

Solution of MHD Flow past a Vertical Porous Plate through a Porous Medium under Oscillatory Suction

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ABSTRACT

The model of mass transfer on free convective flow of a viscous incompressible electrically conducting fluid past vertically porous plate through a porous medium with time dependant permeability and oscillatory suction in presence of a transverse magnetic field is considered. Perturbation technique is obtained the solution for velocity field and concentration distribution analytically. The effects of the flow parameters on the velocity field and concentration distribution are presented with the aid of figures. Also, the skin friction and the rate of mass transfer are calculated with the aid of tables.

Keywords: MHD Flow; Mass Transfer; Porosity; Oscillation Suction

1. Introduction

The phenomena of MHD flow with unsteady oscillatory free convective flows play an important role in aerospace technology and in chemical engineering turbo-machinery. Such flow arise either due to unsteady motion of boundary or due to oscillatory free system velocity. Several authors have studied the free convection and mass transfer flow of a viscous fluid through porous medium. They considered that, the permeability of the porous medium is assumed to be constant, while the porosity of the medium may be not necessary constant because the porous material containing the fluid is non-homogenous medium. Then, the permeability of the porous medium may be not necessary constant. The effect of mass transfer on free convective hydromagnetic oscillatory flow past an infinite vertical unsteady flow of non-Newtonian fluid has been estimated by Koullias *et al.* [1]. Ashgar *et al.* [2] have reported the periodic unsteady flows of non-Newtonian fluid. The flow of non-Newtonian fluid induced due to the oscillations of a porous plate has studied by Hayat *et al.* [3]. Singh and Gupta [4] have investigated the MHD free convective flow of a viscous fluid through a porous medium bounded by an oscillating porous plate in the slip flow regime with mass transfer. Das *et al.* [5] analyzed the mass transfer effects on unsteady flow past

an accelerated vertical porous plate with suction employing numerical methods. Raptis [6] analyzed the unsteady free convection flow through a porous medium. Hossain and Begum [7] discussed the effect of mass transfer and free convection on the flow past a vertical plate. Unsteady hydromagnetic convective flow past an infinite vertical porous flat plate in a porous medium has been discussed by Das and his coworkers [8]. The objective of the present study is to analyzed the effects of permeability variation and oscillatory suction velocity in free convective and mass transfer flow of a viscous incompressible fluid past an infinite vertical porous plate through a porous medium when the plate is subjected to a time dependent suction velocity normal to the plate in the presence of a uniform transverse magnetic field. The solutions for velocity field and concentration distribution are obtained by perturbation technique analytically. The effects of the flow parameters on the velocity field and concentration distribution are presented with the aid of figures. Also, the skin friction and the rate of mass transfer are calculated with the aid of tables.

2. Mathematical Analysis

The unsteady free convective flow of a viscous incompressible electrically conducting fluid past an infinite

vertical porous plate through a porous medium with time dependant permeability and oscillatory suction in presence of a transverse magnetic field is considered. The y -axis is taken along the infinite vertical porous plate and x -axis is perpendicular to it and the velocity in this direction is u . In view of these, the analysis of the problem is based on the following assumptions:

- All the fluid properties are constant;
- The plate as well as the fluid is assumed to be at the same temperature and the concentration of the species is raised or lowered;
- The magnetic Reynolds number is small so that the induced magnetic field can be neglected in comparison to the applied magnetic field;
- The permeability of the porous medium is assumed to be $K'(t') = K_p (1 + \varepsilon e^{i\omega t'})$;
- The suction velocity is assumed to be $v(t') = -v_o (1 + \varepsilon e^{i\omega t'})$.

With foregoing assumptions and taking the usual Boussinesq's approximation into account as well as following non-dimensional quantities:

$$u = \frac{u}{v_o}, y = \frac{v_o y'}{v}, t = \frac{v_o^2 t'}{4\nu}, S_c = \frac{\nu}{D}, \omega = \frac{4\nu\omega'}{v_o^2},$$

$$K_p = \frac{v_o^2 K'}{v_o^2}, M^2 = \frac{\sigma\nu B_o^2}{\rho\nu_o^2}, C = \frac{C' - C_\infty}{C_w - C_\infty}, \quad (1)$$

$$G_c = \frac{(C_w - C_\infty)vg\beta}{v_o^3}.$$

Therefore, the non-dimensional form of the governing equations for momentum and concentration are:

$$u_{yy} + f(t)u_y + CG_c = \frac{u_t}{4} + \left(M^2 + \frac{1}{f(t)} \right) u, \quad (2)$$

$$C_{yy} + f(t)S_c C_y = \frac{S_c C_t}{4}, \quad (3)$$

where $f(t) = (1 + \varepsilon e^{i\omega t})$ Subjected to the following boundary conditions:

$$u(0, t) = C(0, t) = f(t), \text{ and} \quad (4)$$

$$\lim_{y \rightarrow \infty} u(y, t) = \lim_{y \rightarrow \infty} C(y, t) = 0.$$

