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Numerical Approach of Network Problems in Optimal Mass Transportation

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ABSTRACT

In this paper, we focus on the theoretical and numerical aspects of network problems. For an illustration, we consider the urban traffic problems. And our effort is concentrated on the numerical questions to locate the optimal network in a given domain (for example a town). Mainly, our aim is to find the network so as the distance between the population position and the network is minimized. Another problem that we are interested is to give an numerical approach of the Monge and Kantorovitch problems. In the literature, many formulations (see for example [1-4]) have not yet practical applications which deal with the permutation of points. Let us mention interesting numerical works due to E. Oudet begun since at least in 2002. He used genetic algorithms to identify optimal network (see [5]). In this paper we introduce a new reformulation of the problem by introducing permutations σ. And some examples, based on realistic scenarios, are solved.

Keywords: Optimal Mass Transportation; Network; Urban Traffic; Monge-Kantorovich Problem; Global Optimization

1. Introduction

In this paper we present some models of urban planning. These models are examples of applications in mass transportation theory. They describe how to optimize the design of urban structures and their management under realistic assumptions. The paper is organized as follows: in Section 2 we present at first some urban planning models and preliminaries. The Section 3 is devoted to the approximation of the models; and numerical simulations that are our main results. Finally, summary and conclusions are presented in Section 4.

2. Preliminaries and Mathematical Modeling

Given two distributions μ and ν on $\mathbb{R}^d$ with equal total mass, the classical generic Monge transportation problem consists in finding among all the maps $T:\mathbb{R}^d \to \mathbb{R}^d$ verifying $\mu\left(T^{-1}(B)\right)=\nu(B)$ for any measurable set in $\mathbb{R}^d$, those which solve the minimization problem:

$$\min_T I(T),\ I(T)=\int_{\mathbb{R}^d} c\left(x,T(x)\right)\mathrm{d}\mu(x).$$

These maps are said to be transportation maps; they transport a measure μ (quantity) to a measure ν.

For the existence of solutions, we recommend to see [6-10]. We invite the reader to see the books written in this topic by Villani [11,12] for additional information. In particularly Sudakov have studied in [13] the existence of optimal map transportation when $c(x,y)=\|x-y\|$ and μ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^d$.

For studying the many cases where the Monge transportation problem doesn't give a solution, Kantorovich considered the relaxed version of the Monge problem. In this framework, the transportation problem consists in finding among all admissible measures $\gamma:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^+$ having μ and ν as marginals, those which solve the minimization problem

$$MK(\mu,\nu,c) = \min\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d} c(x,y)\mathrm{d}\gamma(x,y) : \pi_1^{\#}\gamma=\mu, \pi_2^{\#}\gamma=\nu\right\}.$$

The meaning of the following expressions $\pi_1^{\#}\gamma=\mu$ and $\pi_2^{\#}\gamma=\nu$ is explained respectively as follows: $\gamma\left[\mathbb{R}^d\times A\right]=\nu[A]$ and $\gamma\left[A\times\mathbb{R}^d\right]=\mu[A]$ for any measurable subset A of $\mathbb{R}^d$.

The Monge-Kantorovich problem obtained, depends only on the two distributions μ and ν, and the cost c which may be a function of the path connecting x to y.

When the unknowns of the problem are the distributions μ and ν, the Monge-Kantorovich mass transportation problemcan be interpreted as an optimal urban design problem. When the unknown is the transportation network, we have an irrigation problem, that is an optimal design of public transportation networks.

We also mention the dynamic formulation of mass transportation given in [4,14,15] and generalized in [16]. In this framework, we consider that:

- the space of measures acting is a time-space domain $Q=[0,1]\times\Omega$ where the urban area Ω is a bounded Lipschitz open subset with outward normal vector n_Ω;
- the mass density $\rho(t,x)$ at the position x and time t is a Borel measure supported on $\bar{Q}$, *i.e.* $\rho\in\mathcal{M}_b(\bar{Q},\mathbb{R}^+)$;
- the velocity field $v(t,x)$ of a particle at (t,x) is a Borel vectorial measures supported on $\bar{Q}$;
- the velocity field $\xi(t,x)$ of the flow at (t,x) is a Borel vectorial measures supported on $\bar{Q}$ $\left(i.e.\ \xi\in\mathcal{M}_b(\bar{Q},\mathbb{R}^d)\right)$ and defined by $\xi(t,x)=\rho(t,x)v(t,x)$.

