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A Conventional Approach for the Solution of the Fifth Order Boundary Value Problems Using Sixth Degree Spline Functions

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ABSTRACT

In this communication we have used Bickley's method for the construction of a sixth order spline function and apply it to solve the linear fifth order differential equations of the form $y^{v}(x)+g(x)y(x)=r(x)$ where $g(x)$ and $r(x)$ are given functions with the two different problems of different boundary conditions. The method is illustrated by applying it to solve some problems to demonstrate the application of the methods discussed.

Keywords: Cubic Spline; Tridiagonal; Conventional Approach

1. Introduction

In the recent past, several authors have considered the application of cubic spline functions for the solution of two point boundary value problems. Bickley [1] has considered the use of cubic spline for solving second order two point boundary value problems. The essential feature of his analysis is that it leads to the solution of a set of linear equations whose matrix coefficients are of upper Heisenberg form. Bickley uses a special notation other than the conventional one for the representation of the cubic spline, for a detailed discussion one may refer to E. A. Boquez and J. D. A. Walker [2], M. M. Chawla [3], and P. S. Ramachandra Rao [4-7]. We used Bickley's method for the construction of a sixth degree spline and apply it to the linear fifth order differential equation with two different problems with different boundary conditions. The work has been illustrated through examples with $h = 0.5$ and $h = 0.25$.

2. Cubic Spline-Bickley's Method

Suppose the interval $[x_0, x_n]$ is divided in to n subintervals with knots $x_0, x_1, x_2, \cdots, x_n$ starting at x_0, the function $u(x)$ in the interval $[x_0, x_n]$ is represented by a cubic spline in the form

$$S_j(x)=a_j+b_j(x-x_j)+c_j(x-x_j)^2+d_j(x-x_j)^3 \quad (1)$$

Proceeding in to the next interval $[x_1, x_2]$, we add a term $d_1(x-x_1)^3$; proceeding in to the next interval $[x_2, x_3]$, we add another term $d_2(x-x_2)^3$ and so until we reach x_n. Thus the function $S_j(x)$ is represented in the form for $j=0$

$$S_j(x)=a_j+b_j(x-x_j)+c_j(x-x_j)^2+\sum_{i=0}^{n-1}d_i(x-x_i)^3 \quad (1.1)$$

$$S_j'(x)=b_j+2c_j(x-x_j)\sum_{i=0}^{n-1}3d_i(x-x_i)^2 \quad (1.2)$$

$$S_j''(x)=2c_j+\sum_{i=0}^{n-1}6d_i(x-x_i) \quad (1.3)$$

2.1. The Two-Point Second Order Boundary Value Problem

First, we consider the linear differential equation

$$P(x)u''+q(x)u'+r(x)u=v(x) \quad (1.4)$$

With the boundary conditions

$$\alpha_0 u+\beta_0 u'=\gamma_0 \text{ at } x=x_0,$$
$$\alpha_n u+\beta_n u'=\gamma_n \text{ at } x=x_n \quad (1.5)$$

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The number of coefficients in (1.1) is (n + 3). The satisfaction of the differential equation by the spline function at the (n + 1) nodes gives (n + 1) equations in the (n + 3) unknowns. Also the end conditions (1.5) give us two more equations in the unknowns. Thus we get (n + 3) equations in (n + 3) unknowns $a_0, b_0, c_0, d_0, d_1, \cdots, d_{n-1}$. after determining these unknowns we substitute them in (1.1) and thus we get the cubic spline approximation of $u(x)$. Putting $x = x_0, x_1, x_2, \cdots, x_n$ in the spline function thus determined, we get the solution at the nodes. The system of equations to be satisfied by the coefficients $a_0, b_0, c_0, d_0, d_1, \cdots, d_{n-1}$ are derived below.

