

$$\begin{aligned} & \text{Optimal System and Invariant Solutions on} \\ & \left( \left( U_{yy}(t, s, y) - U_t(t, s, y) \right) y - 2sU_{sy}(t, s, y) \right) y \\ & + \left( s^2 + 1 \right) U_{ss}(t, s, y) + 2sU_s = 0 \end{aligned}$$

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## ABSTRACT

The purpose of this paper is to find the invariant solutions of the reduction of the Navier-Stokes equations  $\left( \left( U_{yy}(t, s, y) - U_t(t, s, y) \right) y - 2sU_{sy}(t, s, y) \right) y + (s^2 + 1)U_{ss}(t, s, y) + 2sU_s = 0$  where  $s = z/y$ . This equation is constructed from the Navier-Stokes equations rising to a partially invariant solutions of the Navier-Stokes equations. Group classification of the admitted Lie algebras of this equation is obtained. Two-dimensional optimal system is constructed from classification of their subalgebras. All invariant solutions corresponding to these subalgebras are presented.

**Keywords:** Optimal System; Invariant Solutions; Partially Invariant Solutions; Navier-Stokes Equations

## 1. Introduction

An invariant solution of a differential equation is a solution of the differential equation which is also an invariant surface of a group admitted by the differential equation. Invariant solution can be found by solving an algebraic equation derived from the given differential equation and the infinitesimals of an admitted Lie group of transformations. Constructing of invariant solutions consists of some steps: choosing a subgroup of the admitted group, finding a representation of solution, substituting the representation into the studied system of equations and the study of compatibility of the obtained (reduced) system of equations.

This paper is devoted to use the basic Lie symmetry method for finding the admitted Lie group of the reduction of the Navier-Stokes equations,

$$\begin{aligned} & \left( \left( U_{yy}(t, s, y) - U_t(t, s, y) \right) y - 2sU_{sy}(t, s, y) \right) y \\ & + (s^2 + 1)U_{ss}(t, s, y) + 2sU_s = 0 \end{aligned} \quad (1)$$

where  $U$  is a dependent variable and  $t, s = z/y, y$  are independent variables. This equation is constructed from

the Navier-Stokes equations. Subgroups for studying are taken from the part of optimal system of subalgebras considered for the gas dynamics equations [1]. The proposed research will deal with two-dimensional optimal system of subalgebras for the reduction of the Navier-Stokes Equations (1). It is determined for symmetry algebras obtained through classification of their subalgebras. All invariant solutions are presented. They can return to new solutions of the Navier-Stokes equations.

## 2. Invariant and Partially Invariant Solutions

One of the main goals of application of group analysis to differential equation is construction of representations of solutions. Solutions whose representations are obtained with the help of the admitted group are called invariant or partially invariant solutions. The notion of invariant solution was introduced by Sophus Lie [2]. The notion of a partially invariant solution was introduced by Ovsianikov [3]. This notion of partially invariant solutions generalizes the notion of an invariant solution, and extends the scope of applications of group analysis for con-

structing exact solutions of partial differential equations. The algorithm of finding invariant and partially invariant solutions consists of the following steps.

Let  $L^r$  be a Lie algebra with the basis  $X_1, \dots, X_r$ . The universal invariant  $J$  consists of  $s = m + n - r_*$  functionally independent invariants

$$J = (J^1(x, u), J^2(x, u), \dots, J^{m+n-r_*}(x, u)),$$

where  $n, m$  are the numbers of independent and dependent variables, respectively and  $r_*$  is the total rank of the matrix composed by the coefficients of the generators  $X_i, (i = 1, 2, \dots, r)$ . If the rank of the Jacobi matrix

$$\frac{\partial(J^1, \dots, J^{m+n-r_*})}{\partial(u^1, \dots, u^m)}$$

is equal to  $q$ , then one can choose the first  $q \leq m$  invariants  $J^1, \dots, J^q$  such that the rank of the Jacobi matrix

$$\frac{\partial(J^1, \dots, J^q)}{\partial(u_1, \dots, u_m)}$$

is equal to  $q$ . A partially invariant solution is characterized by two integers:  $\sigma \geq 0$  and  $\delta \geq 0$ . These solutions are also called  $H(\sigma, \delta)$ -solutions. The number  $\sigma$  is called the rank of a partially invariant solution. This number gives the number of the independent variables in the representation of the partially invariant solution. The number  $\delta$  is called the defect of a partially invariant solution. The defect is the number of the dependent functions which can not be found from the representation of partially invariant solution. The rank  $\sigma$  and the defect  $\delta$  must satisfy the conditions

$$\begin{aligned} \sigma + \delta + n - r_* &\geq 0, \delta \geq 0, \rho \leq \sigma < n, \\ \max\{r_* - n, m - q, 0\} &\leq \delta \leq \min\{r_* - 1, m - 1\}, \end{aligned}$$

where  $\rho$  is the maximum number of invariants which depends on the independent variables only. Note that for invariant solutions,  $\delta = 0$  and  $q = m$ .

