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Time Delay Induced Oscillation: An Example on a Class of *n* Coupled Van Der Pol Oscillators Model with Delays

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ABSTRACT

In this paper, a class of *n* coupled van der Pol oscillator model with delays is considered. By employing an analysis approach, some sufficient conditions to guarantee the existence of stability and oscillations for the model are obtained. Examples are provided to demonstrate the results.

Keywords: *n* Coupled Van Der Pol Oscillator Model; Characteristic Equation; Delay; Oscillation

1. Introduction

In recent years, there has been an increasing interest in studying van der Pol coupled oscillator models. Synchronization, which is defined as an adjustment of rhythms due to a weak interaction, is one of the most interesting features displayed by coupled oscillators. It occurs in physics, chemistry, engineering, biology, social sciences, etc. [1-6]. For a delayed feedback model as follows:

$$x''(t) + \varepsilon \left(x(t)^2 - 1\right)x'(t) + x(t) = \varepsilon g(t - \tau), \quad (1)$$

where $g(0) = 0$, $g'(0) = k \neq 0$, Jiang and Wei found that there were Bagdanov-Takens bifurcation, triple zero and Hopf-zero singularities for the equation under some restrictive conditions [7]. The following is a model of a self-excited system:

$$\begin{cases} x_1''(t) + \varepsilon \left(x_1^2(t) - 1\right)x_1'(t) + x_1(t) = \alpha y_1(t - \tau_2) - x_1(t), \\ y_1''(t) + \varepsilon \left(y_1^2(t) - 1\right)y_1'(t) + y_1(t) = \alpha x_1(t - \tau_2) - y_1(t). \end{cases} \quad (2)$$

Zhang and Gu [8] have shown that there exist the stability switches and a sequence of Hopf bifurcations occur at zero equilibrium when the delay varies. Recently, Zhang *et al*. [9] have considered a system of coupled electric circuit in a ring with *n* symmetrical van der Pol oscillators. By the Kirchhoff's law, considering the time delay between the signal transmission of the oscillators, and after the normalization of the state variables and parameters, the model looks like this:

$$\begin{cases} x_k'(t) = y_k(t) + cx_k(t) - \left[\dfrac{cx_k(t)}{3}\right]^3 - \delta\big[2x_k(t - \tau)\big) \\ \quad -x_{k-1}(t - \tau) - x_{k+1}(t - \tau)\big], \\ y_k'(t) = -x_k(t) - py_k(t), \\ \quad \text{where } k(\text{mod } n), k = 1, 2, \cdots, n, \end{cases} \quad (3)$$

where parameters $c, p,\ \tau$ and δ are positive constants. By choosing the delay as the bifurcating parameter, the authors have proposed some results of the Hopf bifurcations occurring at the zero equilibrium as the delay increases. Using the symmetric functional differential equation theories, the authors also have exhibited the multiple Hopf bifurcations, and their spatio-temporal patterns: mirror-reflecting waves, standing waves, and discrete waves.

However, if the time delay and the parameters $c, p,\ \tau$ and δ are different in each oscillator, in other words, if the system is not symmetric which may draw closer to reality, then we get the following system:

$$\begin{cases} x_k'(t) = y_k(t) + c_k x_k(t) - \left[\dfrac{c_k x_k(t)}{3}\right]^3 - \delta_k\big[2x_k(t - \tau_k)\big) \\ \quad -x_{k-1}(t - \tau_{k-1}) - x_{k+1}(t - \tau_{k+1})\big], \\ y_k'(t) = -x_k(t) - p_k y_k(t), \\ \quad \text{where } k(\text{mod } n), k = 1, 2, \cdots, n. \end{cases}$$
$$(4)$$

We shall point out that with the method of bifurcation it will become very difficult to discuss the properties of

the solutions of system (4) due to the complexity of parameters. In this paper, we use the analysis method to discuss the oscillatory behavior of the system (4). Simple criteria to guarantee the existence of oscillation of the system have been proposed.