Substituting (1.1), (1.2), (1.3) in (1.4), at $x = x_m$ we get

$$\begin{aligned} & ar_m + b\left[r_m(x_m - x_0) + q_m\right] \\ & + c\left[r_m(x_m - x_0)^2 + 2q_m(x_m - x_0) + 2p_m\right] \\ & + \sum_{i=0}^{m-1} d_i\left[r_m(x_m - x_i)^3 + 3q_m(x_m - x_i)^2 + 6p_m(x_m - x_i)\right] \\ & = S_m, \\ & m = 0, 1, 2, \cdots, n \end{aligned} \tag{1.6}$$

where $p_m = p(x_m)$ and so on. Applying boundary conditions in (1.5), we get

$$\begin{aligned} & \alpha_0 a + \beta_0 b = \gamma_0 \\ & \alpha_n a_0 + \left[\alpha_n(x_n - x_0) - \beta_n\right] b_0 \\ & + \left[\alpha_n(x_n - x_0)^2 - 2\beta_n(x_n - x_0)\right] c_0 \\ & + \sum_{m=0}^{n-1}\left[\alpha_n(x_n - x_0)^3 - 3\beta_m(x_n - x_m)^2\right] d_m = \gamma_n \end{aligned} \tag{1.7}$$

If these equations are taken in the order (1.7), (1.6) with $m = n, n-1, \cdots, 0$, the matrix of the coefficients of the unknowns $d_{n-1}, d_{n-2}, \cdots, d_1, d_0, c_0, b_0, a_0$ is of the Heisenberg form, namely an upper triangle with a single lower sub-diagonal. The forward elimination is then simple, with only one multiplier at each step and the back substitution is correspondingly easy.

2.2. Construction of the Sixth Degree Spline

Suppose the interval $[x_0, x_n]$ is divided in to "n" subintervals with knots $x_0, x_1, x_2, \cdots, x_n$. Starting at x_0, the function $y(x)$ in the interval $[x_0, x_1]$ is represented by a sixth degree spline

$$\begin{aligned} y(x) = {} & a + b(x - x_0) + c(x - x_0)^2 + d(x - x_0)^3 \\ & + e(x - x_0)^4 + g(x - x_0)^5 + h_0(x - x_0)^6 \end{aligned}$$

Proceeding in to the next interval $[x_1, x_2]$, we add a term $h_1(x - x_1)^6$, Proceeding in to the next interval $[x_2, x_3]$ we add another term $h_2(x - x_2)^6$ and so until we reach x_n. Thus the function $y(x)$ is represented in the form

$$\begin{aligned} y(x) = {} & a + b(x - x_0) + c(x - x_0)^2 + d(x - x_0)^3 \\ & + e(x - x_0)^4 + g(x - x_0)^5 + \sum_{i=0}^{n-1} h_i(x - x_i)^6 \end{aligned} \tag{1.8}$$

It can be seen that $y(x)$ and its first five derivatives are continuous across nodes.

3. Fifth Order Boundary Value Problem

We consider the linear fifth order differential equation

$$y^{(5)}(x + f(x)y(x)) = r(x) \tag{1.9}$$

With the boundary conditions

$$\begin{aligned} & y(x_0) = \alpha, y(x_n) = \beta, y'(x_0) = \alpha', \\ & y'(x_n) = \beta', y''(x_0) = \alpha'' \end{aligned} \tag{1.10}$$

We get (n + 6) equations in (n + 6) unknowns a, b, c, $d, e, g, h_0, h_1, h_2, \cdots, h_{n-1}$. After determining these unknowns we substitute them in (1.8) and thus we get the sixth degree spline approximation of $y(x)$. Putting $x = x_1, x_2, x_3, \cdots, x_n$ in the spline function thus determined, we get the solution at the nodes. The system of equations to be satisfied by the coefficients $a, b, c, d, e, g, h_0, h_1, h_2, \cdots, h_{n-1}$ are derived below. From (1.8) we get

$$\begin{aligned} y^{(5)}(x) = {} & 120g + 720h_0(x - x_0) + 720h_1(x - x_1) \\ & + 720h_2(x - x_2) + 720h_3(x - x_3) \\ & + \cdots + 720h_{n-1}(x - x_{n-1}) \end{aligned} \tag{1.11}$$

using (1.8) & (1.11) in the differential Equation (1.9) at the nodes x_m takes of the form

$$\begin{aligned} & af_m + bf_m(x_m - x_0) + cf_m(x_m - x_0)^2 + df_m(x_m - x_0)^3 \\ & + ef_m(x_m - x_0) + g\left[f_m(x_m - x_0)^5 + 120\right] \\ & + \sum_{i=0}^{m-1} h_i\left[f_m(x_m - x_0)^6 + 720(x_m - x_i)\right] = r_m \\ & m = 0, 1, 2, \cdots, n \end{aligned} \tag{1.12}$$