For constructing a representation of a  $H(\sigma, \delta)$ -solution one needs to choose  $l = m - \delta$  invariants and separate the universal invariant in two parts:

$$\bar{J} = (J^1, \dots, J^l), \bar{J}^{\%} = (J^{l+1}, J^{l+2}, \dots, J^{m+n-r_*}).$$

The number  $l$  satisfies the inequality  $1 \leq l \leq q \leq m$ . The representation of the  $H(\sigma, \delta)$ -solution is obtained by assuming that the first  $l$  coordinates  $\bar{J}$  of the universal invariant are functions of the invariants  $\bar{J}$ :

$$\bar{J} = W(\bar{J}). \tag{2}$$

Equation (2) form the invariant part of the representation of a solution. The next assumption about a partially

invariant solution is that Equation (2) can be solved for the first  $l$  dependent functions, for example,

$$u^i = \phi^i(u^{l+1}, u^{l+2}, \dots, u^m, x), (i = 1, \dots, l). \tag{3}$$

It is important to note that the functions  $W^i, (i = 1, \dots, l)$  are involved in the expressions for the functions  $\phi^i, (i = 1, \dots, l)$ . The functions  $u^{l+1}, u^{l+2}, \dots, u^m$  are called superfluous. The rank and the defect of the  $H(\sigma, \delta)$ -solution are  $\delta = m - l$  and  $\sigma = m + n - r_* - l = \delta + n - r_*$ , respectively.

Note that if  $\delta = 0$ , the above algorithm is the algorithm for finding a representation of an invariant solution. If  $\delta \neq 0$ , then Equation (3) do not define all dependent functions. Since a partially invariant solution satisfies the restrictions (2), this algorithm cuts out some particular solutions from the set of all solutions.

After constructing the representation of an invariant or partially invariant solution (3), it has to be substituted into the original system of equations. The system of equations obtained for the functions  $W$  and superfluous functions  $u^k, (k = l + 1, 2, \dots, m)$  is called the reduced system. This system is overdetermined and requires an analysis of compatibility. Compatibility analysis for invariant solutions is easier than for partially invariant solutions. Another case of partially invariant solutions which is easier than the general case occurs when  $\bar{J}$  only depends on the independent variables

$$J^{l+1} = J^{l+1}(x), J^{l+2} = J^{l+2}(x), \dots, J^{m+n-r_*} = J^{m+n-r_*}(x).$$

In this case, a partially invariant solution is called regular, otherwise it is irregular. The number  $\sigma - \rho$  is called the measure of irregularity.

The process of studying compatibility consists of reducing the overdetermined system of partial differential equations to an involutive system. During this process different subclasses of  $H(\sigma, \delta)$  partially invariant solutions can be obtained. Some of these subclasses can be  $H_1(\sigma_1, \delta_1)$ -solutions with subalgebra  $H_1 \subset H$ . In this case  $\sigma_1 \geq \sigma, \delta_1 \leq \delta$ . The study of compatibility of partially invariant solutions with the same rank  $\sigma_1 = \sigma$ , but with smaller defect  $\delta_1 < \delta$  is simpler than the study of compatibility for  $H(\sigma, \delta)$ -solutions. In many applications, there is a reduction of a  $H(\sigma, \delta)$ -solution to a  $H_1(\sigma, 0)$  solution. In this case the  $H(\sigma, \delta)$ -solution is called reducible to an invariant solution. The problem of reduction to an invariant solution is important since invariant solutions are usually studied first.

### 3. The Reduction of the Navier-Stokes Equations

The reduction of the Navier-Stokes equations to partial differential equation in three independent variables is described. Unsteady motion of incompressible viscous

fluid is governed by the Navier-Stokes equations

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \Delta \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0, \quad (4)$$

where  $\mathbf{u} = (u_1, u_2, u_3) = (u, v, w)$  is the velocity field,  $p$  is the fluid pressure,  $\nabla$  is the gradient operator in the three-dimensional space  $\mathbf{x} = (x_1, x_2, x_3) = (x, y, z)$  and  $\Delta$  is the Laplacian. A group classification of the Navier-Stokes equations in the three-dimensional case was done in [4]. The Lie group admitted by the Navier-Stokes equations is infinite. Its Lie algebra can be presented in the form of the direct sum  $L^\infty \oplus L^5$ , where the infinite-dimensional ideal  $L^\infty$  is generated by the operators

$$X_{\phi_i} = \phi_i(t) \partial_{x_i} + \phi_i'(t) \partial_{u_i} - \phi_i''(t) x_i \partial_p, \quad X_\psi = \psi(t) \partial_p$$

with arbitrary functions  $\phi_i(t), (i=1,2,3)$  and  $\psi(t)$ . The subalgebra  $L^5$  has the following basis:

$$Y = 2t \partial_t + x_i \partial_{x_i} - u_i \partial_{u_i} - 2p \partial_p, \quad Z_0 = \partial_t, \\ Z_{ik} = x_i \partial_{x_k} - x_k \partial_{x_i} + u_i \partial_{u_k} - u_k \partial_{u_i}, \quad (i < k \leq 3).$$

The Galilean algebra  $L^{10}$  is contained in  $L^\infty \oplus L^5$ . Several articles [5-11] are devoted to invariant solutions of the Navier-Stokes equations. While partially invariant solutions of the Navier-Stokes equations have been less studied, there has been substantial progress in studying such classes of solutions of inviscid gas dynamics equations [12-19].

In this section analysis of compatibility of regular partially invariant solutions with defect 1 and rank 1 of the subalgebras  $\{\partial_x, t \partial_x + \partial_u, \partial_t, t \partial_t + x \partial_x + y \partial_y + z \partial_z\}$  is given. Note that the generator  $t \partial_t + x \partial_x + y \partial_y + z \partial_z$  is not admitted by the Navier-Stokes equations. The groups are taken from the optimal system constructed for the gas dynamics equations [20].

The Navier-Stokes equations are used in the component form:

$$u_t + uu_x + vu_y + wu_z = -p_x + u_{xx} + u_{yy} + u_{zz}, \quad (5)$$

$$v_t + uv_x + vv_y + wv_z = -p_y + v_{xx} + v_{yy} + v_{zz}, \quad (6)$$

$$w_t + uw_x + vw_y + ww_z = -p_z + w_{xx} + w_{yy} + w_{zz}, \quad (7)$$

$$u_x + v_y + w_z = 0. \quad (8)$$

The dependent variables  $u, v, w$  and  $p$  are functions of the space variables  $x, y, z$  and time  $t$ .

