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Certain *pl*(*m*, *n*)-Kummer Matrix Function of Two Complex Variables under Differential Operator

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ABSTRACT

The main aim of this paper is to define and study of a new matrix functions, say, the *pl*(*m*, *n*)-Kummer matrix function of two complex variables. The radius of regularity, recurrence relation and several new results on this function are established when the positive integers p is greater than one. Finally, we obtain a higher order partial differential equation satisfied by the *pl*(*m*, *n*)-Kummer matrix function and some special properties.

Keywords: Hypergeometric Matrix Function; *pl*(*m*, *n*)-Kummer Matrix Function; Matrix Differential Equation; Differential Operator

1. Introduction

Many Special matrix functions appear in connection with statistics [1], mathematical physics, theoretical physics, group representation theory, Lie groups theory [2] and orthogonal matrix polynomials are closely related [3-5]. The hypergeometric matrix function has been introduced as a matrix power series and an integral representation and the hypergeometric matrix differential equation in [6-9] and the explicit closed form general solution of it has been given in [10]. The author has earlier studied the Kummer's and Horn's H_2 matrix function of two complex variables under differential operators [11-13]. In [14-16], extension to the matrix function framework of the classical families of p-Kummer's matrix function, p and q-Appell matrix function and Humbert matrix function have been proposed.

Throughout this paper for a matrix A in $C^{N\times N}$, its spectrum $\sigma(A)$ denotes the set of all the eigenvalues of A. If A is a matrix in $C^{N\times N}$, its two-norm denoted by $\|A\|_2$ is defined by [17]

$$\|A\|_2=\sup_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2}$$

where for a vector y in C^N, $\|y\|_2=\left(y^{\mathrm{T}}y\right)^{\frac{1}{2}}$ is the Euclidean norm of y.

If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, defined in an open set Ω of the complex plane, and if A and B are a matrix in $C^{N\times N}$ with $\sigma(A)\subset\Omega$ and $\sigma(B)\subset\Omega$ also and if $AB=BA$, then from the properties of the matrix functional calculus [18], it follows that

$$f(A)g(B)=g(B)f(A). \tag{1.1}$$

The reciprocal gamma function denoted by $\Gamma^{-1}(z)=\frac{1}{\Gamma(z)}$ is an entire function of the complex variable z. Then for any matrix A in $C^{N\times N}$, the image of $\Gamma^{-1}(z)$ acting on A denoted by $\Gamma^{-1}(A)$ is a well defined matrix. Furthermore, if

$$A+nI \text{ is invertible for every non-negative integer } n \tag{1.2}$$

where I is the identity matrix in $C^{N\times N}$, then $\Gamma(A)$ is invertible, its inverse coincides with $\Gamma^{-1}(A)$ and one gets [6]

$$\begin{aligned}(A)_n&=A(A+I)(A+2I)\cdots(A+(n-1)I)\\&=\Gamma(A+nI)\Gamma^{-1}(A);\ n\geq 1;\ (A)_0=I.\end{aligned} \tag{1.3}$$

Jódar and Cortés have proved in [6], that

$$\Gamma(A)=\lim_{n\to\infty}(n-1)!\left[(A)_n\right]^{-1}n^A. \tag{1.1}$$

2. On *pl*(*m*, *n*)-Kummer Matrix Function

We We define the *pl*(*m*, *n*)-Kummer matrix function ${}^P\Phi_2(A;B;z,w)$ of two complex variables in the form

$$
\begin{aligned}
&{}^{p}\Phi_2(A;B;z,w)\\
&=\sum_{l(m,n)\geq 0}\frac{(A)_{l(m,n)}\left[(B)_{l(m,n)}\right]^{-1}}{(pl(m,n))!}z^m w^n \qquad (2.1)\\
&=\sum_{l(m,n)\geq 0}U_{m,n}(z,w)
\end{aligned}
$$

where

$$U_{m,n}(z,w)=V_{m,n}z^m w^n,$$

$$V_{m,n}=\frac{(A)_{l(m,n)}\left[(B)_{l(m,n)}\right]^{-1}}{(pl(m,n))!},$$

$l(m,n)=\frac{1}{2}(m+n+1)(m+n)+n$ [19] and p, m and n are non-negative integer numbers. Notice that $l(m,n)$ is a non-negative integer number.

