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Iterative Solution Methods for a Class of State and Control Constrained Optimal Control Problems

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ABSTRACT

Iterative methods for solving discrete optimal control problems are constructed and investigated. These discrete problems arise when approximating by finite difference method or by finite element method the optimal control problems which contain a linear elliptic boundary value problem as a state equation, control in the righthand side of the equation or in the boundary conditions, and point-wise constraints for both state and control functions. The convergence of the constructed iterative methods is proved, the implementation problems are discussed, and the numerical comparison of the methods is executed.

Keywords: Constrained Optimal Control Problem; Saddle Point Problem; Finite Element Method; Iterative Algorithm

1. Sample Examples of the State and Control Constrained Optimal Control Problems

We give two sample examples of the elliptic optimal control problems. The corresponding existence theory, methods of the approximation and more examples can be found, e.g., in [1,2] (see also the bibliography therein).

Consider the homogeneous Dirichlet boundary value problem for Poisson equation which plays a role of the state problem in first sample example:

$$-\Delta y = f + \chi_0 u \text{ in } \Omega, y(x) = 0 \text{ on } \partial\Omega. \quad (1)$$

Above $\Omega \subset \mathbb{R}^2$ is a bounded domain with piecewise smooth boundary $\partial\Omega$, $\chi_0 \equiv \chi_{\Omega_0}$ is the characteristic function of a subdomain $\Omega_0 \subseteq \Omega$, f is a fixed function, while u is a variable control function. For all $f, u \in L_2(\Omega)$ there exists a unique weak solution y of the boundary value problem (1) from Sobolev space $H_0^1(\Omega)$. We impose the point-wise constraints for both state function y and control function u:

$$\begin{aligned} Y_{ad}^{dif} &= \left\{ y \in H_0^1(\Omega) : \underline{y} \le y(x) \le \overline{y} \text{ in } \Omega \right\}, \\ U_{ad}^{dif} &= \left\{ u \in L_2(\Omega_0) : \underline{u} \le u(x) \le \overline{u} \text{ in } \Omega_0 \right\} \end{aligned} \quad (2)$$

with $-\infty \le \underline{y} < \overline{y} \le +\infty$ and $-\infty \le \underline{u} < \overline{u} \le +\infty$.

Suppose that (1) and (2) are not contradictory in the sense that

$$\begin{aligned} &\text{the set } K = \left\{ (y,u) \in Y_{ad}^{dif} \right. \\ &\left. \times U_{ad}^{dif} \text{ satisfy state Equation (1)} \right\} \text{is not empty.} \end{aligned} \quad (3)$$

Let the objective functional be defined by the equality

$$I_1^{dif}(y,u) = \frac{1}{2}\int_{\Omega_1} (y - y_d)^2 \, \mathrm{d}x + \frac{1}{2}\int_{\Omega_0} u^2 \mathrm{d}x$$

with a given function $y_d \in L_2(\Omega_1), \Omega_1 \subseteq \Omega$. The optimal control problem

$$\min_{(y,u)\in K} I_1^{dif}(y,u). \quad (4)$$

has a unique solution (y,u) if assumption (3) is fulfilled.

In the second sample example we take as the state problem mixed boundary value problem for Poisson equation

$$\begin{aligned} &-\Delta y = f \text{ in } \Omega, y(x) = 0 \text{ on } \Gamma_D, \frac{\partial y}{\partial n} = u \text{ on } \Gamma_N, \\ &\text{where } \partial\Omega = \Gamma_D \cup \Gamma_N \text{ and meas } \Gamma_D \ne 0, \end{aligned} \quad (5)$$

and the objective functional of the form

$$I_2^{dif}(y,u) = \frac{1}{2}\int_{\Gamma_{ob}} \left(\frac{\partial y}{\partial n} - q_d \right)^2 \mathrm{d}\Gamma + \frac{1}{2}\int_{\Gamma_N} u^2 \mathrm{d}\Gamma.$$