Invariants of the Lie group corresponding to subalgebra generated by  $\{\partial_x, t \partial_x + \partial_u, \partial_t, t \partial_t + x \partial_x + y \partial_y + z \partial_z\}$  are  $v, w, p, z/y$ .

The representation of the regular partially invariant solution is

$$v = V(s), \quad w = W(s), \quad p = P(s), \quad (9)$$

where  $s = z/y$ . For the function  $u = u(t, x, y, z)$  there

is no restrictions. Substituting the representation of partially invariant solution (9) into the Navier-Stokes Equations (5)-(8), we obtain

$$u_t + uu_x + Vu_y + Wu_z - (u_{xx} + u_{yy} + u_{zz}) = 0, \quad (10)$$

$$((W - sV)V' - sP')_y - ((s^2 + 1)V'' + 2sV') = 0, \quad (11)$$

$$((W - sV)W' + P')_y - ((s^2 + 1)W'' + 2sW') = 0, \quad (12)$$

$$yu_x - (sV' + W') = 0. \quad (13)$$

Since  $V$  and  $W$  only depend on  $s$ , Equations (11) and (12) can be split with respect to  $y$ :

$$(W - sV)V' - sP' = 0, \quad (W - sV)W' + P' = 0, \quad (14)$$

$$(s^2 + 1)V'' + 2sV' = 0, \quad (s^2 + 1)W'' + 2sW' = 0. \quad (15)$$

Solving Equations (15), we have

$$V = C_1 \arctan(s) + C_2, \quad W = C_3 \arctan(s) + C_4.$$

Multiplying the first equation by  $s$  and combining it with the second Equation of (14), we obtain

$$(W - sV)(V' + sW') = 0.$$

Let  $W - sV = 0$ , then  $C_1 = C_2 = C_3 = C_4 = 0$ . This means that  $V = 0, W = 0$  and hence  $P = C_5$ . Substituting  $V$  and  $W$  in Equation (13), we have  $u_x = 0$ . It means that  $u$  depend on  $t, y, z$  or  $u = U(t, s, y)$ . Equation (10) becomes

$$((U_{yy} - U_t)_y - 2sU_{sy})_y + (s^2 + 1)U_{ss} + 2sU_s = 0. \quad (16)$$

Thus, there is a solution of the Navier-Stokes equations of the type

$$u = U(t, s, y), \quad v = 0, \quad w = 0, \quad p = C_5,$$

where the function  $U(t, s, y)$  satisfies Equation (16).

If  $V' + sW' = 0$ , then  $V = C_2, W = C_4$ . In this case  $P = C_5$ . Note that the Galilei transformation applied to  $V$  and  $W$ , also change  $s$ . Substituting  $V$  and  $W$  in Equation (13), we have  $u_x = 0$  or  $u = U(t, s, y)$ . Equation (10) becomes

$$((U_{yy} - C_2U_y - U_t)_y - 2sU_{sy})_y + (s^2 + 1)U_{ss} \\ + ((C_2y + 2)s - C_4y)U_s = 0. \quad (17)$$

Thus, there is a solution of the Navier-Stokes equations of the type

$$u = U(t, s, y), \quad v = C_2, \quad w = C_4, \quad p = C_5,$$

where the function  $U(t, s, y)$  satisfies Equation (17).

These solutions are partially invariant solution with respect to the group which are not admitted Lie algebra  $\{\partial_x, t \partial_x + \partial_u, \partial_t, t \partial_t + x \partial_x + y \partial_y + z \partial_z\}$ .

### 4. Admitted Group of Equation (16)

In this section, the Lie group admitted by Equation (16) is studied. It was obtained from the Navier-Stokes equations and gives rise to a partially invariant solutions of the Navier-Stokes equations

$$\begin{aligned} & \left( (U_{yy}(t,s,y) - U_t(t,s,y))y - 2sU_{sy}(t,s,y) \right) y \\ & + (s^2 + 1)U_{ss}(t,s,y) + 2sU_s = 0 \end{aligned}$$

where the function  $U$  depends on  $t, s, y$  and  $s = z/y$ .

Assume that the generator has a representation of the form

$$\begin{aligned} X = & \xi^t(t,s,y,U)\partial_t + \xi^s(t,s,y,U)\partial_s \\ & + \xi^y(t,s,y,U)\partial_y + \zeta^U(t,s,y,U)\partial_U. \end{aligned}$$

The second prolongation of the operator  $X$  is

$$\begin{aligned} X^{(2)} = & X + \zeta^{U_t}(t,s,y,U)\partial_{U_t} + \zeta^{U_s}(t,s,y,U)\partial_{U_s} \\ & + \zeta^{U_y}(t,s,y,U)\partial_{U_y} + \zeta^{U_{tt}}(t,s,y,U)\partial_{U_{tt}} \\ & + \zeta^{U_{ts}}(t,s,y,U)\partial_{U_{ts}} + \zeta^{U_{ty}}(t,s,y,U)\partial_{U_{ty}} \\ & + \zeta^{U_{sy}}(t,s,y,U)\partial_{U_{sy}} + \zeta^{U_{ss}}(t,s,y,U)\partial_{U_{ss}} \\ & + \zeta^{U_{yy}}(t,s,y,U)\partial_{U_{yy}}. \end{aligned}$$

