad (6)$$

Hence,

$$\omega^6 + a_4\omega^4 + a_5\omega^2 + a_6 = 0, \qquad (7)$$

where

$$a_4 = p_1^2 - 2p_2 - q_1^2,$$
$$a_5 = p_2^2 - 2p_1 p_3 + 2q_1 q_3 - q_2^2,$$
$$a_6 = p_3^2 - q_3^2.$$

Supposing $f(\omega) = \omega^6 + a_4\omega^4 + a_5\omega^2 + a_6$, let $\omega^2 = z$, then

$$f(z) = z^3 + a_4 z^2 + a_5 z + a_6. \qquad (8)$$

It is easy to show that $a_6 = p_3^2 - q_3^2 < 0$. Then, we have $f(0) < 0$ and $f(\infty) \to +\infty$. Thus Equation (8) has at least one positive root z_1, Equation (7) has at least one positive root, denoted by $\omega_1 = \sqrt{z_1}$.

Now, we turn to the bifurcation analysis. We use the delay τ as bifurcation parameter. We view the solutions of Equation (4) as functions of the bifurcation parameter τ. Let $\lambda(\tau) = \gamma(\tau) + i\omega(\tau)$ be the eigenvalue of Equation (4) such that for some initial value of the bifurcation parameter τ_1, we have $\gamma(\tau_1) = 0$, and $\omega(\tau_1) = \omega_1$ (we assume $\omega_1 > 0$). From Equations (5) and (6) we have

$$\tau_1 = \frac{1}{\omega_1}\arccos\left(\frac{(p_1 q_1 - q_2)\omega_1^4 + (p_2 q_2 - p_3 q_1 - p_1 q_3)\omega_1^2}{q_2^2\omega_1^2 + (q_3 - q_1\omega_1^2)^2}\right.$$
$$\left. + \frac{p_3 q_3}{q_2^2\omega_1^2 + (q_3 - q_1\omega_1^2)^2}\right) + \frac{2j\pi}{\omega_1}, j = 0, 1, 2, \cdots$$

Also, we can have $\left.\dfrac{d\operatorname{Re}\lambda(\tau)}{d\tau}\right|_{\tau=\tau_1} > 0$. By continuity, the real part of $\lambda(\tau)$ becomes positive when $\tau > \tau_1$ and the steady state becomes unstable. A Hopf bifurcation occurs when τ passes through the critical value τ_1 [18].

2.2. The Delayed SIR Model with the Standard Incidence Rate

When $\alpha = 0$, we have the standard incidence rate $\beta S(t-\tau)I(t-\tau)e^{-\mu\tau}$. Then, we can get the following model:

$$\begin{cases} \dfrac{dS(t)}{dt} = b(S(t) + I(t) + R(t)) - \beta S(t-\tau) \\ \qquad \cdot I(t-\tau)e^{-\mu\tau} - \mu S(t), \\ \dfrac{dI(t)}{dt} = \beta S(t-\tau)I(t-\tau)e^{-\mu\tau} - (\mu + \rho + \delta)I(t), \\ \dfrac{dR(t)}{dt} = \rho I(t) - \mu R(t). \end{cases} \qquad (9)$$

Define the basic reproduction number by

$$\bar{R}_0 = \frac{\mu(\mu + \rho + \delta)}{b(\mu + \rho)}$$

Theorem 3. If $\bar{R}_0 < 1$, the solution of system (9) is $(\bar{S}(t), \bar{I}(t), \bar{R}(t)) \to (\infty, 0, 0)$ with $t \to \infty$. When $\bar{R}_0 > 1$, the system has a unique endemic equilibrium $P^* = (S^*, I^*, R^*)$, where:

$$S^* = \frac{\mu(\mu + \rho + \delta)e^{\mu\tau}}{\beta},$$

$$I^* = \frac{\mu(b-\mu)(\mu + \rho + \delta)e^{\mu\tau}}{\beta(\mu(\mu + \rho + \delta) - b(\mu + \rho))},$$

$$R^* = \frac{\rho(b-\mu)(\mu + \rho + \delta)e^{\mu\tau}}{\beta(\mu(\mu + \rho + \delta) - b(\mu + \rho))}.$$

Proof: When $t \to \infty$, consider two case: $I(t) = 0$ and $I(t) > 0$.

If $I(t) = 0$, we can get $R(t) = 0$ from the third Equation in (9), then we have

$$\frac{dS(t)}{dt} = (b-\mu)S(t).$$

When $t \to \infty$, we have $S(t) \to \infty$. Then, we can get that system (9) always has $(\bar{S}(t), \bar{I}(t), \bar{R}(t))$ $(\bar{S}(t), \bar{I}(t), \bar{R}(t)) \to (\infty, 0, 0)$ with $t \to \infty$.