3. Solutions of the Problem

In order to solve the problem, we assume the solutions of the following form because the amplitude $\varepsilon (\ll 1)$ of the permeability variation is very small:

$$u(y, t) = u_o(y) + \varepsilon u_1(y) e^{i\omega t}, \quad (5)$$

$$C(y, t) = C_o(y) + \varepsilon C_1(y) e^{i\omega t}. \quad (6)$$

Substitute Equations (5) and (6) into Equations (2) and

(3) and equate the harmonic and non-harmonic terms to obtain the following non-dimensional ordinary differential equations:

$$u_o'' + u_o' + G_c C_o = a_1 u_o, \quad (7)$$

$$u_1'' + u_1' + u_o' + \frac{u_o}{K_p} + G_c C_1 = a_2 u_1, \quad (8)$$

$$C_o'' + S_c C_o' = 0, \quad (9)$$

$$C_1'' + S_c C_1' + S_c C_o' = \frac{i\omega}{4} S_c C_1. \quad (10)$$

where $a_1 = M^2 + \frac{1}{k_p}$ and $a_2 = a_1 + \frac{i\omega}{4}$.

Furthermore, the boundary conditions are reduced to

$$u_o(0) = u_1(0) = C_o(0) = C_1(0) = 1 \text{ and} \quad (11)$$

$$u_o = u_1 = C_o = C_1 \rightarrow 0 \text{ as } y \rightarrow \infty$$

By solving Equations (7)-(10) under boundary conditions (11), we obtain

$$u(y, t) = e^{-m_2 y} + \frac{G_c (e^{-S_c y} - e^{-m_2 y})}{(m_2 - S_c)(m_3 + S_c)}$$

$$+ \varepsilon e^{i\omega t} \left\{ e^{-m_4 y} + \frac{G_c \left(1 - \frac{4i}{\omega} S_c \right) (e^{-m_1 y} - e^{-m_4 y})}{(m_4 - m_1)(m_5 + m_1)} \right.$$

$$+ \frac{G_c (e^{-S_c y} - e^{-m_4 y})}{(m_4 - S_c)(m_5 + S_c)} \left[\frac{4i}{\omega} S_c + \frac{\left(\frac{1}{K_p} - S_c \right)}{(m_2 - S_c)(m_3 + S_c)} \right]$$

$$\left. + \frac{\left(\frac{1}{K_p} - m_2 \right) (e^{-m_2 y} - e^{-m_4 y})}{(m_4 - m_2)(m_5 + m_2)} \left(1 - \frac{G_c}{(m_2 - S_c)(m_3 + S_c)} \right) \right\} \quad (12)$$

$$C(y, t) = e^{-S_c y} + \varepsilon e^{i\omega t} \left\{ e^{-m_1 y} + \frac{4i}{\omega} S_c (e^{-S_c y} - e^{-m_1 y}) \right\}. \quad (13)$$

The real and imaginary parts are separated from Equations (12) and (13). The physical significance is the real parts which are given as follow:

$$u(y, t) = u_o(y) + \varepsilon (M_r \cos(\omega t) - M_i \sin(\omega t)), \quad (14)$$

$$C(y, t) = C_o(y) + \varepsilon (L_r \cos(\omega t) - L_i \sin(\omega t)). \quad (15)$$

For $\omega t = \frac{\pi}{2}$, the transient velocity and concentration distribution are given by

$$u\left(y, \frac{\pi}{2\omega}\right) = u_c(y) - \varepsilon M_i, \tag{16}$$

$$C\left(y, \frac{\pi}{2\omega}\right) = C_c(y) - \varepsilon L_i. \tag{17}$$

The skin friction at the plate in terms of amplitude and phase is given by

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = \tau = \tau_o + \varepsilon |N| \cos(\omega t + \alpha), \tag{18}$$

where

$$\tau_o = -m_2 + \frac{G_c}{m_3 + S_c}, \left(\frac{du_1}{dy}\right)_{y=0} = N = N_r + iN_i, \tag{19}$$

$$\tan(\alpha) = \frac{N_i}{N_r}.$$

The Sherwood number (the mass transfer coefficient) at the plate terms of amplitude and phase is given by

$$-\left(\frac{\partial C}{\partial y}\right)_{y=0} = S_h = S_c + \varepsilon |Q| \cos(\omega t + \gamma), \tag{20}$$

where

$$-\left(\frac{dC_1}{dy}\right)_{y=0} = Q = Q_r + iQ_i, \tan(\gamma) = \frac{Q_i}{Q_r}. \tag{21}$$

The constants $A_1 - A_6$, a_3 , $B_1 - B_6$, $b_1 - b_5$, L_r , L_i , M_r , M_i , $m_1 - m_5$, $N_1 - N_{12}$, N_r , N_i , Q_r , Q_i , R_r and R_i are defined in appendix section.