To these equations we add those obtained from the boundary conditions (1.10), we get

$$a = \alpha \tag{1.13}$$

$$\begin{aligned} & a + b(x_n - x_0) + c(x_n - x_0)^2 + d(x_n - x_0)^3 \\ & + e(x_n - x_0)^4 + g(x_n - x_0)^5 + \sum_{i=0}^{m-1} h_i(x_n - x_i)^6 = \beta \end{aligned} \tag{1.14}$$

$$b = \alpha' \tag{1.15}$$

$$\begin{aligned} & b + 2c(x_n - x_0) + 3d(x_n - x_0)^2 + 4e(x_n - x_0)^3 \\ & + 5g(x_n - x_0)^4 + 6\sum_{i=0}^{n-1} h_i(x_n - x_i)^5 = \beta' \end{aligned} \tag{1.16}$$

$$2c = \alpha'' \tag{1.17}$$

If these equations are taken in the order (1.14), (1.16), (1.12) with $m = n, n-1, \cdots, 0$, (1.17), (1.15) & (1.13) the matrix of the coefficients of the unknowns, $h_{n-1}, h_{n-2}, \cdots, h_1, h_0, g, e, d, c, b, a$ is an upper triangular matrix with two lower sub diagonals. The forward elimination is then simple with only two multipliers at each step, and the back substitution is correspondingly easy.

3.1. Example 1

Consider the following fifth order linear boundary value problem

$$y^{(5)}(x) - y(x) = -(15+10x)e^{(x)},\ 0 \le x \le 1 \tag{1.18}$$

With the boundary conditions

$$y(0) = y(1) = 0,\ y'(0) = 1,\ y'(1) = -e,\ y''(0) = 0 \tag{2}$$

by taking equal subintervals with $h = 0.5$ and $h = 0.25$

1) Solution with $h = 0.5$

The sixth order spline $s(x)$ which approximates $y(x)$ is given by

$$\begin{aligned} s(x) = {} & a + b(x-x_0) + c(x-x_0)^2 + d(x-x_0)^3 \\ & + e(x-x_0)^4 + g(x-x_0)^5 + h_0(x-x_0)^6 \\ & + h_1(x-x_1)^6 \end{aligned} \tag{3}$$

where $x_0 = 0, x_1 = 0.5, x_2 = 1$. We have eight unknowns $a, b, c, d, e, g, h_0, h_1$ and eight conditions to be satisfied by these unknowns are $s(x_0) = 0, s(x_2) = 0, s'(x_0) = 1$,

$$s'(x_2) = -e, s''(x_0) = 0 \tag{4}$$

$$s^{(5)}(x_i) - s(x_i) = -(15+10x_i)e^{x_i} \text{ for } i = 0,1,2 \tag{5}$$

Since $s(x_0) = 0, s'(x_0) = 1, s''(x_0) = 0$ it follows that $a = 0, b = 1, c = 0$ Equation (3) reduces to the form

$$\begin{aligned} s(x) = {} & (x-x_0) + d(x-x_0)^3 + e(x-x_0)^4 \\ & + g(x-x_0)^5 + h_0(x-x_0)^6 + h_1(x-x_1)^6 \end{aligned} \tag{6}$$

also since $s(x_0) = 0, s(x_2) = 0$ and equations of (5) for $i = 0$ & 2 reduces to

$$s^{(5)}(x_0) = -(15+10x_0)e^{x_0} \quad \text{and}$$

$$s^{(5)}(x_2) = -(15+10x_2)e^{x_2}$$

It follows that we have to determine the five unknowns d, e, g, h_0, h_1 in Equation (6), subject to the five conditions

$$\begin{aligned} & s(x_2) = 0, s'(x_2) = -e, s^{(5)}(x_0) = -(15+10x_0)e^{x_0}, \\ & s^{(5)}(x_1) - s(x_1) = -(15+10x_1)e^{x_1}, \\ & s^{(5)}(x_2) = -(15+10x_2)e^{x_2} \end{aligned} \tag{7}$$