2. Preliminaries

For convenience, set $u_{2r-1} = x_k$, $u_{2r}(t) = y_k$, $r = 1, 2, \cdots, n$. Then system (4) can be rewritten as follows:

$$
\begin{cases}
u_1'(t) = u_2(t) + c_1 u_1(t) - \left[\dfrac{c_1 u_1(t)}{3}\right]^3 - \delta_1\big[2u_1(t-\tau_1)) \\
\qquad - u_{2n-1}(t-\tau_{2n-1}) - u_3(t-\tau_3)\big], \\
u_2'(t) = -u_1(t) - p_2 u_2(t), \\
u_3'(t) = u_4(t) + c_3 u_3(t) - \left[\dfrac{c_3 u_3(t)}{3}\right]^3 - \delta_3\big[2u_3(t-\tau_3)) \\
\qquad - u_1(t-\tau_1) - u_5(t-\tau_5)\big], \\
u_4'(t) = -u_3(t) - p_4 u_4(t), \\
\hdashline
u_{2n-1}'(t) = u_{2n}(t) + c_{2n-1} u_{2n-1}(t) - \left[\dfrac{c_{2n-1} u_{2n-1}(t)}{3}\right]^3 \\
\qquad - \delta_{2n-1}\big[2u_{2n-1}(t-\tau_{2n-1}) - u_{2n-1}(t-\tau_{2n-1}) - u_1(t-\tau_1)\big], \\
u_{2n}'(t) = -u_{2n-1}(t) - p_{2n} u_{2n}(t).
\end{cases}
$$
(5)

The nonlinear system (5) can be expressed in the following matrix form:

$$U'(t) = AU(t) + BU(t-\tilde{\tau}) + P(U) \qquad (6)$$

where

$$U(t) = \big[u_1(t), u_2(t), \cdots, u_{2n}(t)\big]^T,$$

$$U(t-\tilde{\tau}) = \big[u_1(t-\tau_1), u_2(t-\tau_2), \cdots, u_{2n}(t-\tau_{2n})\big]^T.$$

Both A and B are $2n$ by $2n$ matrices, $P(U)$ is a $2n$ by 1 vector.

$$
A = \begin{pmatrix}
c_1 & 1 & 0 & & 0 & 0 & 0 \\
0 & -p_2 & 0 & \cdots & 0 & 0 & 0 \\
-1 & 0 & c_3 & & 0 & 0 & 0 \\
\vdots & & \ddots & & & & \vdots \\
0 & 0 & 0 & & -p_{2n-2} & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & c_{2n-1} & 0 \\
0 & 0 & 0 & & 0 & -1 & -p_{2n}
\end{pmatrix},
$$

$$
B = \begin{pmatrix}
-2\delta_1 & 0 & \delta_1 & & 0 & \delta_1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\delta_3 & 0 & -2\delta_3 & & 0 & 0 & 0 \\
\vdots & & & \ddots & & & \vdots \\
0 & 0 & 0 & & 0 & 0 & 0 \\
\delta_{2n-1} & 0 & 0 & \cdots & 0 & -2\delta_{2n-1} & 0 \\
0 & 0 & 0 & & 0 & -1 & 0
\end{pmatrix},
$$

$$P(U) = \left(-\frac{c_1^3 u_1^3}{27}\ \ 0\ -\frac{c_3^3 u_3^3}{27}\ \ 0 \cdots -\frac{c_{2n-1}^3 u_{2n-1}^3}{27}\ \ 0\right)^T.$$

Definition 1. A solution of system (4) is called oscillatory if the solution is neither eventually positive nor eventually negative. If $U(t) = \big[u_1(t), u_2(t), \cdots, u_{2n}(t)\big]^T$ is an oscillatory solution of system (5), then each component of $U(t)$ is oscillatory.

We adopt the following norms of vectors and matrices [10]:

$$\|u(t)\| = \sum |u_i(t)|,$$

$$\|A\| = \max_i \sum_{j=1}^{2n} |a_{ij}|,$$

$$\|B\| = \max_i \sum_{j=1}^{2n} |b_{ij}|.$$

The measure $\mu(A)$ of the matrix A is defined by $\mu(A) = \lim_{\theta \to 0^+} \dfrac{I + \theta A - 1}{\theta}$, which for the chosen norms reduces to $\mu(A) = \max_{1 \le i \le 2n} \left[a_{ii} + \sum_{j=1, j\neq i} |a_{ij}|\right].$