Here n is the unit vector of outward normal, $\Gamma_{ob} \subseteq \Gamma_D$ and $q_d(x) \in L_2(\Gamma_{ob})$. Let the constraints for y and u be

$$\begin{aligned} Y_{ad}^{dif} &= \left\{ y \in H_0^1(\Omega) : \underline{y} \le y(x) \le \overline{y} \text{ in } \Omega \right\}, \\ W_{ad}^{dif} &= \left\{ u \in L_2(\Gamma_N) : \underline{u} \le u(x) \le \overline{u} \text{ on } \Gamma_N \right\}. \end{aligned}$$

Suppose again that the set

$$M=\left\{(y,u)\in Y_{ad}^{dif}\times W_{ad}^{dif}\ \text{satisfy state Equation (5)}\right\}$$

is not empty. Then the optimal control problem

$$\min_{(y,u)\in M} I_2^{dif}(y,u) \quad (6)$$

has a unique solution (y,u).

2. Finite Element Approximation of the Optimal Control Problems

We briefly describe the approximation of the problems (4) and (6) by a finite element method (cf. [3]). Suppose that the domains Ω, Ω_0 and Ω_1 are polygons and construct a conforming triangulation of Ω into triangle and/or rectangle finite elements e_i. Let the triangulation be consistent with the subdomains Ω_0 and Ω_1 and the partition of the boundary into the parts Γ_D, Γ_N and Γ_{ob} in the following sense: every subdomain consists of the integer number of the elements e_i and every part of the boundary consists of the integer number of the sides of e_i. Construct the finite element spaces which consist of the continuous functions, piecewise linear on the triangles (P_1-elements) and piecewise bilinear on the rectangles (Q_1-elements), and satisfy Dirichlet boundary conditions. The integrals over domains and curves we approximate by the composed quadrature formulaes using the simplest 3-points quadrature formulaes for the triangle elements e_i and 4-points formulaes for the rectangle elements e_i. Note, that such kind of the approximation of a 2nd order elliptic equation in the case of rectangular Ω and rectangular elements e_i coincides with a finite difference approximation of this equation.

We apply the described above approximation procedure for the sample optimal control problems and obtain the mesh optimal control problems which are the finite dimensional problems for the vectors of nodal values[1] of the corresponding mesh functions. Namely, the approximations of the state problems (1) and (5) are the discrete state equations of the form

$$Ly=Mf+Su,$$

where $L\in\mathbb{R}^{N_y\times N_y}$ is a positive definite stiffness matrix, $M\in\mathbb{R}^{N_y\times N_y}$ is a diagonal mass matrix, and $S\in\mathbb{R}^{N_y\times N_u}$ is a rectangular matrix. The discrete objective function approximating I_1^{dif} or I_2^{dif} is a quadratical function

$$J(y,u)=\frac{1}{2}(M_y y,y)+\frac{1}{2}(M_u u,u)-(g,y)$$

with a symmetric and positive definite matrix $M_u\in\mathbb{R}^{N_u\times N_u}$, symmetric and positive semidefinite matrix $M_y\in\mathbb{R}^{N_y\times N_y}$, and given vector $g\in\mathbb{R}^{N_y}$. Finally, the sets of constraints for the vectors y and u are

$$Y_{ad}=\left\{y\in\mathbb{R}^{N_y}:\underline{y}\le y_i\le\overline{y},\forall i\right\},$$
$$U_{ad}=\left\{u\in\mathbb{R}^{N_u}:\underline{u}\le u_i\le\overline{u},\forall i\right\}.$$

Further we denote by $\theta:\mathbb{R}^{N_y}\to\overline{\mathbb{R}}=\mathbb{R}\bigcup\{+\infty\}$ and $\varphi:\mathbb{R}^{N_u}\to\overline{\mathbb{R}}$ the indicator functions of the sets Y_{ad} and U_{ad}, respectively. Resulting mesh optimal control problem which approximates (4) or (6) can be written in the following form:

$$\min_{Ly=Mf+Su} J(y,u),\text{where } J(y,u) =\frac{1}{2}(M_y y,y)+\frac{1}{2}(M_u u,u)-(g,y)+\theta(y)+\phi(u). \quad (7)$$

Below we list the main properties of the matrices and functions in problem (7) which are used for its theoretical investigation and proving the convergence of the corresponding iterative methods:

L and M_u are positive definite matrices;
M_y is apositive semi-definite matrix;
matrix S has full row rank; (8)
θ and φ are conex, lower semicontinuous and properfunctions.