For simplicity, we can write the ${}^{p}\Phi_2(A\pm I;B;z,w)$ in the form ${}^{p}\Phi_2(A\pm)$, ${}^{p}\Phi_2(A;B\pm I;z,w)$ in the form ${}^{p}\Phi_2(B\pm)$ and ${}^{p}\Phi_2(A\pm I;B\pm I;z,w)$ in the form ${}^{p}\Phi_2(A\pm,B\pm)$.

We begin the study of this function by calculating its radius of regularity R of such function for this purpose we recall relation (1.3.10) of [19] and keeping in mind that $1\leq\sigma_{m,n}\leq 2^{\frac{m+n}{2}}$. Hence

$$
\begin{aligned}
\frac{1}{R}&=\lim_{m+n\to\infty}\sup\left(\frac{\|V_{m,n}\|}{\sigma_{m,n}}\right)^{\frac{1}{m+n}}=\lim_{m+n\to\infty}\sup\left(\frac{\left\|(A)_{l(m,n)}\left[(B)_{l(m,n)}\right]^{-1}\right\|}{(pl(m,n))!\,\sigma_{m,n}}\right)^{\frac{1}{m+n}}\\
&=\lim_{m+n\to\infty}\sup\left\|\sqrt{2\pi\left(A+(l(m,n)-1)I\right)}\left(\frac{\left(A+(l(m,n)-1)I\right)}{e}\right)^{\left(A+(l(m,n)-1)I\right)}\right\|^{\frac{1}{m+n}}\\
&\times\left\|\left[\sqrt{2\pi\left(B+(l(m,n)-1)I\right)}\left(\frac{\left(B+(l(m,n)-1)I\right)}{e}\right)^{\left(B+(l(m,n)-1)I\right)}\right]^{-1}\right\|^{\frac{1}{m+n}}\\
&\times\left\|\left[\sqrt{2p\pi l(m,n)}\left(\frac{pl(m,n)}{e}\right)^{pl(m,n)}\right]^{-1}\frac{1}{\sigma_{m,n}}\right\|^{\frac{1}{m+n}}\\
&\leq\lim_{m+n\to\infty}\sup\left\|\left(\frac{\left(A+(l(m,n)-1)I\right)}{e}\right)^{(A-I)}\left(\frac{\left(A+(l(m,n)-1)I\right)}{e}\right)^{l(m,n)I}\right\|^{\frac{1}{m+n}}\\
&\times\left\|\left(\frac{\left(B+(l(m,n)-1)I\right)}{e}\right)^{-B+I}\left(\frac{\left(B+(l(m,n)-1)I\right)}{e}\right)^{-l(m,n)I}\left(\frac{(pl(m,n))}{e}\right)^{-(pl(m,n))}\right\|^{\frac{1}{m+n}}\\
&\leq\lim_{m+n\to\infty}\sup\left\|\left(\frac{\left(A+(l(m,n)-1)I\right)}{l(m,n)}\right)^{l(m,n)I}\right\|^{\frac{1}{m+n}}\times\left\|\left(\frac{\left(B+(l(m,n)-1)I\right)}{l(m,n)}\right)^{-l(m,n)I}\left(\frac{e}{pl(m,n)}\right)^{pl(m,n)}\right\|^{\frac{1}{m+n}}\\
&\leq\lim_{m+n\to\infty}\sup\left\|\left(I+\frac{A-I}{l(m,n)}\right)^{l(m,n)I}\right\|^{\frac{1}{m+n}}\times\left\|\left(I+\frac{B-I}{l(m,n)}\right)^{-l(m,n)I}\left(\frac{e}{pl(m,n)}\right)^{pl(m,n)}\right\|^{\frac{1}{m+n}}=0.
\end{aligned}
$$

where

$$\sigma_{m,n}=\begin{cases}\left(\frac{m+n}{m}\right)^{\frac{m}{2}}\left(\frac{m+n}{n}\right)^{\frac{n}{2}}, & m,n\neq 0;\\ 1, & m,n=0.\end{cases}$$