Note, that the linearization of system (5) about zero is the following:

$$
\begin{cases}
u_1'(t) = u_2(t) + c_1 u_1(t) - \delta_1\big[2u_1(t-\tau_1)) \\
\qquad - u_{2n-1}(t-\tau_{2n-1}) - u_3(t-\tau_3)\big], \\
u_2'(t) = -u_1(t) - p_2 u_2(t), \\
u_3'(t) = u_4(t) + c_3 u_3(t) - \delta_3\big[2u_3(t-\tau_3)) \\
\qquad - u_1(t-\tau_1) - u_5(t-\tau_5)\big], \\
u_4'(t) = -u_3(t) - p_4 u_4(t), \\
\hdashline
u_{2n-1}'(t) = u_{2n}(t) + c_{2n-1} u_{2n-1}(t) \\
\qquad - \delta_{2n-1}\big[2u_{2n-1}(t-\tau_{2n-1}) - u_{2n-1}(t-\tau_{2n-1}) - u_1(t-\tau_1)\big], \\
u_{2n}'(t) = -u_{2n-1}(t) - p_{2n} u_{2n}(t).
\end{cases}
$$
(7)

Or the matrix form

$$U'(t) = AU(t) + BU(t-\tilde{\tau}) \qquad (8)$$

Obviously, if the equilibrium point of system (7) (or (8)) is unstable, then this equilibrium point for system (5) (or (6)) is still unstable. Thus, in order to discuss the instability of the equilibrium point of system (5) (or (6)),

we only need to focus on the instability of the equilibrium point of system (7) (or (8)). We first have

Lemma 1. If for given parameter values of c_i, p_i, and δ_i, the determinant of matrix $(A + B)$ is not zero, then system (7) (or (8)) has a unique equilibrium point.

Proof. An equilibrium point $U^* = \left[u_1^*, u_2^*, \cdots, u_{2n}^*\right]^T$ is a constant solution of the following algebraic equation

$$AU^* + BU^* = (A + B)U^* = 0 \qquad (9)$$

Clearly, U^* is an equilibrium point. Since the determinant of matrix $(A + B)$ is not zero, then Equation (9) has a unique equilibrium point, that is, the zero point.

3. Oscillation Analysis

Theorem 1. If linearized system (7) has a unique equilibrium point for given parameter values of c_i, p_i, and δ_i ($i = 1, 2, \cdots, 2n$), assume that the following conditions hold for system (7) (or (8)):

(C1) $\mu(A) + B \neq 0$;

(C2) Both

$$\left(\|B\|\tau e\right)\exp\left(-\tau|\mu(A)|\right) > 1,$$

and

$$\left(\|B\|\tau^* e\right)\exp\left(-\tau^*|\mu(A)|\right) > 1$$

hold, where

$$\tau = \min\left\{\tau_1, \tau_3, \cdots, \tau_{2n-1}\right\},$$

and

$$\tau^* = \max\left\{\tau_1, \tau_3, \cdots, \tau_{2n-1}\right\}.$$

Then the unique equilibrium point of system (7) is unstable, implying that the unique equilibrium point of system (5) is also unstable. That is, system (5) generates permanent oscillations.

Proof. First consider the special case of system (7) as $\tau_i = \tau$ ($i = 1, 3, \cdots, 2n - 1$):

$$\begin{cases}
u_1'(t) = u_2(t) + c_1 u_1(t) - \delta_1\left[2u_1(t-\tau)\right. \\
\qquad \left. -u_{2n-1}(t-\tau) - u_3(t-\tau)\right], \\
u_2'(t) = -u_1(t) - p_2 u_2(t), \\
u_3'(t) = u_4(t) + c_3 u_3(t) - \delta_3\left[2u_3(t-\tau)\right. \\
\qquad \left. -u_1(t-\tau) - u_5(t-\tau)\right], \\
u_4'(t) = -u_3(t) - p_4 u_4(t), \\
\text{-----------------------} \\
u_{2n-1}'(t) = u_{2n}(t) + c_{2n-1} u_{2n-1}(t) \\
\qquad -\delta_{2n-1}\left[2u_{2n-1}(t-\tau) - u_{2n-1}(t-\tau) - u_1(t-\tau)\right], \\
u_{2n}'(t) = -u_{2n-1}(t) - p_{2n} u_{2n}(t).
\end{cases} \qquad (10)$$

If the unique equilibrium point is unstable for system (10), in other words, the trivial solution of (10) is unstable, then consider (10), for some $t^* > 0$, we have

$$\frac{\mathrm{d}|u_i(t)|}{\mathrm{d}t} \leq a_{ii}|u_i(t)| + \sum_{j=1, j\neq i}^{2n}|a_{ij}||u_j(t)| \\
+ \sum_{j=1, j\neq i}^{2n}|b_{ij}||u_j(t-\tau)| \qquad (11)$$

where $a_{ii} = c_i$ or $-p_i$, $a_{ij} = 1$ or -1 ($i = 1, 2, \cdots, 2n$).