The coefficients of the prolonged operator are defined by formulae

$$\begin{aligned} \zeta^{U_i} &= D_i(\zeta^U) - U_j D_i(\xi^{x_j}); \quad i, j = 1, 2, 3 \\ \zeta^{U_{i_1, i_2}} &= D_{i_2}(\zeta^{U_{i_1}}) - U_{i_1, j} D_{i_2}(\xi^{x_j}); \quad i, j = 1, 2, 3 \\ D_i &= \frac{\partial}{\partial x_i} + U_i \frac{\partial}{\partial U} + U_{ij} \frac{\partial}{\partial U_j} + \dots; \quad i, j = 1, 2, 3. \end{aligned}$$

Here we used the notations  $x_1 = t, x_2 = s, x_3 = y$  and for the derivatives

$$U_i = D_i(U), U_{ij} = D_j(U_i).$$

The determining equations are

$$X^{(2)}F \Big|_{[F=0]} = 0. \tag{18}$$

All necessary calculations here were carried out on a computer using the symbolic manipulation program REDUCE.

The result of the calculations is the admitted Lie group with the basis of the generators:

$$\begin{aligned} X_1 = & \partial_t, X_2 = \frac{1}{y}\partial_s, X_3 = \frac{s}{y}\partial_s - \partial_y, \\ X_4 = & 2t\partial_t + y\partial_y - U\partial_U, X_5 = \frac{2t}{y}\partial_s - syU\partial_U, \end{aligned} \tag{19}$$

$$X_6 = (s^2 + 1)\partial_s - sy\partial_y, X_7 = \frac{2ts}{y}\partial_s - 2t\partial_y + yU\partial_U,$$

$$X_8 = 4t^2\partial_t + 4ty\partial_y - (4t + (s^2 + 1)y^2)U\partial_U,$$

$$X_9 = U\partial_U, X_{10} = b(t,s,y)\partial_U,$$

where  $b(t,s,y)$  is an arbitrary solution of

$$\left( (b_{yy} - b_t)y - 2sb_{sy} \right) y + (s^2 + 1)b_{ss} + 2sb_s = 0.$$

### 5. Optimal System of Subalgebras

To obtain all different invariant solutions, we make recourse to the concept of optimal system of subalgebras. This concept follows from the fact that, given a Lie algebra  $L$  of order  $r > 1$  with  $G$  the corresponding group of transformations, if two subalgebras of  $L$  are similar, *i.e.*, they are connected with each other by a transformation of  $G$ , then their corresponding invariant solutions are connected with each other by the same transformation. Therefore, in order to construct all the non similar  $s$ -dimensional subalgebras of  $L$ , it is sufficient to put into one class all similar subalgebras of a given dimension, say  $s < r$ , and select a representative from each class. The set of all representatives of these classes is called optimal system of  $s$ -dimensional subalgebras of  $L$ . The classification of subalgebras can be done relatively easy for small dimensions. The problem of finding the optimal system is the same as the problem of classifying the orbits of the adjoint transformations. Two-dimensional subalgebras of the optimal system of the Lie algebra spanned by the generators  $X_1, \dots, X_9$  are constructed in [21].

The list of two-dimensional subalgebras of the optimal system of the algebra  $L^9$  is presented in the **Table 1**, where  $\varepsilon = \pm 1$  and  $\alpha, \beta, \gamma$  are arbitrary constants.

**Table 1. Two-dimensional subalgebras of the optimal system of the algebra  $L^9$ .**

N	Generator	N	Generator
1	2, 3	11	1, 2 + $\alpha 4 + 6 + \beta 9$
2	2, 7	12	8, $\alpha 4 + 6 + \beta 9$
3	5, 7	13	8, $\alpha 4 + 5 + 6 + \beta 9$
4	2, 3 + $\varepsilon 7$	14	4 + $\alpha 9, 6 + \beta 9$
5	1, 4 + $\alpha 9$	15	6 + $\alpha 9, \varepsilon 1 + 8 + \beta 9$
6	2, 4 + $\alpha 9$	16	2 + 7, 3 + 5 + $\alpha 7$
7	5, 4 + $\alpha 9$	17	2 + $\alpha 9, 1 + \beta 7 + \gamma 9$
8	8, 4 + $\alpha 9$	18	5 + $\alpha 9, \alpha 3 + 8 + \gamma 9$
9	1, 2 + 4 + $\alpha 9$	19	2 + $\varepsilon 5, -1 + 8 + \alpha 9$
10	1, $\alpha 4 + 6 + \beta 9$	20	1 - 8 + $\alpha 9, 1 + 4 + \alpha 9$

### 6. Invariant Solutions of the Equation (1)

One of the advantages of the symmetry analysis is the possibility to find solutions of the original differential equation by solving reduced equations. The reduced equations are obtained by introducing suitable new variables, determined as invariant functions with respect to the infinitesimal generators. Constructing of invariant solutions consists of some steps: choosing a subgroup of the admitted group, finding a representation of solution, substituting the representation into the studied system of equations and the study of compatibility of the obtained (reduced) system of equations.

Invariant solutions of the Equation (1) are presented in this section. Analysis of invariant solutions is presented in details for four examples.