Summarizing, the following result has been established.

Theorem 2.1. *Let A and B be matrices in $C^{N\times N}$ such that $B+l(m,n)I$ are invertible for all integer $l(m,n)\geq 0$. Then, the pl(m, n)-Kummer matrix function is an entire function.*

For $p=1$, we have

$$\frac{1}{R}=\lim_{m+n\to\infty}\sup\left(\frac{\|V_{m,n}\|}{\sigma_{m,n}}\right)^{\frac{1}{m+n}}$$

$$=\lim_{m+n\to\infty}\sup\left(\frac{\left\|(A)_{l(m,n)}\left[(B)_{l(m,n)}\right]^{-1}\right\|}{l(m,n)!\,\sigma_{m,n}}\right)^{\frac{1}{m+n}}=0$$

i.e., the *l*(*m*, *n*)-Kummer matrix function is an entire function.

Some matrix recurrence relations are carried out on the *pl*(*m*, *n*)-Kummer matrix function. In this connection the following matrix contiguous functions relations follow, directly by increasing or decreasing one in original relation

$$\begin{aligned}&{}^{p}\Phi_2\left(A+;B;z,w\right)\\&=\sum_{l(m,n)\geq 0}\frac{(A+I)_{l(m,n)}\left[(B)_{l(m,n)}\right]^{-1}}{\left(pl(m,n)\right)!}z^m w^n\\&=\sum_{l(m,n)\geq 0}\frac{A^{-1}\left(A+l(m,n)I\right)(A)_{l(m,n)}\left[(B)_{l(m,n)}\right]^{-1}}{\left(pl(m,n)\right)!}z^m w^n\\&=A^{-1}\sum_{l(m,n)\geq 0}\left(A+l(m,n)I\right)U_{m,n}(z,w).\end{aligned}\tag{2.2}$$

Similarly

$$\begin{aligned}&{}^{p}\Phi_2\left(A-;B;z,w\right)\\&=\sum_{l(m,n)\geq 0}(A-I)\left[\left(A+\left(l(m,n)-1\right)I\right)\right]^{-1}U_{m,n}(z,w),\\&{}^{p}\Phi_2\left(A;B+;z,w\right)\\&=\sum_{l(m,n)\geq 0}B\left[\left(B+l(m,n)I\right)\right]^{-1}U_{m,n}(z,w),\\&{}^{p}\Phi_2\left(A;B-;z,w\right)\\&=\sum_{l(m,n)\geq 0}(B-I)^{-1}\left(B+\left(l(m,n)-1\right)I\right)U_{m,n}(z,w).\end{aligned}\tag{2.3}$$

By the same way, we have

$$\begin{aligned}&{}^{p}\Phi_2\left(A+;B+\right)=A^{-1}B\sum_{l(m,n)\geq 0}\left(A+l(m,n)I\right)\left[\left(B+l(m,n)I\right)\right]^{-1}U_{m,n}(z,w),\\&{}^{p}\Phi_2\left(A+;B-\right)=A^{-1}(B-I)^{-1}\sum_{l(m,n)\geq 0}\left(A+l(m,n)I\right)\left(B+\left(l(m,n)-1\right)I\right)U_{m,n}(z,w),\\&{}^{p}\Phi_2\left(A-;B+\right)=B(A-I)\sum_{l(m,n)\geq 0}\left[\left(A+\left(l(m,n)-1\right)I\right)\right]^{-1}\left[\left(B+l(m,n)I\right)\right]^{-1}U_{m,n}(z,w),\\&{}^{p}\Phi_2\left(A-;B-\right)=(A-I)(B-I)^{-1}\sum_{l(m,n)\geq 0}\left[\left(A+\left(l(m,n)-1\right)I\right)\right]^{-1}\left(B+\left(l(m,n)-1\right)I\right)U_{m,n}(z,w).\end{aligned}\tag{2.4}$$