Hence for $t > t^* + \tau$, we have

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} \leq \mu(A)v(t) + Bv(t-\tau) \qquad (12)$$

where $v(t) = \sum_{i=1}^{2n}|u_i(t)|$. Consider the scalar delay differential equation

$$\frac{\mathrm{d}w(t)}{\mathrm{d}t} = \mu(A)w(t) + \|B\|w(t-\tau) \qquad (13)$$

with $v(s) = w(s)$, $s \in \left[t^*, t^* + \tau\right]$. Based on the comparison theory of differential equation we get $v(s) \leq w(s)$, for $t > t^* + \tau$. Thus, if the trivial solution $w(t)$ is unstable, then the trivial solution $v(t)$ is also unstable. It is not difficult to see that the trivial solution of (13) is unstable. Otherwise, the characteristic equation associated with (13) given by

$$\lambda = \mu(A) + \|B\|e^{-\lambda\tau} \qquad (14)$$

will have a real negative root, say λ^*, and

$$|\lambda^*| \geq \|B\|e^{-\lambda^*\tau} - \mu(A) \qquad (15)$$

This means that

$$\begin{aligned}
1 &\geq \frac{\|B\|\exp\left(\tau|\lambda^*|\right)}{|\lambda^*| + |\mu(A)|} \\
&\geq \frac{\tau\|B\|\exp\left(-\tau|\mu(A)|\right)\exp\left(|\lambda^*|\tau + |\mu(A)|\tau\right)}{\left(|\lambda^*| + |\mu(A)|\right)\tau}
\end{aligned} \qquad (16)$$

where we use the inequality $e^x \geq ex$ for $x > 0$. The last inequality of (16) contradicts condition (C2). Thus the trivial solution of (13) is oscillatory. Now for time delays $\tau \leq \tau_i \leq \tau^*$ ($i = 1, 3, \cdots, 2n - 1$), condition (C2) still holds. Therefore, the trivial solution of system (7) is oscillatory, implying the oscillations of the trivial solution about the equilibrium point of system (5).

Noting that $\exp\left(-\tau|\mu(A)|\right)$ tends to zero as τ tends to infinity. Thus, Theorem 1 holds only for finite τ. In the following we provide a criterion for suitably large τ holds.

Theorem 2. If linearized system (7) has a unique equilibrium point for given parameter values of c_i, p_i, and δ_i ($i = 1, 2, \cdots, 2n$), assume that there exists a suitably large positive number K such that the following conditions hold for system (7) (or (8)):

(C3) $\quad K - \mu(A) - \|B\| e^{-K\tau} > 0 \qquad (17)$

Then the unique equilibrium point of system (7) is unstable, implying that the unique equilibrium point of system (5) is also unstable. That is, system (5) generates permanent oscillations.

Proof. We will prove that the trivial solution of system (7) is unstable. It is sufficient to show that the characteristic equation associated with (13) given by (14) has a real positive root under the stated condition (C3). Since the Equation (14) is a transcendental equation, the characteristic values may be complex numbers. We claim that there exists a real positive root from condition (C3). Set

$$g(\lambda) = \lambda - \mu(A) - \|B\| e^{-\lambda\tau} \qquad (18)$$

then $g(\lambda)$ is a continuous function of λ. Note that delay $\tau > 0$, $\mu(A) > 0$, $\|B\|$ is bounded, then $g(0) = -\mu(A) - \|B\| < 0$. On the other hand, from condition (C3) we have $g(K) = K - \mu(A) - \|B\| e^{-K\tau} > 0$. Therefore, there exists $\tilde{\lambda} \left(0 < \tilde{\lambda} < K \right)$ such that $g(\tilde{\lambda}) = 0$ from the continuity of $g(\lambda)$. In other words, λ is a real positive characteristic root of (14). The proof is completed.