from (6)

$$\begin{aligned} s'(x) = {} & 1 + 3d(x-x_0)^2 + 4e(x-x_0)^3 + 5g(x-x_0)^4 \\ & + 6h_0(x-x_0)^5 + 6h_1(x-x_1)^5 \end{aligned} \tag{8}$$

and

$$s^{(5)}(x) = 120g + 720h_0(x-x_0) + 720h_1(x-x_1) \tag{9}$$

Substituting (6), (8), (9) in (7) we get the system of equations

$$\begin{aligned} & d + e + g + h_0 + (0.015625)h_1 = -1, \\ & 3d + 4e + 5g + 6h_0 + (0.1875)h_1 = -3.718281828, \\ & 8g = -1, \\ & (0.125)d + (0.0625)e - (119.96875)g \\ & -(359.984375)h_0 = 32.4744216, \\ & (-120)g - (720)h_0 - (360)h_1 = 67.95704571 \end{aligned} \tag{10}$$

Solving these we get

$$d = -0.510480,\ e = -0.314961,\ g = -0.125,$$
$$h_0 = -0.048785,\ h_1 = 0.049531$$

Substituting these values in (6) we get

$$\begin{aligned} s(x) = {} & (x-x_0) - (0.510480)(x-x_0)^3 \\ & - (0.314961)(x-x_0)^4 - (0.125)(x-x_0)^5 \\ & - (0.048785)(x-x_0)^6 + (0.049531)(x-x_1)^6 \end{aligned} \tag{11}$$

$$\begin{aligned} y(x_1) = s(x_1) = {} & (h) - (0.510480)(h^3) - (0.314961)(h^4) \\ & - (0.125)(h^5) - (0.048785)(h^6) \end{aligned}$$

where $h = 0.5$

Therefore $y(x_1) = y(0.5) = 0.411836421$

The analytical solution of the differential equation (1.18) subject to the conditions is given by

$$y(x) = x(1-x)e^x \tag{11.1}$$

The exact value of $y(0.5) = 0.412180317$

It follows that the Absolute error of the numerical value of $y(0.5)$, computed from the spline approximation is 0.00083433 which is very small.

2) Solution with $h = 0.25$

The interval [0,1] is divided in to 4 equal subintervals we denote the knots by x_0, x_1, x_2, x_3, x_4 where $x_0 = 0$, $x_1 = 0.25, x_2 = 0.5, x_3 = 0.75, x_4 = 1$.

The sixth order spline $s(x)$ which approximate $y(x)$ is given by

$$\begin{aligned} s(x) = {} & a + b(x-x_0) + c(x-x_0)^2 + d(x-x_0)^3 \\ & + e(x-x_0)^4 + g(x-x_0)^5 + h_0(x-x_0)^6 \Big| \\ & + h_1(x-x_1)^6 \Big| + h_2(x-x_2)^6 \Big| + h_3(x-x_3)^6 \end{aligned} \tag{12}$$

There are 10 unknowns in $s(x)$ which are to be determined from 10 conditions

$$\begin{aligned} &s(x_0)=0, s(x_4)=0, s'(x_0)=1, s'(x_4)=-e,\\ &s''(x_0)=0, s^{(5)}(x_i)-s(x_i)=-(15+10x_i)e^{x_i} \\ &\text{for } i=0,1,2,3,4 \end{aligned} \tag{13}$$

In view of the conditions $s(x_0)=0, s'(x_0)=1$ and $s''(x_0)=0$ it follows that $a=0, b=1, c=0$ hence The spline $s(x)$ reduces to the form

$$\begin{aligned} s(x)=&(x-x_0)+d(x-x_0)^3+e(x-x_0)^4\\ &+g(x-x_0)^5+h_0(x-x_0)^6\left|+h_1(x-x_1)^6\right| \\ &+h_2(x-x^2)^6\left|+h_3(x-x_3)^6\right| \end{aligned} \tag{14}$$