Minimization problem (7) with the matrices and the functions which satisfy assumptions (8) arise when approximating by finite difference method or by simplest finite element methods with quadrature formulaes the wide class of the optimal control problems which contain a linear boundary value problem as a state equation, control in the right-hand side of the equation or in the boundary conditions, and point-wise constraints for both state and control functions.

Introduce Lagrange function for problem (7):

$$\mathcal{L}(y,u,\lambda)=J(y,u)+(Ly-f-Su,\lambda).$$

Its saddle point (y,u,λ) satisfies the saddle point problem

$$M_y y-L^{\mathrm{T}}\lambda+\partial\theta(y)\ni g, M_u u+S^{\mathrm{T}}\lambda+\partial\varphi(u)\ni 0, \quad Ly-Su=Mf, \quad (9)$$

where $\partial\theta(y)$ and $\partial\varphi(u)$ are the subdifferentials of the corresponding functions.

Theorem 1 *Let the assumptions* (8) *be valid and*

$$\exists(y_0,u_0)\in(dom\theta,dom\varphi):Ly=Su+Mf.$$

Then problem (7) has a unique solution (y,u). If more strict assumption

$$\exists(y_0,u_0)\in(\operatorname{int} dom\theta,\operatorname{int} dom\varphi):Ly=Su+Mf$$

is fulfilled then saddle point problem (9) has a nonempty

[1]Hereafter we use the same notations *y*, *u* for the vectors and for the functions in the differential problems. This doesn't lead to a confuse because later we consider only finite-dimensional problems.

set of solutions (y,u,λ).

3. Transformations of the Primal Saddle Point Problem and Preconditioned Uzawa Method

Matrix $A_0=\begin{pmatrix} M_y & 0 \\ 0 & M_u \end{pmatrix}$ which multiplies by the vector (y,u) in system (9) is only positive semidefinite. This prevents the usage of the dual iterative methods (such as Uzawa method) for solving (9). We transform this saddle point problem to an equivalent one with a positive definite matrix by using the last equation in system (9). Consider two equivalent to (9) saddle point problems:

$$\begin{aligned}&(M_y+rM)y-rML^{-1}Su-L^{\mathrm{T}}\lambda+\partial\theta(y)\ni g\\&+rML^{-1}Mf, M_u u+S^{\mathrm{T}}\lambda+\partial\varphi(u)\ni 0, Ly-Su=Mf.\end{aligned} \quad (10)$$

$$\begin{aligned}&(M_y+rL)y-rSu-L^{\mathrm{T}}\lambda+\partial\theta(y)\ni g+rMf,\\&M_u u+S^{\mathrm{T}}\lambda+\partial\varphi(u)\ni 0.\ Ly-Su=Mf.\end{aligned} \quad (11)$$

Lemma 1 *Matrices* $\begin{pmatrix} M_y+rM & -rML^{-1}S \\ 0 & M_u \end{pmatrix}$ *and* $\begin{pmatrix} M_y+rL & -rS \\ 0 & M_u \end{pmatrix}$ *are positive definite for* $0<r<r_i, i=1,2,$ *where constants* r_i *don't depend on the dimension of the finite element space* (*or*, *on the mesh step*).

Now, we can apply the preconditioned Uzawa methods for solving problem (10) and problem (11):

$$\begin{aligned}&(M_y+rM)y^{k+1}+\partial\theta(y^{k+1})\ni g+rML^{-1}Mf\\&+rML^{-1}Su^k+L^{\mathrm{T}}\lambda^k,\\&M_u u^{k+1}+\partial\varphi(u^{k+1})\ni -S^{\mathrm{T}}\lambda^k,\\&LM^{-1}L^{\mathrm{T}}\frac{\lambda^{k+1}-\lambda^k}{\tau}+Ly^{k+1}-Su^{k+1}=Mf.\end{aligned} \quad (12)$$

$$\begin{aligned}&(M_y+rL)y^{k+1}+\partial\theta(y^{k+1})-rSu^k\ni g+rf+L^{\mathrm{T}}\lambda^k,\\&M_u u^{k+1}+\partial\varphi(u^{k+1})\ni -S^{\mathrm{T}}\lambda^k,\\&(L^{\mathrm{T}}+SM_u^{-1}S^{T})\frac{\lambda^{k+1}-\lambda^k}{\tau}+Ly^{k+1}-Su^{k+1}=f.\end{aligned} \quad (13)$$

The choice of the preconditioners is based on the properties of the matrices in problems (10) and (11) (see the corresponding theory in [4]).

