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Scholz's Third Conjecture: A Demonstration for Star Addition Chains

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ABSTRACT

This paper presents a brief demonstration of Scholz's third conjecture [1] for n numbers such that their minimum chain addition is star type [2]. The demonstration is based on the proposal of an algorithm that takes as input the star-adding chain of a number n, and returns a string in addition to $x = 2^n - 1$ of length equal to $l(n) + n - 1$. As for any type addition chain star of a number n, this chain is minimal demonstrating the Scholz's third Conjecture for such numbers.

Keywords: Addition Chain; Exponentiation; Short Chain; Scholz's Conjecture

1. Basic Definitions

Definition 1. Let $S_e = \{a_i\}$ denote a finite sequence of natural numbers. We will call it an addition chain of a natural number ***e*** if it satisfies:

$1 = a_0 < a_1 < \cdots < a_r = e$

$a_i = a_j + a_k, 0 \le k \le j < i,$ with $0 < i \le r$.

Definition 2. Let $S_e = \{a_i\}$ denote a finite sequence of natural numbers. We will call it a star addition chain of a natural number ***e*** if it satisfies:

1) $1 = a_0 < a_1 < \cdots < a_r = e$

2) $a_i = a_{i-1} + a_k, 0 \le k \le i,$ with $0 < i \le r$.

Definition 3. Let $S_e = \{a_i\} = \{1 = a_0, a_1, \cdots, a_r = e\}$ denote an addition chain of a number ***e***, the highest index of the sequence ***r*** is called length of the chain ***S****e*, and it is represented by $l(S_e)$.

Definition 4. The **minimum length** of all addition chains of a natural number ***e*** is denoted by ***l***(***e***), that is:

$l(e) = \min\{l(S_e) | S_e \text{ is an addition chain of } e\}$

2. Basic Properties

Proposition 1. Let $S_n = \{a_0 = 1, a_0 = 2, \cdots, a_k = n\}$ denote an addition chain of ***n***; then, $l(S_n) = \|S_n\| - 1$.

Proof:

Clearly $\|S_n\| = k + 1$, since the terms' sub-indexes start at zero and end at ***k***. Now, by definition, the length of the addition chain is the last sub-index, which implies $l(S_n) = k = (k+1) - 1 = \|S_n\| - 1$.

Q.E.D.

Proposition 2.

Let $S_n^* = \{a_i\} = \{1 = a_0 < a_1 < \cdots < a_p = n\}$ denote a star addition chain of ***n***, then:

$S_{(2^n - 1)} = \{b_{ij}\}$ where

$$b_{ij} = \begin{cases} 2^{a_i} - 1; & \text{for } j = 0, i = 0, \cdots p \\ 2^j\left(2^{a_i} - 1\right); & 1 \le j \le a_{i+1} - a_i; i = 0, \cdots p - 1 \end{cases} \quad (1)$$

It defines a star addition chain at $2^n - 1$.

Proof:

Let $S_n^* = \{a_i\}$ denote an addition chain of ***n*** of type *****, of length ***p***, then the sequence defined in (1) fulfills the following properties:

1) Its first element is $b_{0,0} = 2^{a_0} - 1 = 2^1 - 1 = 1$

2) Its last element is $b_{p,0} = 2^{a_p} - 1 = 2^n - 1$

For each $0 < i < p$ and $j > 0$ the following is true:

$$b_{i,j} = 2^j\left(b_{i,0}\right) = 2\left(2^{j-1}\right)\left(b_{i,0}\right) = 2b_{i,j-1} = b_{i,j-1} + b_{i,j-1}$$

That is, $b_{i,j}$ is of the star type for $j > 0$, since it is equal to the sum repeated from the previous to it in the sequence.

Now we will prove that the elements $b_{i,0}$ for $0 < i \le p$ are of the star type, since we have already proved that it is equal to 1 for the case $i = 0$.