4. Numerical Simulation and Discussion

The problem of free convective and mass transfer flow of

a viscous incompressible electrically conducting fluid past an infinite vertical porous plate through a porous medium when with time dependent permeability and oscillatory suction in presence of a uniform transverse magnetic field has been considered. Perturbation technique is obtained the solution for velocity field and concentration distribution analytically. The effects of the flow parameters such as magnetic parameter M , Grashof number for mass transfer G_c , porosity parameter K_p and Schmidt number S_c have been studied analytically and presented with the help of **Figures 1-4**, respectively. The effects of Schmidt number S_c on the concentration distribution have been showed in **Figure 5**. Furthermore, the effects of the flow parameters on skin friction and the rate of mass transfer have been reported with the help of **Tables 1-7**.

The effect of magnetic field parameter M on the velocity profiles of the flow field keeping other parameters of the flow field constant is showed in **Figure 1**. The magnetic field parameter is taken in increasing order and it is found to slow down the velocity of the flow field to a considerable amount due to the magnetic pull of the Lorentz force acting on the flow field. This parameter shows overturn effect.

The effect of Schmidt number S_c on the velocity profiles of the flow field keeping other parameters of the flow field constant is showed in **Figure 2**. The Schmidt number is taken in increasing order and it is reveal that the presence of heavier diffusion species has a delaying effect on velocity of the flow field. This parameter shows reverse effect.

The effect of Grashof number for mass transfer G_c on the velocity profiles of the flow field keeping other parameters of the flow field constant is showed in **Figure 3**. It is noted that the Grashof number for mass transfer is accelerated the velocity of the flow field at all points.

The effect of porosity parameter K_p on the velocity

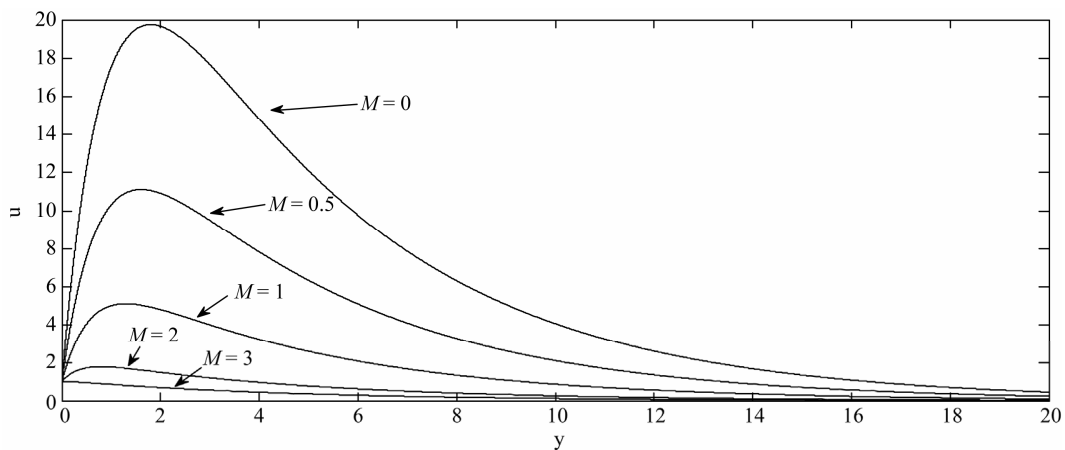


Figure 1. Effect of M on velocity profiles with $G_c = 10$, $K_p = 10$, $S_c = 0.22$, $\varepsilon = 0.002$, $\omega = 5$ and $\omega t = \frac{\pi}{2}$.