Based on the property of characteristic roots of matrices A and B, we immediately have

Theorem 3. If linearized system (8) as $\tau_i = \tau$ $(i = 1, 3, \cdots, 2n - 1)$ has a unique equilibrium point for given parameter values of c_i, p_i, and δ_i $(i = 1, 2, \cdots, 2n)$, let $\alpha_1, \alpha_2, \cdots, \alpha_{2n}$, and $\beta_1, \beta_2, \cdots, \beta_{2n}$ be characteristic roots of matrices A and B, respectively. Suppose that the following conditions hold for the system (7) (or (8)):

(C4) There exists some real number pair (α_j, β_j), $j \in \{1, 2, \cdots, 2n\}$ satisfying $\alpha_j > 0$. Then the trivial solution of (7) is unstable, implying that the unique equilibrium point of system (5) is also unstable. That is, system (5) generates permanent oscillations.

Proof. Consider the special case of system (7) as $\tau_i = \tau$ $(i = 1, 3, \cdots, 2n - 1)$ and we get the following matrix form:

$$U'(t) = AU(t) + BU(t - \tau). \qquad (19)$$

Since α_i and β_i $(i = 1, 2, \cdots, 2n)$ are characteristic roots of matrices A and B, respectively, then the characteristic equation of (19) can be expressed as follows:

$$\prod_{i=1}^{2n} \left(\lambda - \alpha_i - \beta_i e^{-\lambda\tau} \right) = 0. \qquad (20)$$

Hence, we are led to an investigation of the nature of the roots for some j

$$\lambda = \alpha_j - \beta_j e^{-\lambda\tau}. \qquad (21)$$

Since $\alpha_j > 0$ and β_j is a real number, if $\beta_j > 0$, obviously, Equation (19) has a positive root $\lambda = \bar{\lambda}$, where $\bar{\lambda} > \alpha_j$. If $\beta_j < 0$, noting that $e^{-\lambda\tau}$ tends suffi-

ciently small as τ tends suitably large and $\lambda > 0$. We can also find a positive root $\lambda = \bar{\lambda}$ for Equation (19), where $0 < \bar{\lambda} < \alpha_j$. So the trivial solution of (19) is unstable, implying that the unique equilibrium point of system (5) is also unstable.

We point out that Theorem 3 for different τ_i is still holds, and each criterion of the above theorem is a sufficient condition.

4. Simulation Result

We select parameter values $c_1 = 0.1$, $c_3 = 0.2$, $c_5 = 0.3$; $\delta_1 = 0.1$, $\delta_3 = 0.2$, $\delta_5 = 0.3$; $p_2 = 2$, $p_4 = 3$, $p_6 = 4$, and $\tau_1 = 2.5$, $\tau_3 = 5$, $\tau_5 = 4.5$, respectively. Consider the following three-node case:

$$
\begin{cases}
u_1'(t) = u_2(t) + 0.1u_1(t) - \left[\dfrac{0.1u_1(t)}{3} \right]^3 - 0.1 \left[2u_1(t - \tau_1) \right. \\
\qquad\qquad \left. - u_5(t - \tau_5) - u_3(t - \tau_3) \right], \\
u_2'(t) = -u_1(t) - 2u_2(t), \\
u_3'(t) = u_4(t) + 0.2u_3(t) - \left[\dfrac{0.2u_3(t)}{3} \right]^3 - 0.2 \left[2u_3(t - \tau_3) \right. \\
\qquad\qquad \left. - u_1(t - \tau_1) - u_5(t - \tau_5) \right], \\
u_4'(t) = -u_3(t) - 3u_4(t), \\
u_5'(t) = u_6(t) + 0.3u_5(t) - \left[\dfrac{0.3u_5(t)}{3} \right]^3 - 0.3 \left[2u_5(t - \tau_5) \right. \\
\qquad\qquad \left. - u_3(t - \tau_3) - u_1(t - \tau_1) \right], \\
u_6'(t) = -u_5(t) - 4u_6(t).
\end{cases}
$$

$$(22)$$

Thus, A and B are six by six matrices as follows:

$$
A = \begin{pmatrix}
0.1 & 1 & 0 & 0 & 0 & 0 \\
-1 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.2 & 1 & 0 & 0 \\
0 & 0 & -1 & -3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.3 & 0 \\
0 & 0 & 0 & 0 & -1 & -4
\end{pmatrix},
$$