If $I(t) \neq 0$, from the third equation of system (9), we get

$$R^* = \frac{\rho}{\mu}I^*,$$

and from second equation, we have

$$S^* = \frac{\mu(\mu + \rho + \delta)e^{\mu\tau}}{\beta}.$$

Then, we get $I^* = \dfrac{\mu(b-\mu)(\mu + \rho + \delta)e^{\mu\tau}}{\beta(\mu(\mu + \rho + \delta) - b(\mu + \rho))}$.

To make sure $S^* \geq 0$, $I^* \geq 0$, $R^* \geq 0$. We must have $\mu(\mu + \rho + \delta) - b(\mu + \rho) > 0$. This implies that there exists the endemic equilibrium with $\bar{R}_0 > 1$.

In the next, we will analyze the stability of the endemic equilibrium P^* with $\bar{R}_0 > 1$.

Theorem 4. If $\bar{R}_0 > 1$ and when $\tau = 0$, the endemic equilibrium P^* is stable, and when $\tau > 0$, P^* is unstable.

Proof. When $\tau = 0$, the characteristic equation of system (9) read as:

$$\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0$$

It is easy to show $c_1 > 0$, $c_2 > 0$ and $c_3 > 0$. According to Hurwitz criterion, we can know the system is

stable with $\tau = 0$.

When $\tau > 0$, use the same method as system (3). Suppose it has a purely imaginary root $\omega i (\omega > 0)$, then we can get:

$$f(z) = z^3 + c_4 z^2 + c_5 z + c_6, \qquad (10)$$

where $c_6 = p_3^2 - q_3^2$. Then, we have $f'(0) < 0$ and $f'(\infty) \to +\infty$. Thus Eq.(10) has at least one positive root z_2. Also, we can have $\left. \dfrac{\mathrm{d}\,\mathrm{Re}\,\lambda(\tau)}{\mathrm{d}\tau} \right|_{\tau=\tau_2} > 0$. A Hopf bifurcation occurs when τ passes through the critical value τ_2.

3. Simulations and Sensitivity Analysis

In this section, we present some numerical results of system (3) and (9) at different values τ of supporting the theoretical analysis in Section 2.

When $R_0 < 1$, we know system (3) has $(S(t), I(t), R(t)) \to (\infty, 0, 0)$ with $t \to \infty$; and when $\bar{R}_0 < 1$ system (9) has $(\bar{S}(t), \bar{I}(t), \bar{R}(t)) \to (\infty, 0, 0)$. In **Figures 1(a)** and **(b)**, we give appropriate parameter values of system (3) and system (9) with $\tau = 0$.

From **Figure 1**, we know system (3) (and system (9)) has $(S(t), I(t), R(t)) \to (\infty, 0, 0)$ (and $(\bar{S}(t), \bar{I}(t), \bar{R}(t)) \to (\infty, 0, 0)$). The system (3) (or system (9)) exist unique endemic equilibrium with $R_0 > 1$ (or $\bar{R}_0 > 1$).

When $\tau = 0$, we give the same parameters for endemic equilibrium in system (3) and (9) as follows:

$$b = 0.8, \mu = 0.6, \rho = 0.2, \delta = 0.7, \beta = 0.3.$$

In **Figure 2**, the numerical simulations support the analysis given in Section 2. The endemic equilibriums of the delayed SIR epidemic model with the saturated incidence rate and the bilinear incidence rate are locally stable without delay.

Compare with system (9), we can know the proportion of $S(t)$ in system (3) is higher, and the proportions of $I(t)$, $R(t)$ are lower.

When $\tau > 0$, we give the parameters with different τ for endemic equilibrium in system (3) as follows:

$$b = 0.8, \mu = 0.75, \rho = 0.2, \delta = 0.7, \beta = 0.3, \alpha = 0.01.$$

In **Figure 3(a)**, we give $\tau = 0.1$, we find endemic equilibrium in system (3) is stable. When $\tau = 0.85$, from **Figure 3(b)**, we find it is unstable; from **Figure 3(c)**, we find endemic equilibrium in system (3) exists a periodic solution.