Lemma 2 *The iterative methods* (12) *and* (13) *converge if* $\tau\in(0,\tau_i), i=1,2$, *where* τ_i *don't depend on mesh step.*

The implementation of every step of (12) includes solution of two systems of linear equations with matrices L and L^{T} and solution of two inclusions with multivalued operators $M_y+rM+\partial\theta$ and $M_u+\partial\varphi$. The matrices in the discrete model problems are diagonal while the functions are separable:

$$\theta(y)=\sum_{i=1}^{N_y}\theta_i(y_i), \varphi(u)=\sum_{i=1}^{N_u}\varphi_i(u_i).$$

Due to these properties $M_y+rM+\partial\theta$ and $M_u+\partial\varphi$ are diagonal operators, and the solution of the inclusions with these operators reduce to an easy problem of the orthogonal projection on the corresponding sets Y_{ad} and U_{ad}.

The implementation of every step of (13) includes solution of the system of linear equations with matrix $L^{\mathrm{T}}+SM_u^{-1}S^{\mathrm{T}}$ and solution of the inclusion with operator $M_y+rL+\partial\theta$, which corresponds to a mesh variational inequality of second order. This is more time consuming problem than solution of a linear system, but the numerical tests demonstrates the preference of method (13) in some particular cases. The methods of solving the variational inequalities can be found in the books [5-7].

4. Block SOR-Method for the Problem with Penalization of the State Equation

Let D be a symmetric and positive matrix while $\varepsilon>0$ be a regularization parameter. Consider the following regularization of problem (7):

$$\begin{aligned}&\min_{y,u}\left(\frac{1}{2}(M_y y_\varepsilon, y_\varepsilon)+\frac{1}{2}(M_u u_\varepsilon, u_\varepsilon)-(g,y)\right.\\&\left.+\theta(y_\varepsilon)+\varphi(u_\varepsilon)+\frac{1}{2\varepsilon}\|Ly_\varepsilon-f-Su_\varepsilon\|^2_{D^{-1}}\right).\end{aligned} \quad (14)$$

Theorem 2 *Let the assumptions* (8) *be valid. Then problem* (14) *has a unique solution* $(y_\varepsilon,u_\varepsilon)$. *If* (y,u,λ) *is a solution of saddle point problem* (9), *then the following estimate holds*:

$$\|y_\varepsilon-y\|^2+\|u_\varepsilon-u\|^2\le c\varepsilon\|\lambda\|_D^2 \quad (15)$$

with a constant c independent on ε and on mesh size. Problem (14) is equivalent to the following system of the inclusions:

$$\begin{aligned}&\left(M_y+\frac{1}{\varepsilon}L^{\mathrm{T}}D^{-1}L\right)y-\frac{1}{\varepsilon}L^{\mathrm{T}}D^{-1}Su\\&+\partial\theta(y)\ni g-\frac{1}{\varepsilon}L^{\mathrm{T}}D^{-1}f,\\&\left(M_u+\frac{1}{\varepsilon}S^{\mathrm{T}}D^{-1}S\right)u-\frac{1}{\varepsilon}S^{\mathrm{T}}D^{-1}Ly\\&+\partial\varphi(u)\ni\frac{1}{\varepsilon}D^{-1}f\end{aligned} \quad (16)$$

with a positive definite and symmetric matrix. Different iterative methods can be used for solving problem (14) or equivalent problem (16) (see, e.g., [5,6] and the bibliography therein). We solve system (16) by block SOR-method:

$$\frac{1}{\sigma}\left(M_y+\frac{1}{\varepsilon}L^{\mathrm{T}}D^{-1}L\right)y^{k+1}+\partial\theta\left(y^{k+1}\right)\ni F_1^k,$$
$$\frac{1}{\sigma}\left(M_u+\frac{1}{\varepsilon}S^{\mathrm{T}}D^{-1}S\right)u^{k+1}+\partial\varphi\left(u^{k+1}\right)\ni F_2^k, \tag{17}$$

where $\sigma\in(0,2)$ is a relaxation parameter and $F_1^k=F_1\left(u^k,y^k\right),F_2^k=F_2\left(u^k,y^{k+1}\right)$.