$$
B = \begin{pmatrix}
-0.2 & 0 & 0.1 & 0 & 0.1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0.2 & 0 & -0.4 & 0 & 0.2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0.3 & 0 & 0.3 & 0 & -0.6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

and

$$
P(U) = \left(-\frac{0.001u_1^3}{27} \quad 0 \quad -\frac{0.008u_3^3}{27} \quad 0 \quad -\frac{0.027u_5^3}{27} \quad 0 \right)^T.
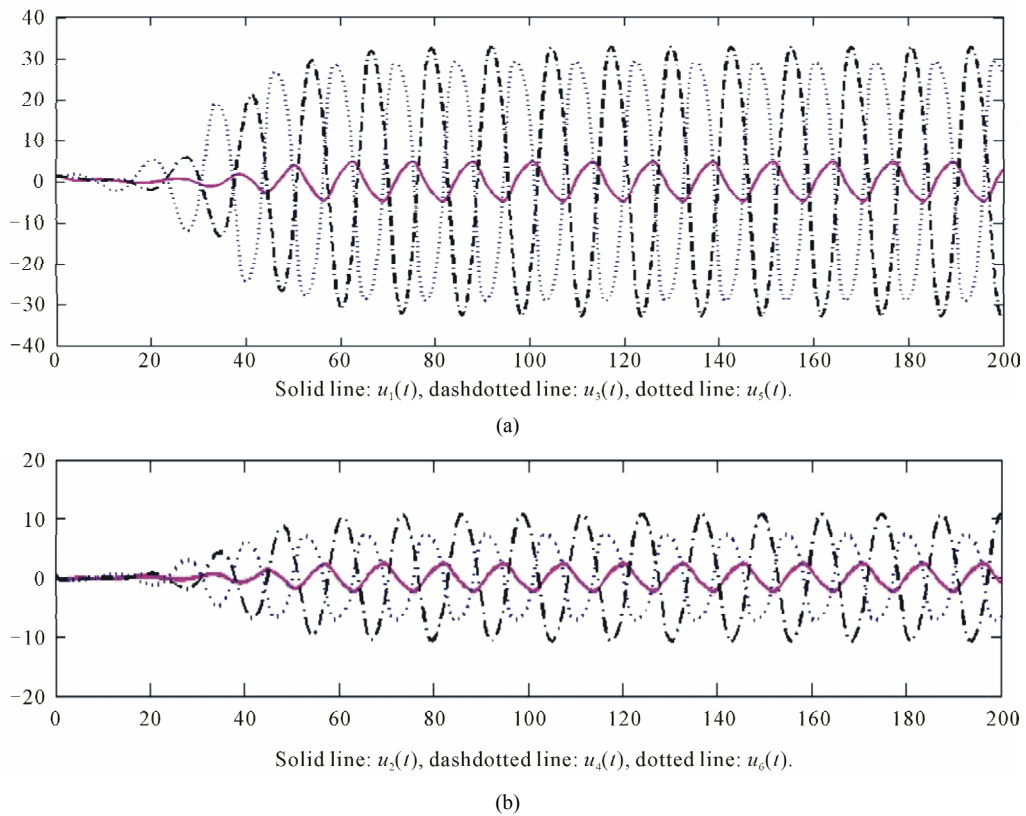$$