If we don't consider the parameter measure, it means that $\alpha = 0$, the incidence rate will become to the standard incidence rate, then we have system (9). We give parameters with different τ for endemic equilibrium in system (9) as follows:
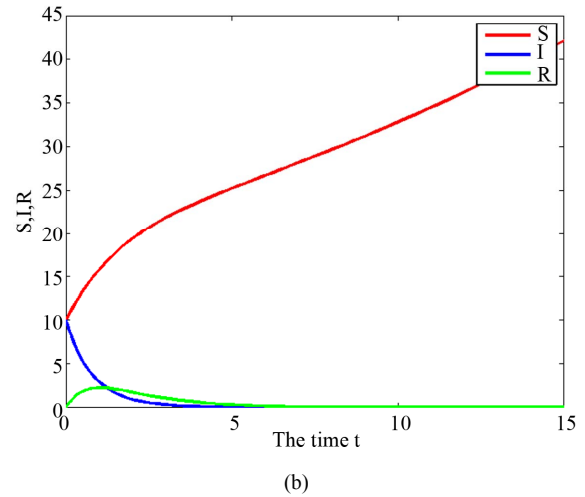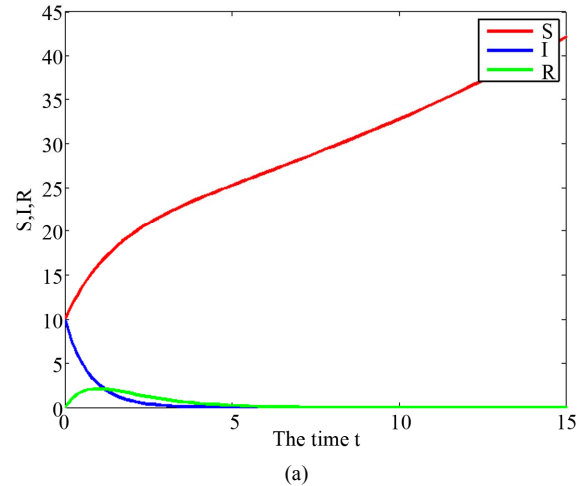

(a)

(b)

Figure 1. (a) System (3) has $(S(t), I(t), R(t)) \to (\infty, 0, 0)$ **with** $t \to \infty$ **and** $R_0 < 1$; **(b) System (9) has** $(S(t), I(t), R(t)) \to (\infty, 0, 0)$ **with** $t \to \infty$ **and** $R_0 < 1$.

$$b = 0.8, \mu = 0.7, \rho = 0.2, \delta = 0.7, \beta = 0.3.$$

In **Figure 4(a)**, we give $\tau = 0.1$, we find it is stable; from **Figure 4(b)**, we find it is unstable with $\tau = 0.85$; and from **Figure 4(c)**, we find the endemic equilibrium in system (9) exists a periodic solution.