By definition, we obtain from (1) that $b_{i,0} = 2^{a_i} - 1 = 2^{a_{i-1}+a_k} - 1$ for any $0 \le k \le i-1$, since $\{a_j\}$ is of the star type

$b_{i,0} = 2^{a_k} 2^{a_{i-1}} - 1 = 2^{a_k}\left(2^{a_{i-1}} - 1\right) + 2^{a_k} - 1 = 2^{a_k} b_{i-1,0} + b_{k,0}$

For $b_{i-1,j}$, j varies between $1 \le j \le a_i - a_{i-1}$; as $\{a_j\}$ is of star type, $a_i = a_{i-1} + a_k$.

From where $j \le a_i - a_{i-1} = a_k$; a_k is the maximum value of j for $b_{i-1,j}$, which proves that $b_{i,0} = b_{i-1,a_k} + b_{k,0}$; where b_{i-1,a_k} is the maximum value of ***j*** corresponding to $b_{i-1,j}$, that is, the former to $b_{i,0}$, which completes our demonstration: the sequence $\{b_{ij}\}$ is a star addition chain of $2^n - 1$.

Q.E.D.

Proposition 3. The length of the addition chain of $2^n - 1$. defined by:

$S_{(2^n-1)} = \{b_{ij}\}$ where

$$b_{ij} = \begin{cases} 2^{a_i} - 1; & \text{for } j = 0, i = 0, \cdots, p \\ 2^j\left(2^{a_i} - 1\right); & 1 \le j \le a_{i+1} - a_i; i = 0, \cdots, p-1 \end{cases}$$

Induced by the star addition chain $S_n^* = \{a_i\}$, it has length: $l\left(S^*_{2^n-1}\right) = l\left(S_n^*\right) + n - 1$.

Proof:

Let S_n^* denote a star sequence of ***n***; we will assume without loss of generality that $l\left(S_n^*\right) = p$, then the sequence $S^*_{2^n-1}$ has $p+1$ odd values, which corresponds to the $b_{i,0}$ where $p = l\left(S_n^*\right)$.

The even elements of $S^*_{2^n-1}$ are given by the differences of $a_{i+1} - a_i$ for each ***i*** from zero until ***p* − 1**, the said sum of values is equal to:

$$\sum_{i=0}^{p-1}\left(a_{i+1} - a_i\right)$$
$$= \left(a_1 - a_0\right) + \left(a_2 - a_1\right) + \cdots + \left(a_{p-1} - a_{p-2}\right) + \left(a_p - a_{p-1}\right)$$
$$= a_p - a_0 = n - 1;$$

since $a_0 = 1$ and $a_p = n$.

The number of elements of $\left\|S^*_{2^n-1}\right\| = p + 1 + n - 1 = p + n$ as $l\left(S^*_{2^n-1}\right) = \left\|S^*_{2^n-1}\right\| - 1$ (*Proposition* 1)

From where $l\left(S^*_{2^n-1}\right) = p + n - 1 = l\left(S_n^*\right) + n - 1$; since $l\left(S_n^*\right) = p$.

Q.E.D.

3. Scholz's Third Conjecture: A Demonstration for Star Addition Chains

Theorem. Let $S_n^* = \{a_i\} = \{1 = a_0 < a_1 < \cdots < a_p = n\}$ denote a minimal star addition chain of ***n***, then $l\left(2^n - 1\right) \le l(n) + n - 1$.

Proof:

As S_n^* is a minimal addition chain and is also of the star type, *Proposition* 2 guarantees us the existence of an addition chain at $2^n - 1$, Proposition 3 guarantees us that that chain has a length equal to $l(n) + n - 1$, which proves that $l\left(2^n - 1\right) \le l(n) + n - 1$.

Q.E.D.

At UACyTI's website www.uacyti.uagro.net/3aconjetura an implementation in PHP of this algorithm can be found. It has a star addition chain of a natural number ***n*** as input, then it verifies that it is truly a star addition chain; if it is not, input is rejected, if it is, it generates the star addition chain of $x = 2^n - 1$ of length $l\left(2^n - 1\right) \le l(n) + n - 1$.

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