Figure 1. Oscillations of the solutions about the equilibrium point with time delays values: 2.5, 5, 4.5. (a) Oscillatory behavior of $u_1(t)$, $u_3(t)$, $u_5(t)$ about the equilibrium point, delays: 2.5, 5, 4.5; (b) Oscillatory behavior of $u_2(t)$, $u_4(t)$, $u_6(t)$ about the equilibrium point, delays: 2.5, 5, 4.5.
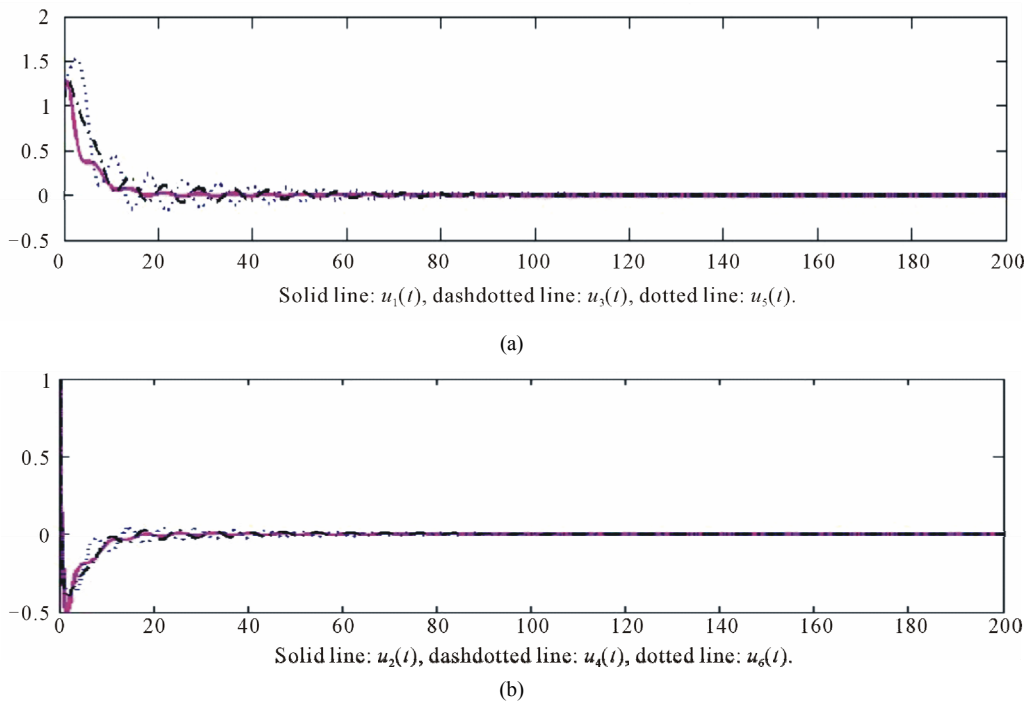

Figure 2. Convergence of the solutions about the equilibrium point with time delays values: 1.5, 1.7, 1.8. (a) Convergence of the $u_1(t)$, $u_3(t)$, $u_5(t)$ about the equilibrium point, delays: 1.5, 1.7, 1.8; (b) Convergence of the $u_2(t)$, $u_4(t)$, $u_6(t)$ about the equilibrium point, delays: 1.5, 1.7, 1.8.

In this case the characteristic roots of A are: $\alpha_1 = 0.0533$, $\alpha_2 = -1.2702$, $\alpha_3 = -0.1510$, $\alpha_4 = -2.6490$, $\alpha_5 = -0.6298$, $\alpha_6 = -3.7533$, and the characteristic roots of B are: $\beta_1 = -0.4268$, $\beta_2 = -0.7732$, $\beta_3 = 0$, $\beta_4 = 0$, $\beta_5 = 0$, $\beta_6 = 0$. There is a real number pair ($\alpha_1 = 0.0533$, $\beta_1 = -0.4268$). From Theorem 3, system (22) is oscillatory (**Figure 1**). However, it is worth emphasizing that this oscillation is due to time delays. In other words, delay induced oscillation of system (22) under the above parameter values. Indeed, the trivial solution of this system is stable if without time delays. One can see that the characteristic values of matrix $C = A + B$ when $\tau_i = 0$ are: $\gamma_1 = -3.7047$, $\gamma_2 = -2.5707$, $\gamma_3 = -0.2060$, $\gamma_4 = -0.8943$, $\gamma_5 = -1.1122 + 0.3801i$, $\gamma_6 = -1.1122 - 0.3801i$. Also $u_i^3(t)$ is a higher order infinitesimal as $u_i(t)$ tends to zero. So the unique equilibrium point of system $U'(t) = CU(t) + P(U(t))$ is stable according to the property of the solution of ordinary differential equation. Also the time delay must approach a certain quantity, for the system to generate oscillations. **Figure 2** indicates that the system is still stable when $\tau = 1.5$.

5. Conclusion

In this paper, we use the analysis method to discuss the oscillatory behavior of a nonsymmetric system. Simple criteria to guarantee the existence of oscillation of the system were proposed. We have discussed the effect of time delays in the system. They can induce oscillation. Computer simulations indicate the theory's accuracy.

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