4. Discussions

From the numerical simulations, we show that the endemic equilibrium is locally stable without time delay. In **Figure 2**, we know it is more effective that taking measures of the inhibition effect from the behavioral change of $I(t)$ reduce the infective proportion. From **Figures 3** and **4**, when $\tau = 0.1$, the endemic equilibrium in system (3) and (9) is locally asymptotically stable; when $\tau = 0.85$, the endemic equilibrium in system (3) and (9) exists periodic solutions. It showed that endemic equilib-
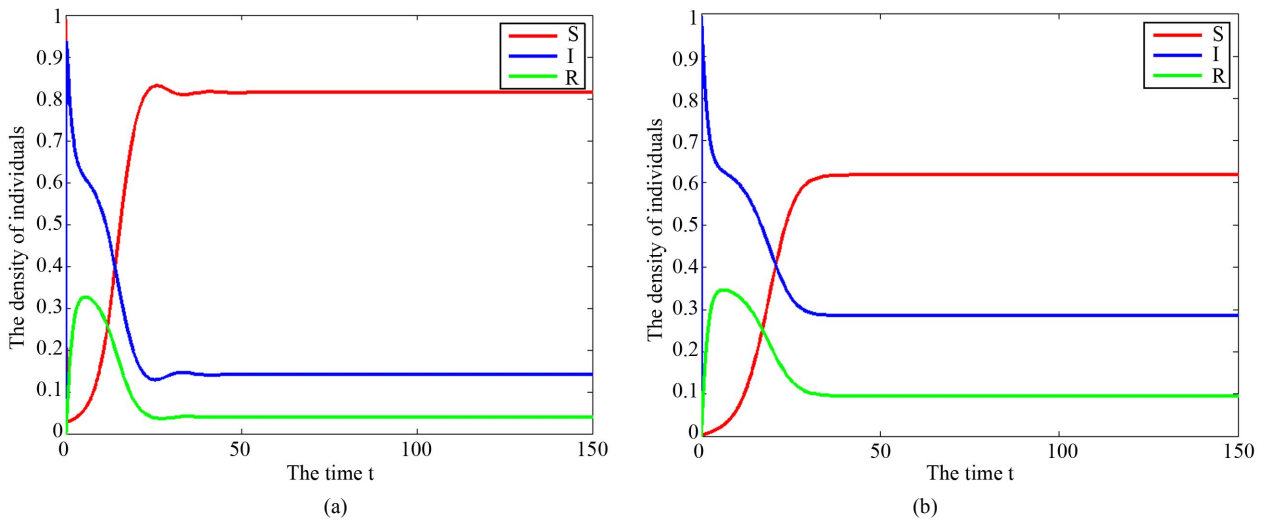
Figure 2. (a) The endemic equilibrium is locally stable with $\tau = 0$ in system (3). (b) The endemic equilibrium is locally stable with $\tau = 0$ in system (9).
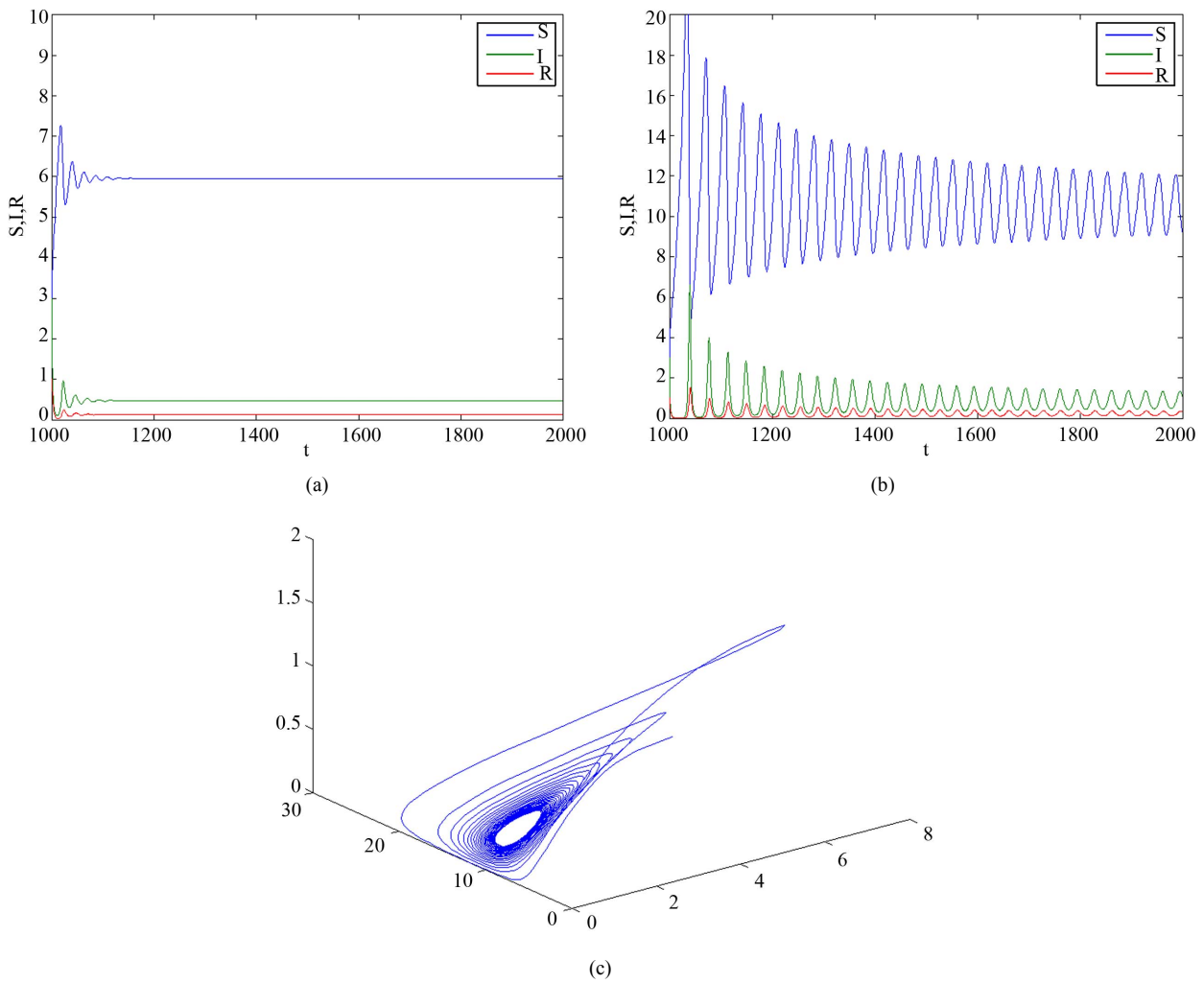