#### 6.1. Subalgebra 3: {5, 7}

The basis of this subalgebra is

$$X_5 = \frac{2t}{y} \partial_s - syU \partial_U, X_7 = \frac{2ts}{y} \partial_s - 2t \partial_y + yU \partial_U.$$

In order to find an invariant solution, one needs to find a universal invariant of this subalgebra. Let a function

$$f = f(t, s, y, U)$$

be an invariant of the generator  $X_5$ . This means that

$$\frac{2t}{y} f_s - syU f_U = 0.$$

The general solution of this equation is

$$f = F(t, y, \hat{U}), \hat{U} = U e^{\frac{(sy)^2}{4t}}.$$

After substituting it into the equation  $(X_7)f = 0$ , one obtains the equation

$$\frac{2ts}{y} F_s - 2t F_y + y \hat{U} F_{\hat{U}} = 0.$$

The characteristic system of the last equation is

$$\frac{dy}{-2t} = \frac{d\hat{U}}{y\hat{U}}.$$

Thus, the universal invariant of this subalgebras consists of invariants

$$t, \hat{U} e^{\frac{y^2}{4t}}, \hat{U} = U e^{\frac{(sy)^2}{4t}}.$$

Hence, a representation of the invariant solution is

$$U = e^{-\frac{(y)^2(1+s^2)}{4t}} \phi(t)$$

with arbitrary functions  $\phi(t)$ . After substituting this representation into Equation (1), one obtains the ordinary

differential equation

$$t\phi' + \phi = 0.$$

The general solution of the last equation is

$$\phi = C/t$$

where  $C$  is arbitrary constants.

Therefore the invariant solution of the reduction of the Navier-Stokes Equations (1) is

$$U = \left( C e^{-y^2(1+s^2)/4t} \right) / t$$

where  $C$  is arbitrary constants.

#### 6.2. Subalgebra 6: {2, 4 + α9}

The basis of this subalgebra is

$$X_2 = \frac{1}{y} \partial_s, X_4 + \alpha X_9 = 2t \partial_t + y \partial_y + (\alpha - 1) U \partial_U.$$

Let a function

$$f = f(t, s, y, U)$$

be an invariant of the generator  $X_2$ . This means that

$$\frac{1}{y} f_s = 0.$$

It means that function  $f$  is not depend on  $s$ .

The general solution of this equation is

$$f = F(t, y, U).$$

After substituting it into the equation  $(X_4 + \alpha X_9)f = 0$ , one obtains the equation

$$2tF_t + yF_y + (\alpha - 1)UF_U = 0.$$

The characteristic system of the last equation is

$$\frac{dt}{2t} = \frac{dy}{y} = \frac{dU}{(\alpha - 1)U}.$$

Thus, the universal invariant of this subalgebras consists of invariants

$$\frac{y^2}{t}, Uy^{1-\alpha}.$$

Hence, a representation of the invariant solution is

$$U = y^{\alpha-1} \phi(q)$$

with arbitrary functions  $\phi(q)$  and  $q = y^2/t$ . After substituting this representation into Equation (1), one obtains the ordinary differential equation

$$4q^2 \phi'' + q(q + 4\alpha - 2) \phi' + (\alpha^2 - 3\alpha + 2) \phi = 0.$$

The general solution of the last equation is

$$\phi = e^{-\frac{q}{8}q^{-\frac{1-2\alpha}{4}}} \left[ C_1 W_1 \left( \frac{1-2\alpha}{4}, \frac{1}{4}, \frac{q}{4} \right) + C_2 W_2 \left( \frac{1-2\alpha}{4}, \frac{1}{4}, \frac{q}{4} \right) \right].$$

Therefore, the invariant solution of the reduction of the Navier-Stokes Equations (1) is

$$U = t^{\frac{2\alpha-1}{4}} y^{-\frac{1}{2}} e^{-\frac{y^2}{8t}} \left[ C_1 W_1 \left( \frac{1-2\alpha}{4}, \frac{1}{4}, \frac{y^2}{4t} \right) + C_2 W_2 \left( \frac{1-2\alpha}{4}, \frac{1}{4}, \frac{y^2}{4t} \right) \right]$$

where

$$W_1 \left( \frac{1-2\alpha}{4}, \frac{1}{4}, \frac{y^2}{4t} \right), W_2 \left( \frac{1-2\alpha}{4}, \frac{1}{4}, \frac{y^2}{4t} \right)$$

are Whittaker functions and  $C_1, C_2$  are arbitrary constants.

### 6.3. Subalgebra 16: {2 + 7, 3 + 5 + α7}

The basis of this subalgebra consists of the generators

$$\begin{aligned} X_2 + X_7 &= \left( \frac{1+2ts}{y} \right) \partial_s - 2t \partial_y + yU \partial_U, \\ X_3 + X_5 + \alpha X_7 &= \left( \frac{s+(1+\alpha s)2t}{y} \right) \partial_s - (1+2\alpha t) \partial_y + (\alpha - s) yU \partial_U. \end{aligned}$$

Let a function

$$f = f(t, s, y, U)$$

be an invariant of the generator  $X_2 + X_7$ . This means that

$$\left( \frac{1+2ts}{y} \right) f_s - 2t f_y + yU f_U = 0.$$

The characteristic system of the last equation is

$$\frac{yds}{(1+2ts)} = \frac{dy}{-2t} = \frac{dU}{yU} = \frac{dt}{0}.$$

The general solution of this equation is

$$f = F(t, \hat{y}, \hat{U}), \hat{y} = y(2ts+1), \hat{U} = Ue^{\frac{y^2}{4t}}.$$

After substituting it into the equation

$$(X_3 + X_5 + \alpha X_7) f = 0$$

one obtains the equation

$$2t(1+2\alpha t - 4t^2) F_{\hat{y}} + \hat{y} \hat{U} F_{\hat{U}} = 0.$$

The characteristic system of this equation is

$$\frac{d\hat{y}}{2t(1+2\alpha t - 4t^2)} = \frac{d\hat{U}}{\hat{y}\hat{U}} = \frac{dt}{0}.$$

Hence, the universal invariant of this subalgebras consists of invariants

$$t, \hat{U} e^{-\frac{\hat{y}^2}{4t(1+2\alpha t - 4t^2)}}, \hat{y} = y(2ts+1), \hat{U} = Ue^{\frac{y^2}{4t}}.$$

A representation of the invariant solution of this subalgebra has the following form

$$U = e^{\frac{(y(2ts+1))^2}{4t(1+2\alpha t - 4t^2)} - \frac{y^2}{4t}} \phi(t)$$

with an arbitrary function  $\phi(t)$ . After substituting the representation of the invariant solution into Equation (1), the functions  $\phi(t)$  has to satisfy the equation

$$(1+2\alpha t - 4t^2) \phi' + (\alpha - 4t) \phi = 0.$$

The general solution of the last equation is

$$\phi = C / \sqrt{1+2\alpha t - 4t^2}$$

where  $C$  is constant.