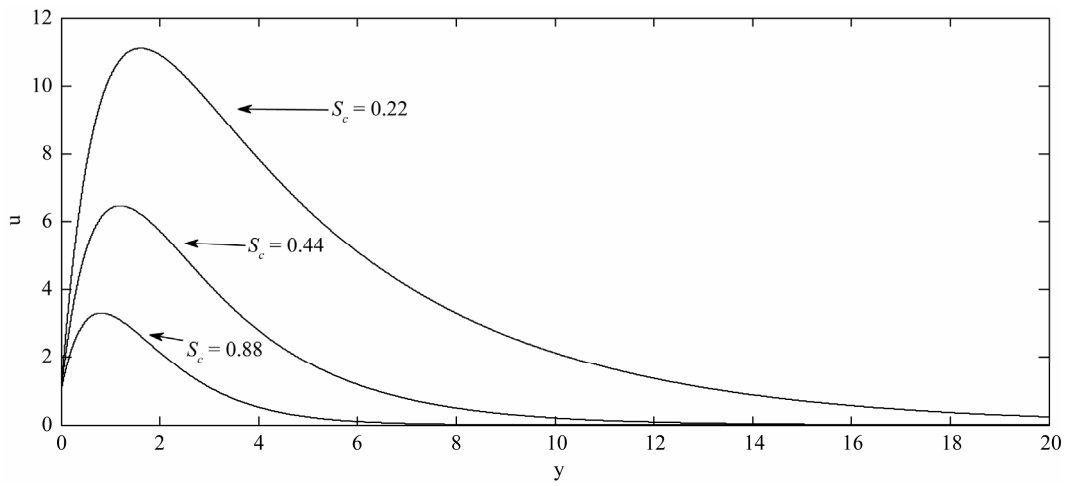


Figure 2. Effect of S_c on velocity profiles with $M = 0.5$, $G_c = 10$, $K_p = 10$, $\varepsilon = 0.002$, $\omega = 5$ and $\omega t = \frac{\pi}{2}$.

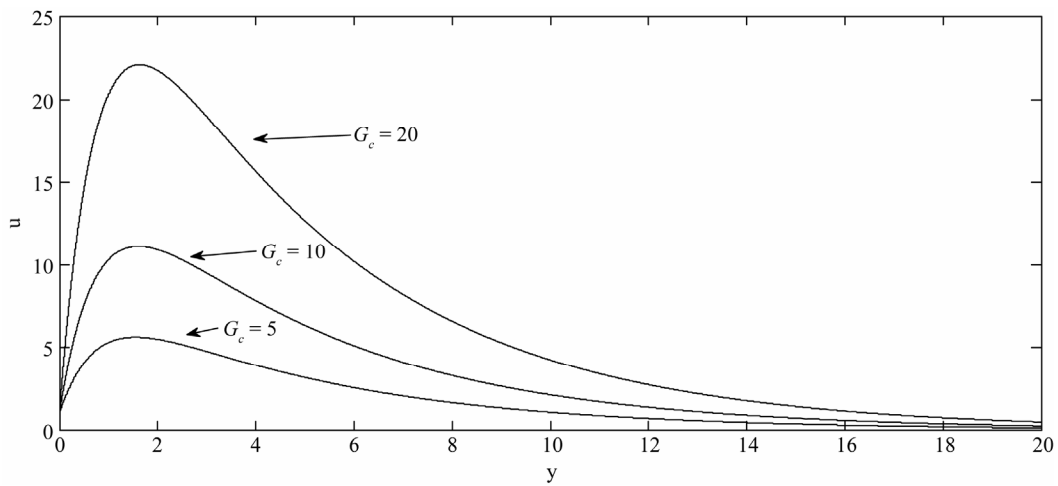


Figure 3. Effect of G_c on velocity profiles with $M = 0.5$, $K_p = 10$, $S_c = 0.22$, $\varepsilon = 0.002$, $\omega = 5$ and $\omega t = \frac{\pi}{2}$.