Figure 3. (a) The endemic equilibrium is asymptotic stable with $\tau = 0.1$ in system (3); (b) The endemic equilibrium is unstable with $\tau = 0.85$ in system (3); (c) The endemic equilibrium exists periodic solution with $\tau = 0.85$ in system (3).
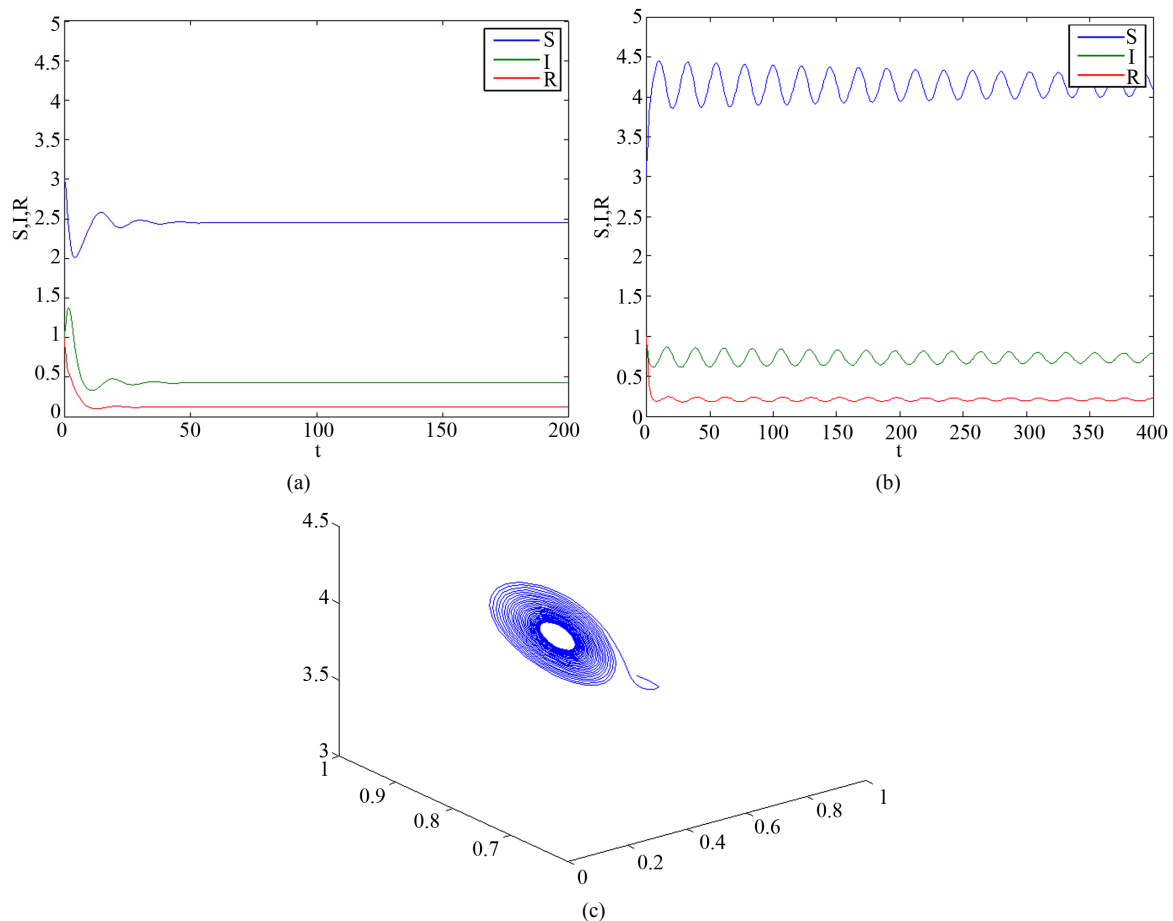
Figure 4. (a) The endemic equilibrium is asymptotic stable with $\tau = 0.1$ **in system (9); (b) The endemic equilibrium is unstable with** $\tau = 0.85$ **in system (9); (c) The endemic equilibrium exists periodic solution with** $\tau = 0.85$ **in system (9).**

rium of system (3) and (9) is locally stable when τ is suitably small. Furthermore, there exist periodic solutions with appropriate τ^* for two models. When $\tau \in \left[0, \tau^* \right)$, the endemic equilibriums is locally asymptotically stable; and unstable when $\tau > \tau^*$, and there exists a Hopf bifurcation.

5. Acknowledgements

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