Therefore the invariant solution of the reduction of the Navier-Stokes Equations (1) is

$$U = \left( C e^{\frac{(y+2tsy)^2}{4t(1+2\alpha t - 4t^2)} - \frac{y^2}{4t}} \right) / \sqrt{1+2\alpha t - 4t^2}$$

where  $C$  is constant.

### 6.4. Subalgebra 17: {2 + α9, 1 + β7 + γ9}

The basis of this subalgebra is

$$\begin{aligned} X_2 + \alpha X_9 &= \frac{1}{y} \partial_s + \alpha U \partial_U, \\ X_1 + \beta X_7 + \gamma X_9 &= \partial_t + \beta \frac{2ts}{y} f_s - 2t \partial_y + (y + \gamma) U \partial_U. \end{aligned}$$

Let a function

$$f = f(t, s, y, U)$$

be an invariant of the generator  $X_2 + \alpha X_9$ . This means that

$$\frac{1}{y} f_s + \alpha U f_U = 0.$$

The general solution of this equation is

$$f = F(t, y, \hat{U}), \hat{U} = Ue^{-\alpha sy}.$$

After substituting it into the equation

$$(X_1 + \beta X_7 + \gamma X_9) f = 0$$

one obtains the equation

$$F_t + \frac{\beta 2ts}{y} F_s - 2\beta t F_y + (\beta y + \gamma) \hat{U} F_{\hat{U}} = 0.$$

Thus the universal invariant of this subalgebras consists of invariants

$$\beta t^2 + y, U e^{-\alpha sy - t(2\beta^2 t^2 + 3(\beta y + \gamma)) / 3}.$$

Hence, a representation of the invariant solution is

$$U = e^{\alpha sy + t(2\beta^2 t^2 + 3(\beta y + \gamma)) / 3} \phi(q)$$

with arbitrary functions  $\phi(q)$  and  $q = \beta t^2 + y$ . After substituting this representation into Equation (1), one obtains the ordinary differential equation

$$\phi'' + (\alpha^2 - \beta q - \gamma) \phi = 0.$$

The general solution of the last equation is

$$\begin{aligned} \phi = & C_1 A_i \left( \frac{-\sqrt[3]{-\beta} (\beta q + \gamma - \alpha^2)}{\beta} \right) \\ & + C_2 B_i \left( \frac{-\sqrt[3]{-\beta} (\beta q + \gamma - \alpha^2)}{\beta} \right) \end{aligned}$$

Therefore, the invariant solution of the reduction of the Navier-Stokes Equations (1) is

$$\begin{aligned} U = e^{\alpha sy + t(2\beta^2 t^2 + 3(\beta y + \gamma)) / 3} & \left[ C_1 A_i \left( \frac{-\sqrt[3]{-\beta} (\beta q + \gamma - \alpha^2)}{\beta} \right) \right. \\ & \left. + C_2 B_i \left( \frac{-\sqrt[3]{-\beta} (\beta q + \gamma - \alpha^2)}{\beta} \right) \right]. \end{aligned}$$

where

$$A_i \left( \frac{-\sqrt[3]{-\beta} (\beta q + \gamma - \alpha^2)}{\beta} \right), B_i \left( \frac{-\sqrt[3]{-\beta} (\beta q + \gamma - \alpha^2)}{\beta} \right)$$

are Airy wave functions and  $C_1, C_2$  are arbitrary constants.

The four examples showed that there are solutions of the Navier-Stokes equations, which are partially invariant with respect to not admitted Lie algebra  $\{t\partial_t + x\partial_x + y\partial_y + z\partial_z\}$ . They can return to new solutions of the Navier-Stokes equations.

The result of the study of invariant solutions of Equation (1) corresponding to the subalgebras of **Table 1** are presented in **Table 2**.

### 7. Conclusion

The admitted algebra of the reduction of the Navier-

Stokes Equations (1) is spanned by the generators (19). The optimal systems of two-dimensional subalgebras of the Lie algebra spanned by generators  $X_1, \dots, X_9$  are obtained: there are 20 classes that have invariant solutions. All invariant solutions corresponding to the optimal system are presented in **Table 2**. Examples given in the manuscript show that this algorithm can be applied to groups, which are not admitted. These possibilities extend an area of using group analysis for constructing exact solutions.