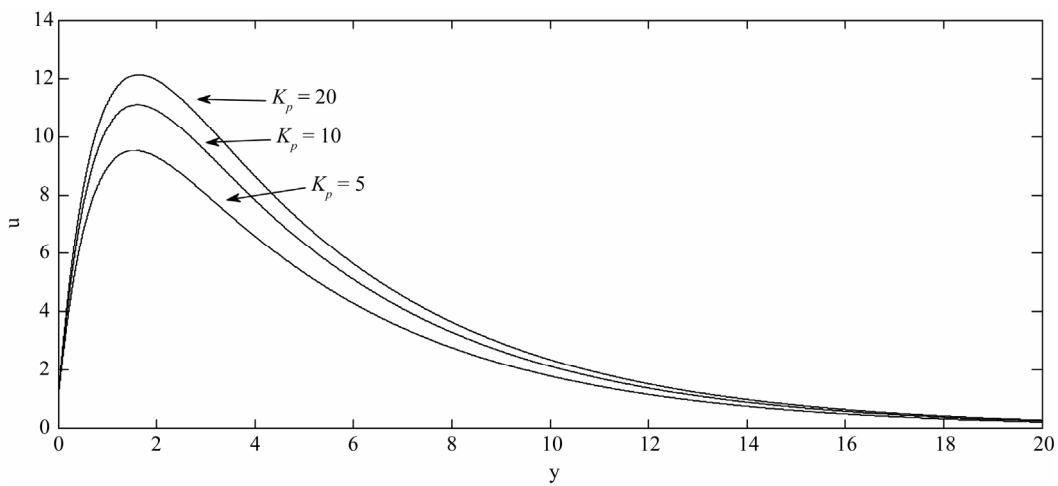


Figure 4. Effect of K_p on velocity profiles with $M = 0.5$, $G_c = 10$, $S_c = 0.22$, $\varepsilon = 0.002$, $\omega = 5$ and $\omega t = \frac{\pi}{2}$.

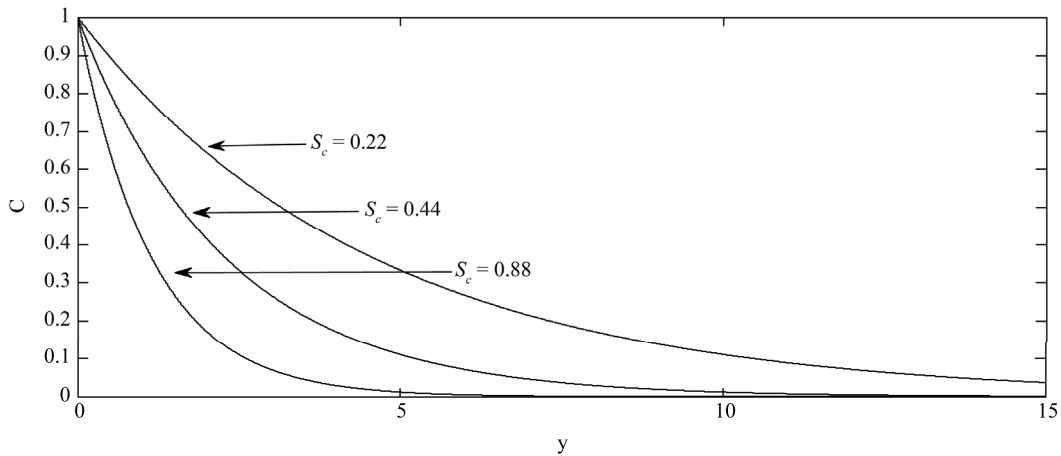


Figure 5. Effect of S_c on concentration distribution with $\varepsilon = 0.002$, $\omega = 5$ and $\omega t = \frac{\pi}{2}$.

Table 1. Values of amplitude, phase and the skin friction coefficients for different values of M with $G_c = 10$, $K_p = 10$, $S_c = 0.22$, $\varepsilon = 0.002$, $\omega = 5$ and $\omega t = \frac{\pi}{2}$.

M	Amplitude ($ N $)	Phase ($\tan(\alpha)$)	τ_s	τ
0.5	15.6917	-0.5289	18.9438	18.9586
1	9.0363	-0.5000	9.6773	9.6854
2	2.8416	-0.4867	2.9524	2.9549
3	0.6636	1.5269	0.0422	0.0433

Table 2. Values of amplitude, phase and the skin friction coefficients for different values of S_c with $M = 0.5$, $G_c = 10$, $K_p = 10$, $\varepsilon = 0.002$, $\omega = 5$ and $\omega t = \frac{\pi}{2}$.

S_c	Amplitude ($ N $)	Phase ($\tan(\alpha)$)	τ_s	τ
0.22	15.6917	-0.5289	18.9438	18.9586
0.44	9.6496	-0.3848	12.7193	12.7262
0.66	6.2993	-0.1921	9.4252	9.4275
0.88	4.4432	0.2405	7.3864	7.3843

profiles of the flow field keeping other parameters of the flow field constant is showed in **Figure 4**. It is observed that the porosity parameter is enhanced the velocity of the flow field at all points.

The concentration distribution decreases at all points of the flow field with the increase of the Schmidt number S_c . Singh *et al.* [9] have applied this result into some foreign mass such as hydrogen ($S_c = 0.22$) helium ($S_c = 0.30$) and ammonia ($S_c = 0.78$).

Table 3. Values of amplitude, phase and the skin friction coefficients for different values of G_c with $M = 0.5$, $K_p = 10$, $S_c = 0.22$, $\varepsilon = 0.002$, $\omega = 5$ and $\omega t = \frac{\pi}{2}$.