Now, we consider the following differential operators

$$D=\frac{1}{2}(D)_2+d_2=\frac{1}{2}\left(D^2+D\right)+d_2$$

where $D=d_1+d_2$, $d_1=z\frac{\partial}{\partial z}$ and $d_2=w\frac{\partial}{\partial w}$.

It is clear that

$$D^{p}\Phi_2\left(A;B;z,w\right)=\left[\frac{1}{2}\left(D^2+D\right)+d_2\right]{}^{p}\Phi_2\left(A;B;z,w\right)=\sum_{l(m,n)\geq 0}\frac{\left[\frac{1}{2}\left((m+n)^2+(m+n)\right)+n\right](A)_{l(m,n)}\left[(B)_{l(m,n)}\right]^{-1}}{\left(pl(m,n)\right)!}z^m w^n$$

$$=\sum_{l(m,n)\geq 0}\frac{\left[\frac{1}{2}\left((m+n)(m+n+1)\right)+n\right](A)_{l(m,n)}\left[(B)_{l(m,n)}\right]^{-1}}{\left(pl(m,n)\right)!}z^m w^n=\sum_{l(m,n)\geq 0}\frac{l(m,n)(A)_{l(m,n)}\left[(B)_{l(m,n)}\right]^{-1}}{\left(pl(m,n)\right)!}z^m w^n.\tag{2.5}$$

So that

$$D^p\Phi_2(A;B;z,w)$$

$$=\sum_{l(m,n)\geq 0}\frac{l(m,n)(A)_{l(m,n)}\left[(B)_{l(m,n)}\right]^{-1}}{(pl(m,n))!}z^m w^n \quad (2.6)$$

$$=\frac{1}{p}\sum_{l(m,n)\geq 0}\frac{(A)_{l(m,n)}\left[(B)_{l(m,n)}\right]^{-1}}{\left[(pl(m,n))-1\right]!}z^m w^n.$$

Putting in this relation $m-1$ and $n+1$ instead of m and n respectively, then

$$l(m-1,n+1)=\frac{1}{2}(m+n)(m+n+1)+n+1=l(m,n)+1$$

and so that we can be written the relation $m-\frac{1}{p}$ and $n+\frac{1}{p}$ instead of m and n yields

$$l\left(m-\frac{1}{p},n+\frac{1}{p}\right)=l(m,n)+\frac{1}{p}$$

$$\text{and}\quad pl\left(m-\frac{1}{p},n+\frac{1}{p}\right)=pl(m,n)+1.$$

Therefore, the power series ${}^p\Phi_2(A;B;z,w)$, as follows

$$D^p\Phi_2(A;B;z,w)$$

$$=\frac{1}{p}\sum_{ll(m,n)\geq 0}\frac{(A)_{l(m,n)+\frac{1}{p}}\left[(B)_{l(m,n)+\frac{1}{p}}\right]^{-1}}{(pl(m,n))!}z^{m-\frac{1}{p}}w^{n+\frac{1}{p}}$$

$$=\frac{1}{p}\left(\frac{w}{z}\right)^{\frac{1}{p}}(A)\left[(B)\right]^{-1}$$

$$\sum_{ll(m,n)\geq 0}\frac{\left(A+\frac{1}{p}I\right)_{l(m,n)}\left[\left(B+\frac{1}{p}I\right)_{l(m,n)}\right]^{-1}}{(pl(m,n))!}z^m w^n$$