**Table 2. Result of invariant solutions of the Equation (1).**

No	Universal invariant	Invariant solution
1	$t, U$	$U = C$
2	$t, U e^{y^2/4t}$	$U = (C e^{-y^2/4t}) / \sqrt{t}$
3	$t, U e^{y^2(1+s^2)/4t}$	$U = (C e^{-y^2(1+s^2)/4t}) / t$
4	1. $t, U e^{y^2/2(2t+1)}$	$U = \frac{C e^{-y^2/2(2t+1)}}{\sqrt{2t+1}}$
	2. $t, U e^{y^2/2(2t-1)}$	$U = \frac{C e^{-y^2/2(2t-1)}}{\sqrt{2t-1}}$
5	$s, U (sy)^{-\alpha}$	$U = C_1 (sy - yi)^{-\alpha-1} + C_2 (sy + yi)^{-\alpha-1}$ $U = t^{(2\alpha-1)/4} y^{-1/2} e^{-y^2/8t} r,$
6	$y^2/t, U y^{1-\alpha}$	$r = C_1 W_1 \left( \frac{1-2\alpha}{4}, \frac{1}{4}, \frac{y^2}{4t} \right) + C_2 W_2 \left( \frac{1-2\alpha}{4}, \frac{1}{4}, \frac{y^2}{4t} \right)$ $U = \frac{t^{(2\alpha-1)/4}}{\sqrt{y} e^{y^2(1+2s^2)/8t}} r,$
7	$y^2/t, U y^{1-\alpha} e^{(sy)^2/4t}$	$r = C_1 W_1 \left( \frac{-1-2\alpha}{4}, \frac{1}{4}, \frac{y^2}{4t} \right) + C_2 W_2 \left( \frac{-1-2\alpha}{4}, \frac{1}{4}, \frac{y^2}{4t} \right)$ $U = t^\alpha e^{-y^2(1+s^2)/4t} r,$
8	$s, U t^{-\alpha} y^{1-\alpha} e^{y^2(1+s^2)/4t}$	$r = C_1 (sy + yi)^{-\alpha-1} + C_2 (sy - yi)^{-\alpha-1}$
9	$(sy+1)/y, U y^{1-\alpha}$	$U = C_1 (sy + 1 - yi)^{-\alpha-1} + C_2 (sy + 1 + yi)^{-\alpha-1}$
10	$q = \sqrt{y^2 + (sy)^2} e^{-\alpha \tan^{-1}(s)},$	$U = q^{\frac{\alpha(\alpha-\beta)}{\alpha^2+1}} e^{(\beta-\alpha) \tan^{-1}(s)} r,$
	$U e^{(\alpha-\beta) \tan^{-1}(s)}$	$r = C_1 \sin \left( \frac{\alpha-\beta}{\alpha^2+1} \ln q \right) + C_2 \cos \left( \frac{\alpha-\beta}{\alpha^2+1} \ln q \right)$

Continued

$$\begin{aligned}
 q &= 2\alpha \tan^{-1} \left( \frac{\alpha + sy + \alpha^2 sy}{1 + y + \alpha^2 y} \right) \\
 & - \ln \left[ (1 + y + \alpha^2 y)^2 \right. \\
 & \left. + (\alpha + sy + \alpha^2 sy)^2 \right], \\
 U e^{(\alpha - \beta) \tan^{-1} \left( \frac{\alpha + sy + \alpha^2 sy}{1 + y + \alpha^2 y} \right)}
 \end{aligned}$$

$$\begin{aligned}
 q &= \frac{\sqrt{y^2(1+s^2)}}{t} e^{\alpha \tan^{-1}(s)}, \\
 U &= \frac{q^{\frac{1-\alpha\beta}{\alpha^2+1}} r}{\sqrt{y^2(1+s^2)} e^{\frac{y^2(1+s^2)}{4t} \beta \tan^{-1}(s)}},
 \end{aligned}$$

$$\begin{aligned}
 q &= \frac{\sqrt{y^2(1+s^2)} U}{e^{\beta \tan^{-1}(s) - y^2(1+s^2)/4t}}, \\
 r &= C_1 \sin \left( \frac{\alpha + \beta}{\alpha^2 + 1} \ln q \right) \\
 & + C_2 \cos \left( \frac{\alpha + \beta}{\alpha^2 + 1} \ln q \right)
 \end{aligned}$$

$$\begin{aligned}
 q &= y^2(1+s^2)^{\frac{1}{2}} t^{-\frac{1}{2}}, \\
 U &= \frac{t^{\frac{\alpha}{2}} e^{-\beta \tan^{-1}(y/s)} r}{\sqrt{y^2(1+s^2)}},
 \end{aligned}$$

$$\begin{aligned}
 U &= \frac{e^{\beta \tan^{-1}(y/s)}}{(y^2(1+s^2))^{\frac{\alpha-1}{2}}} \\
 r &= C_1 W_1 \left( \frac{-\alpha}{2}, \frac{\beta i}{2}, \frac{q^2}{4} \right) \\
 & + C_2 W_2 \left( \frac{-\alpha}{2}, \frac{\beta i}{2}, \frac{q^2}{4} \right)
 \end{aligned}$$

$$\begin{aligned}
 q &= -2\alpha \tan^{-1} \left( \frac{sy + \alpha^2 sy - 2\alpha t}{y + \alpha^2 y + 2t} \right) \\
 & - 2 \ln \left[ ((sy + \alpha^2 sy - 2\alpha t)^2 \right. \\
 & \left. + (y + \alpha^2 y + 2t)^2) t^2 \right], \\
 U &= \frac{t U e^{\frac{y^2(1+s^2)}{4t}}}{e^{(\alpha + \beta) \tan^{-1} \left( \frac{sy + \alpha^2 sy - 2\alpha t}{y + \alpha^2 y + 2t} \right)}}
 \end{aligned}$$

$$\begin{aligned}
 r &= C_1 \sin \left( \frac{(\alpha + \beta) q}{2(\alpha^2 + 1)} \right) \\
 & + C_2 \cos \left( \frac{(\alpha + \beta) q}{2(\alpha^2 + 1)} \right) \\
 U &= \frac{e^{\frac{\beta \tan^{-1}(2t) - q t - \alpha \tan^{-1}(\frac{1}{s})}{2}} r}{\sqrt{y^2(1+s^2)}},
 \end{aligned}$$