From (14)

$$\begin{aligned} s'(x)=&1+3d(x-x_0)^2+4e(x-x_0)^3+5g(x-x_0)^4\\ &+6h_0(x-x_0)^5\left|+6h_1(x-x_1)^5\right| \\ &+6h_2(x-x_2)^5\left|+6h_3(x-x_3)^5\right| \end{aligned} \tag{15}$$

$$\begin{aligned} s^{(5)}(x)=&120g+720h_0(x-x_0)\left|+720h_1(x-x_1)\right| \\ &+720h_2(x-x_2)\left|+720h_3(x-x_3)\right| \end{aligned} \tag{16}$$

Substituting (14), (15), (16) in (13) taken in the order,

$$\begin{aligned} &s(x_4)=0, s'(x_4)=-e,\\ &s^{(5)}(x_i)-s(x_i)=-(15+10x_i)e^{(x_i)}\\ &\text{for } i=4,3,2,1,0 \end{aligned}$$

we get the following system of equations

$$\begin{aligned} &(0.000244140625)h_3+(0.015625)h_2\\ &+(0.177978515)h_1+h_0+g+e+d=-1,\\ &(0.005859375)h_3+(0.1875)h_2+(1.423828125)h_1\\ &+6h_0+5g+4e+3d=-3.718281828,\\ &36h_3+(72)h_2+(108)h_1+(144)h_0+(24)g\\ &=-13.59140914,\\ &(-179.9997559)h_2-(359.984375)h_1\\ &-(539.8220215)h_0-(119.7626953)g\\ &+(0.31640625)e+(0.421875)d=46.88250037,\\ &(-179.9997559)h_1-(359.984375)h_0-(119.96875)g\\ &+(0.0625)e+(0.125)d=32.47442541,\\ &(-179.9997559)h_0-(119.9990234)g+(0.00390625)e\\ &+(0.015625)d=22.22044479, 120g=-15. \end{aligned} \tag{17}$$

From the above system of equations, we notice that the coefficient matrix is an upper triangular matrix with two lower sub diagonals. solving the above equations we get

$$\begin{aligned} &d=-0.502566, e=-0.328795, g=-0.125,\\ &h_0=-0.040165, h_1=-0.017381,\\ &h_2=-0.023829, h_3=-0.033744 \end{aligned} \tag{18}$$

However it may be noticed that from the Equation (17) $g=-15/120$ which when substituted in the remaining equations will give us a 6 × 6 system of equations which may be solved. Substituting (18) in (14) we get the spline Approximation $s(x)$ of $y(x)$. The values of $s(x)$, and The corresponding absolute errors at x_1, x_2, x_3 tabulated in **Table 1**.

The analytical solution of the differential equation (1.18) with the conditions is given by (11.1) is symmetric about the central value. The same aspect is also satisfied by the numerical approximations as is evident from the above table. We found that the approximate values are remarkably accurate.

3.2. Example 2

Consider the following fifth order linear boundary value problem

$$\begin{aligned} &y^{(5)}(x)+xy(x)\\ &=19x\cos(x)+2x^3\cos(x)+41\sin(x)\\ &\quad -2x^2\sin(x), 1\le x\le 1, y(x) \end{aligned} \tag{19}$$

Subject to

$$\begin{aligned} &y(-1)=y(1)=\cos(1),\\ &y'(-1)=-y'(1)=-4\cos(1)+\sin(1),\\ &y''(-1)=3\cos(1)-8\sin(1). \end{aligned} \tag{20}$$

1) Solution with $h=1$

The sixth order spline $s(x)$ which approximates $y(x)$ is given by (3). The equations to be satisfied by the coefficients of the spline function are

$$\begin{aligned} &s(x_0)=\cos(1)=0.540302305,\\ &s(x_2)=\cos(1)=0.540302305,\\ &s'(x_0)=-4\cos(1)+\sin(1)=-1.319738239,\\ &s'(x_2)=4\cos(1)-\sin(1)=1.319738239,\\ &s''(x_0)=3\cos(1)-8\sin(1)=-5.110860961,\\ &s^{(5)}(x_i)+x_is(x_i)=19x_i\cos(x_i)+2x_i^3\cos(x_i)\\ &\qquad +41\sin(x_i)-2x_i^2\sin(x_i), \end{aligned} \tag{21}$$

For $i=0,1,2$

We observe that

$$\begin{aligned} &a=0.540302305, b=-1.319738239,\\ &c=-2.555430481 \end{aligned}$$

Table 1. Approximate solutions and absolute errors for Example 1 with $h = 0.25$.