Theorem 3 ([8]) *Method* (17) *converges for any* $\sigma\in(0,2)$.

The implementation of (17) depends on the choice of the matrix D. Below we consider two variants of the choice.

a) Let $D=LM^{-1}L^{\mathrm{T}}$. In this case the implementation of every iterative step of (17) consists of the solving the following system

$$M_y y+\frac{1}{\varepsilon}My+\partial\theta(y)\ni G_1,$$
$$M_u u+\frac{1}{\varepsilon}\left(L^{-1}S\right)^{\mathrm{T}}M\left(L^{-1}S\right)u+\partial\varphi(u)\ni G_2 \tag{18}$$

with known $G_i=\sigma F_i^k$. For the model problems under consideration the operator $M_y+\frac{1}{\varepsilon}M+\partial\theta$ of the first inclusion has diagonal form, so, its solution reduces the projection on the set Y_{ad}. Further, the matrix in the second inclusion is spectrally equivalent to matrix M_u:

$$M_u\le M_u+\frac{1}{\varepsilon}\left(L^{-1}S\right)^{\mathrm{T}}M\left(L^{-1}S\right)\le\left(1+\frac{c_{stab}}{\varepsilon}\right)M_u.$$

Here c_{stab} is a constant in the stability estimate for a solution of the state equation and it is independent on mesh step. The stationary one-step iterative method with preconditioner M_u converges with the rate of convergence proportional to ε (and doesn't depend on mesh step), and its implementation reduces to the projection on the set U_{ad}.

In the particular case when S is the unit matrix (which corresponds to the distributed in the domain control) the second inclusion in (18) can be transformed to a system of nonlinear equations and an inclusion with diagonal operator. Namely, let $u=\left(M_u+\partial\varphi\right)^{-1}(w)$. Then the auxiliary vector w is a solution of the system of nonlinear algebraic equations

$$LM_y^{-1}L^{\mathrm{T}}w+\frac{1}{\varepsilon}\left(M_u+\partial\varphi\right)^{-1}(w)=LM_y^{-1}L^{\mathrm{T}}G_2$$

with the symmetric and positive definite matrix $LM_y^{-1}L^{\mathrm{T}}$ and monotone, diagonal and Lipshitz-continuous operator $\frac{1}{\varepsilon}\left(M_u+\partial\varphi\right)^{-1}$.

b) In the case of a symmetric matrix L and the unit matrix S the promising choice is $D=L$. Then on every iterative step of (17) we solve the system

$$\left(M_y+\frac{1}{\varepsilon}L\right)y+\partial\theta(y)\ni G_1,$$
$$Lw+\frac{1}{\varepsilon}\left(M_u+\partial\varphi\right)^{-1}(w)=LG_2,$$
$$u=\left(M_u+\partial\varphi\right)^{-1}(w).$$

First inclusion of this system corresponds to a mesh approximation of a second order variational inequality, while the equation for vector w contains the symmetric and positive matrix L and monotone, diagonal and Lipshitz-continuous operator $\frac{1}{\varepsilon}\left(M_u+\partial\varphi\right)^{-1}$.

5. Numerical Example

We solved the optimal control problem with the following state problem and constraints:

$$-\Delta y=f+u,x\in\Omega,y(x)=0,x\in\partial\Omega,$$
$$Y_{ad}=\left\{y(x)\ge 0,x\in\Omega\right\},$$
$$U_{ad}=\left\{\left|u(x)\right|\le 3,x\in\Omega\right\},$$

and the objective functional

$$J(y,u)=\frac{1}{2}\int_{\Omega_1}y^2(x)\mathrm{d}x+\frac{1}{2}\int_{\Omega}u^2(x)\mathrm{d}x,$$

where $\Omega=(0,1)\times(0,1)$ and $\Omega_1=(0,0.7)\times(0,1)$.