$$\begin{aligned}
 1. \quad q &= \frac{y^2(1+s^2)}{4t^2+1}, \\
 U &= \frac{U \sqrt{4t^2+1}}{e^{\frac{\beta \tan^{-1}(2t) - q t - \alpha \tan^{-1}(\frac{1}{s})}{2}}} \\
 r &= C_1 W_1 \left( \frac{\beta i}{4}, \frac{\alpha i}{2}, q i \right) \\
 & + C_2 W_2 \left( \frac{\beta i}{4}, \frac{\alpha i}{2}, q i \right)
 \end{aligned}$$

$$\begin{aligned}
 2. \quad q &= \frac{y^2(1+s^2)}{4t^2-1}, \\
 U &= \frac{U \sqrt{4t^2+1} e^{q t + \alpha \tan^{-1}(\frac{1}{s})}}{\left( \frac{2t-1}{2t+1} \right)^{\frac{\beta}{4}}} \\
 r &= C_1 W_1 \left( \frac{-\beta}{4}, \frac{\alpha i}{2}, q \right) \\
 & + C_2 W_2 \left( \frac{-\beta}{4}, \frac{\alpha i}{2}, q \right)
 \end{aligned}$$

Continued

$$\begin{aligned}
 16 \quad t, U e^{\frac{y^2(1+s^2)}{4t} - \frac{(y+2sy)^2}{4(1+2\alpha t-4t^2)}} \\
 U &= \frac{C e^{\frac{(y+2sy)^2}{4(1+2\alpha t-4t^2)} - \frac{y^2}{4t}}}{\sqrt{1+2\alpha t-4t^2}}
 \end{aligned}$$

$$\begin{aligned}
 U &= e^{\alpha sy + (2\beta^2 t^2 + 3(\beta y + \gamma)) / \beta} r, \\
 17 \quad q &= \beta t^2 + y, \\
 U e^{-\alpha sy - (2\beta^2 t^2 + 3(\beta y + \gamma)) / \beta} \\
 r &= C_1 A_1 \left( \frac{-\sqrt{-\beta}(\beta q + \gamma - \alpha^2)}{\beta} \right) \\
 & + C_2 B_1 \left( \frac{-\sqrt{-\beta}(\beta q + \gamma - \alpha^2)}{\beta} \right)
 \end{aligned}$$

$$\begin{aligned}
 U &= \frac{r e^{\frac{(1-y)^2}{4t}}}{t e^{\frac{24t^2(y^2 + \gamma - 1) + \beta^2 - 12\beta \gamma}{9\alpha t^3}}}, \\
 18 \quad q &= \frac{8ty - \beta}{8t^2}, \\
 U t e^{\frac{(1-y)^2}{4t} - \frac{24t^2(y^2 + \gamma - 1) + \beta^2 - 12\beta \gamma}{9\alpha t^3}} \\
 r &= C_1 A_1 \left( \frac{2\gamma - \beta q - 2}{2\beta^{2/3}} \right) \\
 & + C_2 B_1 \left( \frac{2\gamma - \beta q - 2}{2\beta^{2/3}} \right)
 \end{aligned}$$

$$\begin{aligned}
 U &= \frac{\sqrt{q} e^{\frac{y^2}{4t^2-1} - \frac{y^2 s^2}{2(1+2t)}} (2t-1)^{\frac{\alpha}{4}}}{\sqrt{4t^2-1} (2t+1)^{\frac{\alpha}{4}}} r, \\
 1. \quad q &= \frac{(4t^2-1)^{1/2}}{y}, \\
 r &= C_1 W_1 \left( \frac{1-\alpha}{4}, \frac{1}{4}, \frac{1}{q^2} \right) \\
 & + C_2 W_2 \left( \frac{1-\alpha}{4}, \frac{1}{4}, \frac{1}{q^2} \right)
 \end{aligned}$$

$$\begin{aligned}
 U &= \frac{U \sqrt{4t^2-1} (2t+1)^{\frac{\alpha}{4}}}{e^{\frac{y^2}{4t^2-1} - \frac{(sy)^2}{2(1+2t)}} (2t-1)^{\frac{\alpha}{4}}} \\
 19 \quad 2. \quad q &= \frac{(4t^2-1)^{1/2}}{y}, \\
 U &= \frac{\sqrt{q} e^{\frac{y^2}{4t^2-1} - \frac{y^2 s^2}{2(1+2t)}} (2t-1)^{\frac{\alpha}{4}}}{\sqrt{4t^2-1} (2t+1)^{\frac{\alpha}{4}}} r, \\
 r &= C_1 W_1 \left( \frac{-1-\alpha}{4}, \frac{1}{4}, \frac{1}{q^2} \right) \\
 & + C_2 W_2 \left( \frac{-1-\alpha}{4}, \frac{1}{4}, \frac{1}{q^2} \right)
 \end{aligned}$$

$$\begin{aligned}
 U &= \frac{U \sqrt{4t^2-1} (2t+1)^{\frac{\alpha}{4}}}{e^{\frac{(sy)^2}{2(1-2t)} - \frac{y^2}{4t^2-1}} (2t-1)^{\frac{\alpha}{4}}} \\
 20 \quad 1/s, \\
 U (1+2t)^{-\alpha} (sy)^{1+\alpha} e^{\frac{y^2+(sy)^2}{2(1+2t)}} \\
 r &= C_1 (y + syi)^{-\alpha-1} \\
 & + C_2 (y - syi)^{-\alpha-1}
 \end{aligned}$$

where  $W_1, W_2$  are Whittaker functions,  $A_i, B_i$  are Airy wave functions and  $C, C_1, C_2$  are constants.

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