G_c	Amplitude ($ N $)	Phase ($\tan(\alpha)$)	τ_s	τ
5	7.1117	-0.6758	8.8346	8.8426
10	15.6917	-0.5289	18.9438	18.9585
15	24.3248	-0.4893	29.0531	29.0745
20	32.9695	-0.4709	39.1623	39.1904

Table 4. Values of amplitude, phase and the skin friction coefficients for different values of K_p with $M = 0.5$, $G_c = 10$, $S_c = 0.22$, $\varepsilon = 0.002$, $\omega = 5$ and $\omega t = \frac{\pi}{2}$.

K_p	Amplitude ($ N $)	Phase ($\tan(\alpha)$)	τ_s	τ
5	15.0930	-0.5837	16.6276	16.6428
10	15.6917	-0.5289	18.9438	18.9585
15	15.9409	-0.5025	19.8990	19.9133
20	16.0800	-0.4874	20.4212	20.4353

In **Tables 1-5**, the numerical values of skin friction coefficients τ_s, τ in terms of amplitude ($|N|$) and phase ($\tan(\alpha)$) for different values of M, S_c, G_c, K_p and ω are entered, respectively. It is noted that the value of the phase ($\tan(\alpha)$) increases due to decrease in ω while increases due to increase in M, S_c, G_c and K_p . Also, It is observed that an increasing in G_c and K_p leads to an increasing in the value of amplitude ($|N|$) while an increase of M, S_c and ω leads to a decrease in the value

of $|N|$. Further, it is showed that the skin friction coefficients τ_o and τ increase due to increase in G_c , K_p while decrease due to increase in M , S_c and ω with note that τ_o is not function on ω .

In **Tables 6** and **7**, present the numerical values of amplitude ($|Q|$), phase ($\tan(\gamma)$) and the rate of mass transfer in term of Sherwood number S_h due to the variations in the values of S_c and ω , respectively. It is watched that an increase in S_c results in an increase in

Table 5. Values of amplitude, phase and the skin friction coefficients for different values of ω with $M=0.5$, $G_c=10$,

$K_p=10$, $\epsilon=0.002$, $\omega=5$ and $\omega t = \frac{\pi}{2}$.

ω	Amplitude ($ N $)	Phase ($\tan(\alpha)$)	τ_o	τ
5	15.6917	-0.5289	18.9438	18.9585
10	12.0945	-0.7427	18.9438	18.9583
15	10.2692	-0.9189	18.9438	18.9577
20	9.1398	-1.0936	18.9438	18.9573

Table 6. Values of amplitude, phase and the rate of mass transfer for different values of S_c with $\epsilon=0.002$ and

$\omega t = \frac{\pi}{2}$.

S_c	Amplitude ($ Q $)	Phase ($\tan(\gamma)$)	S_h
0.11	0.3345	0.2060	0.1097
0.22	0.5333	0.2366	0.2194
0.33	0.7218	0.2454	0.3289
0.44	0.9115	0.2457	0.4384
0.55	1.1064	0.2418	0.5476
0.66	1.3089	0.2359	0.6567
0.77	1.5197	0.2292	0.7654
0.88	1.7396	0.2222	0.8741
0.99	1.9687	0.2150	0.9824

Table 7. Values of amplitude, phase and the rate of mass transfer for different values of ω with $\epsilon=0.002$ and $S_c=0.22$.

ω	Amplitude ($ Q $)	Phase ($\tan(\gamma)$)	S_h
5	0.5333	0.2366	0.2194
10	0.6930	0.3118	0.2191
15	0.8194	0.3454	0.2187
20	0.9272	0.3651	0.2183

$|Q|$ and S_h but the phase ($\tan(\gamma)$) increases at $0.11 \leq S_c \leq 0.44$ while it decreases at $S_c \geq 0.55$. An increase in ω results in an increase in $|Q|$ and $\tan(\gamma)$ and a decrease in S_h .

5. Conclusions

The above study shows up the following results of physical interest on the velocity and concentration distribution of the flow field.

- The magnetic field parameter M slows down the velocity of the flow field at all points due to the magnetic pull of the Lorentz force acting on the flow field.
- The Schmidt number S_c reveals that the presence of heavier diffusion species has a delaying effect on velocity of the flow field.
- The Grashof number for mass transfer G_c is accelerated the velocity of the flow field at all points.
- The effect of porosity parameter K_p is enhanced the velocity of the flow field at all points.
- The concentration distribution decreases at all points of the flow field with the increase of the Schmidt number S_c . It means that a greater delaying effect on the concentration distribution is due to the heavier diffusing species of the flow field.
- The value of the phase ($\tan(\alpha)$) increases due to decrease in ω while increases due to increase in M , S_c , G_c and K_p .
- An increasing in G_c and K_p leads to an increasing in the value of amplitude ($|N|$) while an increase of M , S_c and ω leads to a decrease in the value of $|N|$.
- The skin friction coefficients τ_o and τ increase due to increase in G_c , K_p while decrease due to increase in M , S_c and ω with note that τ_o is not function on ω .
- An increase in S_c results in an increase in $|Q|$ and S_h but the phase ($\tan(\gamma)$) increases at $0.11 \leq S_c \leq 0.44$ while it decreases at $S_c \geq 0.55$.
- An increase in ω results in an increase in $|Q|$ and $\tan(\gamma)$ and a decrease in S_h .