Finite difference approximation of this optimal control problem on the uniform grid leads to a minimization problem of the form (7) and the following saddle point problem (particular case of (9)):

$$M_y y+\partial\theta(y)+L\lambda\ni 0,$$
$$u+\partial\varphi(u)-\lambda\ni 0,Ly-u=f.$$

Here symmetric and positive definite matrix L corresponds to the mesh Laplacian with Dirichlet boundary conditions, M_y is a diagonal and positive semidefinite matrix (its positive entries correspond to the grid points in the subdomain Ω_1).

To apply Uzawa method we used the equivalent transformation similar to (11) with parameter $r=1$ (satisfying the assumptions of theorem 1):

$$\left(M_y+L\right)y+\partial\theta(y)-u+L\lambda\ni f,$$
$$u+\partial\varphi(u)-\lambda\ni 0,Ly-u=f.$$

This system was solved by Uzawa method with preconditioner L which is now spectrally equivalent to $L^{\mathrm{T}}+SM_u^{-1}S^{\mathrm{T}}$.

We also solved the mesh optimal control problem by applying SOR-method to the problem with penalized state equation and the choice $D = L$. Corresponding system for y, u and auxiliary vector w reads as:

$$(\varepsilon M_y + L)y + \partial\theta(y) - u \ni -f,$$
$$\varepsilon L w + u - L^{-1} y = L^{-1} f, u + \partial\varphi(u) \ni w.$$

We used block SOR-method with relaxation parameter $\sigma = 1.97$ (found numerically to be close to optimal one). We also used the iterative regularization as follows: calculate the residual vector r^k on the current iteration k and set $\varepsilon \mapsto 10^{-1}\varepsilon$ when $\|r^k\|_{L_2}$ becomes less than 1. Here the norm $\|\cdot\|_{L_2}$ is the mesh analogue of the Lesbegue space norm $L_2(\Omega)$. The smallest value $\varepsilon = 10^{-9}$ was reached started from $\varepsilon = 1$.

In the tables below (**Tables 1** and **2**) k is the number of an iteration, F is the value of the objective function on the current iteration, $\delta y^k = \|y - y^k\|_{L_2}$ and $\delta u^k = \|u - u^k\|_{L_2}$, the calculation results for the 100×100 are presented. We constructed the exact solution (y,u) of the discrete optimal control problem, so, we knew the exact minimum of the cost function $F^* = 14.5293$.

We can conclude that block SOR method had a big advantage in comparison with preconditioned Uzawa method in the accuracy of the calculated state y and control u.

A number of calculations were made for the different state and control constrained optimal control problems on the different meshes. All of them demonstrated the preference of the iterative methods based on the penalty of the state equation in the comparison with the preconditioned Uzawa methods, while that were faster convergent than the gradient methods applied for the problems with the penalization of the state constraints (see [9,10]).

Table 1. Uzawa method.

k	F	δy^k	δu^k
1	8.9435	0.29742	2.967
2	9.6234	0.13092	3.0983
3	9.619	0.13231	3.095
4	9.6496	0.12467	3.0927
5	9.6509	0.12416	3.0845
6	9.6527	0.12349	3.0761
7	9.6535	0.12308	3.0675
8	9.6543	0.12271	3.0589
9	9.6551	0.12237	3.0503
10	9.656	0.12203	3.0417
…	…	…	…
1000	12.6414	0.026754	1.1804
…	…	…	…
10000	13.9096	0.0031814	0.73538

Table 2. Block SOR method.

k	F	δy^k	δu^k
1	41.9339	4.5763	0.077971
2	3.9872	4.439	5.8003
3	36.4713	3.8831	0.51251
4	4.227	3.7676	5.7309
5	31.6414	3.2552	1.0026
6	4.79	3.1618	5.6581
7	27.5472	2.6887	1.3318
8	5.6163	2.6164	5.5933
9	24.0385	2.1819	1.6349
10	6.5528	2.1301	5.5033
…	…	…	…
1000	14.4687	6.9224e−005	0.15564
…	…	…	…
3695	14.525	2.3759e−006	0.0098418

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