$$=\frac{1}{p}\left(\frac{w}{z}\right)^{\frac{1}{p}}(A)\left[(B)\right]^{-1}\Phi_2\left(A+\frac{1}{p}I;B+\frac{1}{p}I;z,w\right)$$

i.e., the $pl(m, n)$-Kummer matrix function is a solution of the matrix differential equation

$$D^p\Phi_2(A;B;z,w)-\frac{1}{p}\left(\frac{w}{z}\right)^{\frac{1}{p}}(A)\left[(B)\right]^{-1}\Phi_2\left(A+\frac{1}{p}I;B+\frac{1}{p}I;z,w\right)=0. \quad (2.7)$$

In this paper, we affect by differential operator D the $pl(m, n)$-Kummer matrix function, successively, then we have

$$\left[D\left(D-\frac{1}{p}\right)\left(D-\frac{2}{p}\right)\cdots\left(D-\frac{p-1}{p}\right)\right]{}^p\Phi_2(A;B;z,w)$$

$$=\sum_{l(m,n)\geq 0}\frac{l(m,n)\left[l(m,n)-\frac{1}{p}\right]\left[l(m,n)-\frac{2}{p}\right]\cdots\left[l(m,n)-\frac{p-1}{p}\right](A)_{l(m,n)}\left[(B)_{l(m,n)}\right]^{-1}}{(pl(m,n))!}z^m w^n$$

$$=\frac{1}{p^p}\sum_{ll(m,n)\geq 0}\frac{(A)_{l(m,n)}\left[(B)_{l(m,n)}\right]^{-1}}{(pl(m,n)-p)!}z^m w^n=\frac{1}{p^p}\sum_{ll(m,n)\geq 0}\frac{(A)_{l(m,n)+1}\left[(B)_{l(m,n)+1}\right]^{-1}}{(pl(m,n))!}z^{m-1}w^{n+1}$$

$$=\frac{w}{z}\frac{1}{p^p}\sum_{l(m,n)\geq 0}\frac{(A)_{l(m,n)+1}\left[(B)_{l(m,n)+1}\right]^{-1}}{(pl(m,n))!}z^m w^n=\frac{w}{z}\frac{1}{p^p}\sum_{l(m,n)\geq 0}\frac{\left[A+l(m,n)I\right]\left[B+l(m,n)I\right]^{-1}(A)_{l(m,n)}\left[(B)_{l(m,n)}\right]^{-1}}{(pl(m,n))!}z^m w^n$$

$$=\frac{w}{z}\frac{1}{p^p}\sum_{l(m,n)\geq 0}\left[A+l(m,n)I\right]\left[B+l(m,n)I\right]^{-1}\frac{(A)_{l(m,n)}\left[(B)_{l(m,n)}\right]^{-1}}{(pl(m,n))!}z^m w^n$$

$$=\frac{w}{z}\frac{1}{p^p}\sum_{l(m,n)\geq 0}\frac{(A)_{l(m,n)}\left[(B)_{l(m,n)}\right]^{-1}}{(pl(m,n))!}z^m w^n$$

$$+\frac{w}{z}\frac{1}{p^p}\sum_{l(m,n)\geq 0}(A-B)\left[B+l(m,n)I\right]^{-1}\frac{(A)_{l(m,n)}\left[(B)_{l(m,n)}\right]^{-1}}{(pl(m,n))!}z^m w^n$$

$$=\frac{w}{z}\frac{1}{p^p}\,{}^p\Phi_2(A;B;z,w)+\frac{w}{z}\frac{1}{p^p}(A-B)B^{-1}\,{}^p\Phi_2(A;B+;z,w)$$

i.e. the (m, n)-Kummer matrix function is a solution to this matrix differential equation

$$\left[D\left(D-\frac{1}{p}\right)\left(D-\frac{2}{p}\right)\cdots\left(D-\frac{p-1}{p}\right)-\frac{w}{z}\frac{1}{p^p}\right]{}^p\Phi_2\left(A;B;z,w\right)-\frac{w}{z}\frac{1}{p^p}\left(A-B\right){}^p\Phi_2\left(A;B+;z,w\right)=0. \tag{2.8}$$