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Nomenclatures

C'	Species concentration;	y'	Distance along y-axis;
C	Non-dimension species concentration;	y	Non-dimension distance along y-axis;
D	Molecular diffusivity;	β	Volumetric coefficient of expansion with species concentration;
G_c	Grashof number for mass transfer;	ε	A small positive constant ($\ll 1$);
G	Acceleration due to gravity;	ρ	Density of fluid;
K'	Permeability of the medium;	ν	Kinematic coefficient of viscosity;
K_p	Permeability/porosity parameter;	ω'	Frequency of oscillation;
M	Magnetic parameter;	ω	Non-dimension frequency of oscillation;
S_c	Schmidt number;	τ	Skin friction;
t'	Time;	σ	Electrical conductivity;
t	Non-dimension time;	∞	Subscript condition far away from the plate;
u'	Velocity component along x-axis;	'	Superscript means $\frac{d}{dy}$ (used in Equations (7)-(10));
u	Non-dimension velocity component along x-axis;	S_h	Sherwood number.
$v(t')$	Suction velocity;		
v_s	Positive constant suction velocity;		

Appendix

$$m_1 = \frac{1}{2} \left[S_c + \sqrt{S_c^2 + i\omega S_c} \right] = A_1 + iB_1,$$

$$m_2 = \frac{1}{2} \left[1 + \sqrt{1 + 4a_1} \right],$$

$$m_3 = \frac{1}{2} \left[-1 + \sqrt{1 + 4a_1} \right],$$

$$m_4 = \frac{1}{2} \left[1 + \sqrt{1 + 4a_2} \right] = A_2 + iB_2,$$

$$m_5 = \frac{1}{2} \left[-1 + \sqrt{1 + 4a_2} \right] = A_3 + iB_3,$$

$$(m_4 - m_1)(m_5 + m_1) = A_4 + iB_4,$$

$$(m_4 - S_c)(m_5 + S_c) = A_5 + iB_5,$$

$$(m_4 - m_2)(m_5 + m_2) = A_6 + iB_6,$$

$$(m_2 - S_c)(m_3 + S_c) = a_3,$$

$$A_1 = \frac{S_c}{2} + \frac{1}{2} \sqrt{\frac{S_c}{2} \left[\sqrt{\sqrt{S_c^2 + \omega^2} + S_c} \right]},$$

$$A_2 = \frac{1}{2} + \frac{1}{2\sqrt{2}} \left[\sqrt{\sqrt{(1 + 4a_1)^2 + \omega^2} + 1 + 4a_1} \right],$$

$$A_3 = A_2 - 1,$$

$$N_3 = b_1 e^{-A_1 y} \left\{ \left(A_4 - \frac{4S_c}{\omega} B_4 \right) \cos B_1 y - \left(\frac{4S_c}{\omega} A_4 + B_4 \right) \sin B_1 y \right\} - b_1 e^{-A_2 y} \left\{ \left(A_4 - \frac{4S_c}{\omega} B_4 \right) \cos B_2 y - \left(\frac{4S_c}{\omega} A_4 + B_4 \right) \sin B_2 y \right\},$$

$$N_4 = b_1 e^{-A_1 y} \left\{ \left(\frac{4S_c}{\omega} A_4 + B_4 \right) \cos B_1 y + \left(A_4 - \frac{4S_c}{\omega} B_4 \right) \sin B_1 y \right\} - b_1 e^{-A_2 y} \left\{ \left(B_4 + \frac{4S_c}{\omega} A_4 \right) \cos B_2 y + \left(A_4 - \frac{4S_c}{\omega} B_4 \right) \sin B_2 y \right\},$$

$$N_5 = b_2 A_5 e^{-S_c y} - b_2 e^{-A_2 y} \{ A_5 \cos B_2 y - B_5 \sin B_2 y \},$$

$$N_6 = b_2 B_5 e^{-S_c y} - b_2 e^{-A_2 y} \{ B_5 \cos B_2 y + A_5 \sin B_2 y \},$$