X	$s(x)$	$y(x)$	Absolute error
0.25	0.240731164	0.240754765	0.00009800292
0.50	0.412091491	0.41218030	0.00021546156
0.75	0.396858187	0.396937503	0.00019981986

also since

$$s(x_0) = 0.540302305,$$
$$s(x_2) = 0.540302305$$

and $x_1 = 0$ the equations of (21) for $i = 0,1,2$ reduces to

$$\begin{aligned} &s^{(5)}(x_0) + x_0(0.540302305) = -44.16371683,\\ &s^{(5)}(x_1) = 0,\\ &s^{(5)}(x_2) + x_2(0.540302305) = 44.16371683 \end{aligned}$$

It follows that we have to determine the 5 unknowns $d, e.\ g, h_0, h_1$ in Equation (3), subject to the five conditions

$$\begin{aligned} &s(x_2) = 0.540302305,\\ &s'(x_2) = 1.319738239,\\ &s^{(5)}(x_0) + x_0(0.540302305) = -44.16371683,\\ &s^{(5)}(x_1) = 0,\\ &s^{(5)}(x_2) + x_2(0.540302305) = 44.16371683 \end{aligned} \quad (22)$$

From (3)

$$\begin{aligned} s'(x) = b + 2c(x-x_0) + 3d(x-x_0)^2 + 4e(x-x_0)^3 \\ + 5g(x-x_0)^4 + 6h_0(x-x_0)^5\Big| + 6h_1(x-x_1)^5 \end{aligned} \quad (23)$$

$$s^{(5)}(x) = 120g + 720h_0(x-x_0)\Big| + 720h_1(x-x_1) \quad (24)$$

Substituting (3), (23), (24) in (22)
We get the system of equations

$$\begin{aligned} &d + 2e + 4g + 8h_0 + (0.125)h_1 = 1.607649801,\\ &12d + 32e + 80g + 192h_0 + 6h_1 = 12.8611984,\\ &120g + 1440h_0 + 720h_1 = 43.62341453,\\ &720h_0 + 120g = 0,\\ &120g = -43.62341453 \end{aligned} \quad (25)$$

Solving these we get

$$\begin{aligned} &d = 2.730596047, e = -0.076768581,\\ &g = -0.383528454, h_0 = 0.060588075,\\ &h_1 = 0.00000106 \end{aligned}$$

also we have

$$\begin{aligned} &a = 0.540302305,\\ &b = -1.319738239,\\ &c = -2.555430481 \end{aligned}$$

Substituting all these values in Equation (3) we get the spline approximation for $y(x)$ which is given by

$$\begin{aligned} s(x) &= (0.540302305) - (1.319738239)(x-x_0)\\ &-(2.555430481)(x-x_0)^2 + (2.730596047)(x-x_0)^3\\ &-(0.076768581)(x-x_0)^4 - (0.363528454)(x-x_0)^5\\ &+(0.060588075)(x-x_0)^6\Big| + (0.00000106)(x-x_1)^6\\ y(x_1) = s(x_1) &= (0.540302305) - (1.319738239)(h)\\ &-(2.555430481)(h)^2 + (2.730596047)(h)^3\\ &-(0.076768581)(h)^4 - (0.363528454)(h)^5\\ &+(0.060588075)(h)^6 \end{aligned} \quad (26)$$

where $h = 1, y(x_1) = y(0) = -0.983978288.$

The analytical solution of (19) with the conditions (20) is given by

$$y(x) = (2x^2 - 1)\cos(x) \quad (27)$$

The exact value of $y(0) = -1$ it follows that the absolute error in the numerical approximation $s(0)$ is found to be (0.016021737) which is very small.