Then

$$\left[D\left(D-\frac{1}{p}\right)\left(D-\frac{2}{p}\right)\cdots\left(D-\frac{p-1}{p}\right)\left(DI+B-I\right)\right]{}^p\Phi_2\left(A;B;z,w\right)$$

$$=\sum_{l(m,n)\ge 0}\frac{l(m,n)\left[l(m,n)-\frac{1}{p}\right]\left[l(m,n)-\frac{2}{p}\right]\cdots\left[l(m,n)-\frac{p-1}{p}\right]\left(B+\left(l(m,n)-1\right)I\right)(A)_{l(m,n)}\left[(B)_{l(m,n)}\right]^{-1}}{\left(pl(m,n)\right)!}z^m w^n$$

$$=\frac{1}{p^p}\sum_{l(m,n)\ge 0}\frac{\left(B+\left(l(m,n)-1\right)I\right)(A)_{l(m,n)}\left[(B)_{l(m,n)}\right]^{-1}}{\left(pl(m,n)-p\right)!}z^m w^n=\frac{1}{p^p}\sum_{l(m,n)\ge 0}\frac{\left(B+l(m,n)I\right)(A)_{l(m,n)+1}\left[(B)_{l(m,n)+1}\right]^{-1}}{\left(pl(m,n)\right)!}z^{m-1}w^{n+1}$$

$$=\frac{w}{z}\frac{1}{p^p}\sum_{l(m,n)\ge 0}\frac{\left(B+l(m,n)I\right)(A)_{l(m,n)+1}\left[(B)_{l(m,n)+1}\right]^{-1}}{\left(pl(m,n)\right)!}z^m w^n=\frac{w}{z}\frac{1}{p^p}\sum_{l(m,n)\ge 0}\frac{\left[A+l(m,n)I\right](A)_{l(m,n)}\left[(B)_{l(m,n)}\right]^{-1}}{\left(pl(m,n)\right)!}z^m w^n$$

$$=\frac{w}{z}\frac{1}{p^p}A\,{}^p\Phi_2\left(A;B;z,w\right).$$

Therefore, the following result has been established.

Theorem 2.2. *Let A and B be matrices in $C^{N\times N}$. Then the pl(m, n)-Kummer matrix function is solution of this matrix differential equation*

$$\left[D\left(D-\frac{1}{p}\right)\left(D-\frac{2}{p}\right)\cdots\left(D-\frac{p-1}{p}\right)\left(DI+B-I\right)-\frac{w}{z}\frac{1}{p^p}A\right]{}^p\Phi_2\left(A;B;z,w\right)=0. \tag{2.9}$$

The $\alpha(D)$ differential operator has been defined by Sayyed [19] in the form

$$\alpha(D)=1+\sum_{k=1}^{N}D^k;\ D^k=DD^{k-1}.$$

From (2.1), (2.3) and (2.5), we obtain

$$\left(DI+B-I\right){}^p\Phi_2\left(A;B;z,w\right)$$

$$=\sum_{l(m,n)\ge 0}\frac{\left(B+\left(l(m,n)-1\right)I\right)(A)_{l(m,n)}\left[(B)_{l(m,n)}\right]^{-1}}{\left(pl(m,n)\right)!}z^m w^n$$

$$=\sum_{l(m,n)\ge 0}\frac{(B-I)(A)_{l(m,n)}\left[(B-I)_{l(m,n)}\right]^{-1}}{\left(pl(m,n)\right)!}z^m w^n$$