$$N_7 = b_3 B_5 e^{-S_c y} - b_3 e^{-A_2 y} \{ B_5 \cos B_2 y + A_5 \sin B_2 y \},$$

$$N_8 = b_3 e^{-A_2 y} \{ A_5 \cos B_2 y - B_5 \sin B_2 y \} - b_3 A_5 e^{-S_c y},$$

$$N_9 = b_4 e^{-A_2 y} \{ A_6 \cos B_2 y - B_6 \sin B_2 y \} - b_4 A_6 e^{-m_2 y},$$

$$N_{10} = b_4 e^{-A_2 y} \{ B_6 \cos B_2 y + A_6 \sin B_2 y \} - b_4 B_6 e^{-m_2 y},$$

$$N_{11} = b_5 A_6 e^{-m_2 y} - b_5 e^{-A_2 y} \{ A_6 \cos B_2 y - B_6 \sin B_2 y \},$$

$$N_{12} = b_5 B_6 e^{-m_2 y} - b_5 e^{-A_2 y} \{ B_6 \cos B_2 y + A_6 \sin B_2 y \},$$

$$b_1 = \frac{G_c}{A_4^2 + B_4^2},$$

$$A_4 = (A_2 - A_1)(A_3 + A_1) + B_1^2 - B_2^2,$$

$$A_5 = (A_2 - S_c)(A_3 + S_c) - B_2^2,$$

$$A_6 = (A_2 - m_2)(A_3 + m_2) - B_2^2,$$

$$B_1 = \frac{S_c}{2\sqrt{2}} \left[\left[\sqrt{\sqrt{S_c^2 + \omega^2} - S_c} \right] \right],$$

$$B_2 = \frac{1}{2\sqrt{2}} \left[\sqrt{\sqrt{(1 + 4a_1)^2 + \omega^2} - 1 - 4a_1} \right],$$

$$B_3 = B_2,$$

$$B_4 = (A_2 - A_1)(B_2 + B_1) + (A_3 + A_1)(B_2 - B_1),$$

$$B_5 = B_2 (A_2 + A_3),$$

$$B_6 = B_5,$$

$$M_r = N_1 + N_3 + N_5 + N_7 + N_9 + N_{11},$$

$$M_i = -N_2 - N_4 - N_6 - N_8 - N_{10} - N_{12},$$

$$L_r = e^{-A_1 y} \left(\cos B_1 y - \frac{4S_c}{\omega} \sin B_1 y \right),$$

$$L_i = \frac{4S_c}{\omega} e^{-S_c y} - e^{-A_1 y} \left(\frac{4S_c}{\omega} \cos B_1 y + \sin B_1 y \right),$$

$$N_1 = e^{-A_2 y} \cos B_2 y,$$

$$N_2 = e^{-A_2 y} \sin B_2 y,$$

$$b_2 = \frac{G_c \left(\frac{1}{k_p} - S_c \right)}{a_3 (A_5^2 + B_5^2)},$$

$$b_3 = \frac{4G_c S_c}{\omega (A_5^2 + B_5^2)},$$

$$b_4 = \frac{G_c \left(\frac{1}{k_p} - m_2 \right)}{a_3 (A_6^2 + B_6^2)},$$

$$b_5 = \frac{\left(\frac{1}{k_p} - m_2 \right)}{A_6^2 + B_6^2},$$

$$\begin{aligned}
N_r &= b_1 \left(A_4 - \frac{4S_c}{\omega} B_4 \right) (A_2 - A_1) & Q_r &= A_1 + \frac{4S_c}{\omega} B_1, \\
&+ b_1 \left(\frac{4S_c}{\omega} A_4 + B_4 \right) (B_2 - B_1) + B_2 (b_2 B_5 - b_3 A_5) & Q_i &= B_1 + \frac{4S_c}{\omega} (S_c - A_1). \\
&+ B_2 B_5 (b_5 - b_4) + (A_2 - S_c) (b_2 A_5 + b_3 B_5) \\
&+ A_6 (A_2 - m_2) (b_5 - b_4) - A_2, \\
N_i &= b_1 \left(A_4 - \frac{4S_c}{\omega} B_4 \right) (B_2 - B_1) \\
&- b_1 \left(\frac{4S_c}{\omega} A_4 + B_4 \right) (A_2 - A_1) + B_2 (b_2 A_5 + b_3 B_5) \\
&+ B_2 A_6 (b_5 - b_4) + (A_2 - S_c) (b_3 A_5 - b_2 B_5) \\
&+ B_5 (A_2 - m_2) (b_4 - b_2) - B_2,
\end{aligned}$$