2) Solution with $h = 0.5$

The interval $[-1,1]$ is divided in to 4 equal subintervals we denote the knots by x_0, x_1, x_2, x_3, x_4 where $x_0 = -1,\ x_1 = -0.5,\ x_2 = 0,\ x_3 = 0.5,\ x_4 = 1$

We assume the spline function $s(x)$ which approximates $y(x)$ in the form is given by (12)

From (12) we have

$$\begin{aligned} s'(x) = b + 2c(x-x_0) + 3d(x-x_0)^2 + 4e(x-x_0)^3\\ + 5g(x-x_0)^4 + 6h_0(x-x_0)^5\Big| + 6h_1(x-x_1)^5\Big|\\ + 6h_2(x-x_2)^5\Big| + 6h_3(x-x_3)^5 \end{aligned} \quad (28)$$

The conditions to be satisfied by $s(x)$ are

$$\begin{aligned} &s(x_0) = \cos(1),\\ &s(x_4) = \cos(1),\\ &s'(x_0) = -4\cos(1) + \sin(1),\\ &s'(x_4) = 4\cos(1) - \sin(1),\\ &s''(x_0) = 3\cos(1) - 8\sin(1),\\ &s^{(5)}(x_i) + x_i s(x_i) = 19x_i\cos(x_i) + 2x_i^3\cos(x_i)\\ &\qquad + 41\sin(x_i) - 2x_i^2\sin(x_i) \end{aligned} \quad (29)$$

for $i = 0,1,2,3,4$ from (29) we find that

Table 2. Approximate solutions and absolute errors for Example 2 with $h = 0.5$.

X	$s(x)$	$y(x)$	Absolute error
−0.5	−0.437521423	−0.43879128	0.002893987
0	−0.997548415	−1	0.002451585
0.5	−0.437518745	−0.43879128	0.002900090

$$a = 0.540302305,$$
$$b = -1.319738239,$$
$$c = -2.555430481$$

using the remaining conditions of (29) in the order,

$$s(x_4) = \cos(1),$$
$$s'(x_4) = 4\cos(1) - \sin(1)$$
$$s^{(5)}(x_i) + x_i s(x_i) = 19x_i \cos(x_i) + 2x_i^3 \cos(x_i) + 41\sin(x_i) - 2x_i^2 \sin(x_i),$$
$$\text{for } i = 4,3,2,1,0$$

that is taking the Equations (12), (16), (28) in (29) & by substituting the values of x_0, x_1, x_2, x_3, x_4

We get the following system of equations

$$\begin{aligned}
&d + 2e + 4g + 8h_0 + (1.423828125)h_1 \\
&+ (0.125)h_2 + (0.001953125)h_3 = 1.607649801, \\
&e + 4g + 8h_0 + (1.423828125)h_1 + (0.125)h_2 \\
&+ (0.001953125)h_3 = 1.607649801, \\
&12d + 32e + 80g + 192h_0 + (45.5625)h_1 + 6h_2 \\
&+ (0.1875)h_3 = 12.8611984, \\
&120g + 1440h_0 + 1080h_1 + 720h_2 + 360h_3 \\
&= 43.62348633, \\
&(1.6875)d + (2.53125)e + (123.796875)g \\
&+ (1085.6953137)h_0 + (720.5)h_1 \\
&+ (360.0078125)h_2 = 31.56767609, \\
&120g + 720h_0 + 360h_1 = 0, \\
&(0.0625)d + (0.03125)e - (119.984375)g \\
&- (359.9921875)h_0 = 28.35237651, \\
&120g = -43.62341453
\end{aligned} \tag{30}$$

Solving (29) we get

$$\begin{aligned}
&d = 2.647519,\ e = 0.010462, \\
&g = -0.363528,\ h_0 = 0.042865, \\
&h_1 = 0.035446,\ h_2 = 0, \\
&h_3 = -0.035445
\end{aligned} \tag{31}$$

Also we have

$$a = 0.540302305,$$
$$b = -1.319738239,$$
$$c = -2.555430481$$

Substituting these values in (12) we get the approximation $s(x)$.

The values of $s(x), y(x)$ and the corresponding absolute errors at x_1, x_2, x_3 are mentioned in **Table 2**.

4. Conclusion

Numerical values obtained by the spline approximation have high accuracy. It has been noticed that the numerical solutions obtained are remarkably accurate and have negligible percentage errors even for values of h as large as 0.5, 1.0.

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