$$=(B-I){}^p\Phi_2\left(A;B-I;z,w\right) \tag{2.10}$$

hence

$$D\,{}^p\Phi_2\left(A;B;z,w\right)=(B-I)\left[{}^p\Phi_2\left(A;B-I;z,w\right)-{}^p\Phi_1\left(A;B;z,w\right)\right] \tag{2.11}$$

and

$$D^2\,{}^p\Phi_2\left(A;B;z,w\right)$$

$$=(B-I)(B-2I){}^p\Phi_2\left(A;B-2I;z,w\right)$$

$$-\left[(B-I)(B-2I)+(B-I)^2\right]{}^p\Phi_2\left(A;B-I;z,w\right) \tag{2.12}$$

$$+(B-I)^2\,{}^p\Phi_2\left(A;B;z,w\right).$$

Thus by mathematical induction, we have the following general form

$$\alpha(D){}^p\Phi_2\left(A;B;z,w\right)=\left(1+\sum_{k=1}^{N}D^k\right){}^p\Phi_2\left(A;B;z,w\right)$$

$$={}^p\Phi_2\left(A;B;z,w\right)+\sum_{k=1}^{N}\prod_{j=1}^{k}(B-jI){}^p\Phi_2\left(A;B-jI;z,w\right)$$

$$-\left[\prod_{j=1}^{k}(B-jI)+\prod_{j=1}^{k-1}(B-jI)\sum_{k=1}^{N-1}(B-jI)\right]$$

$$\cdot{}^p\Phi_2\left(A;B-(j-1)I;z,w\right)$$

$$+\left[\prod_{j=1}^{k-1}(B-jI)\sum_{j=1}^{k-1}(B-jI)+\prod_{j=1}^{k-2}(B-jI)\left(\sum_{j=1}^{k-2}(B-jI)\right)^2\right.$$

$$+\sum_{j=1}^{k-3}(B-jI)\left(B-(j+1)I\right)$$

$$\left.+\sum_{j=1}^{k-4}(B-jI)\left(B-(j+1)I\right)+\cdots\right]$$

$${}^p\Phi_2\left(A;B-(j-2)I;z,w\right)+\cdots$$

$$+(-1)^k(B-I)^k\,{}^p\Phi_2\left(A;B;z,w\right). \tag{2.13}$$

where N is a finite positive integer.

Special cases: we can be written the matrix function $\Phi_2(A;-;z,w)$ in the form

$$^{p}\Phi_2(A;-;z,w)=\sum_{ll(m,n)\geq 0}\frac{(A)_{l(m,n)}}{(pl(m,n))!}z^m w^n \quad (2.14)$$

we see that

$$D^p\Phi_2(A;-;z,w)=\frac{w}{z}(DI+A)\,{}^{p}\Phi_2(A;-;z,w).$$

i.e., the ${}^{p}\Phi_2(A;-;z,w)$ is a solution to this matrix differential equation

$$\left[D\left(1-\frac{w}{z}\right)I-\frac{w}{z}A\right]{}^{p}\Phi_2(A;-;z,w)=0. \quad (2.15)$$

Also

$$D^p\Phi_2(A;-;z,w)$$
$$=\frac{w}{z}(DI+A)\,{}^{p}\Phi_2(A;-;z,w)$$
$$+\frac{w^2}{z^2}(DI+A)(DI+A+I)\,{}^{p}\Phi_2(A;-;z,w).$$

i.e., the ${}^{p}\Phi_2(A;-;z,w)$ is a solution for the matrix partial differential equations

$$\left[D^2I-\frac{w}{z}(DI+A)-\frac{w^2}{z^2}(DI+A)(DI+A+I)\right]$$
$${}^{p}\Phi_2(A;-;z,w)=0.$$

The results of this paper are variant, significant and so it is interesting and capable to develop its study in the future. One can use the same class of differential operators for some other function of several complex variables. Hence, new results and further applications can be obtained.

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