The Monge-Kantorovich mass transportation problem consists in solving the following optimization problem:

$$\min\left\{\Psi(\rho,\xi):\rho\in\mathcal{M}_b(\bar{Q},\mathbb{R}^+),\ \xi\in\mathcal{M}_b(\bar{Q},\mathbb{R}^d)\right\} \quad (1)$$

with the constraints:

$$\begin{cases}-\partial_t\rho-div_x\xi=0 \text{ on } [0,1]\times\partial\bar{\Omega},\\ \rho(0,x)=\rho_0(x)\ \rho(1,x)=\rho_1(x),\\ \xi\cdot n_\Omega=0 \text{ on } [0,1]\times\partial\Omega.\end{cases} \quad (2)$$

where Ψ is an integral functional on the $\mathbb{R}^{1+d}$-valued measures defined on $\bar{Q}$. Note that (2) is the continuity equation of our mass transportation model.

2.1. Optimal Urban Design

In the models of optimal design of an urban area we considered that

- the urban area Ω is a well known regular compact subset of $\mathbb{R}^d$;
- the total population and the total production are fixed data of the problem;
- only the density of residents μ and the density of services ν are unknowns data of the problem.

The aim is to find the density of residents μ and the density of services ν minimizing the transportation cost.

Principally there are two models for studying the optimal urban design. The first one takes into account the following facts:

- there is a transportation cost for moving from the residential areas to the services poles;
- people do not desire to live in areas where the density of population is too high;
- services need to be concentrated as much as possible, in order to increase efficiency and decrease management costs.

The transportation cost will be described through a Monge-Kantorovich mass transportation model.

In particularly, we will take it as the p-Wasserstein distance defined by:

$$W_p(\mu,\nu)=\left[\min\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|^p\,\mathrm{d}\gamma(x,y):\right.\right.$$
$$\left.\left.\pi_1^{\#}\gamma=\mu,\pi_2^{\#}\gamma=\nu\right\}\right]^{\frac{1}{p}}.$$

Taking into account the total unhappiness of residents due to high density of population, we define a penalization functional of the form

$$H(\mu)=\begin{cases}\int_\Omega h(u(x))\mathrm{d}x & \text{if } \mu=u\mathrm{d}x\\ +\infty & \text{otherwise,}\end{cases}$$

where u is the density of the population, $h:[0,+\infty]\to[0,+\infty]$ is supposed to be convex, null at the origin and super linear (that is $\frac{h(t)}{t}\to+\infty$ as $t\to+\infty$). The increasing and diverging function $t\to\frac{h(t)}{t}$ represents the unhappiness to live in an area with population density t.

Thus we define a functional $G(\nu)$ which penalizes sparse services. This functional is of the form:

$$G(\nu)=\begin{cases}\sum_i g(a_i) & \text{if } \nu=\sum_i a_i\delta_{x_i}\\ +\infty & \text{otherwise,}\end{cases}$$

where $g:[0,+\infty]\to[0,+\infty]$ is supposed to be concave, null with infinite slope at the origin (*i.e.* $\frac{g(t)}{t}\to+\infty$ as $t\to0^+$). Every single term $g(a_i)$ in the sum above represents the cost for building and managing a service pole of size a_i located at the point x_i of Ω. The Report $P(a_i)=\frac{a_i}{gP(a_i)}$ is the productivity of pole of size a_i.

So in the first model, the optimal urban design problem becomes the following optimization problem:

$$\min\{W_p(\mu,\nu)+H(\mu)+G(\nu): \mu,\nu \text{ probabilities on } \Omega\} \quad (3)$$

In the second model the population transportation is considered as a flow, that is a vector field $\xi:\Omega\to\mathbb{R}^2$. The equilibrium condition is achieved when the emerging flow is the excess of the demand in K, *i.e.*

$$\int_{\partial K}\xi\cdot n_K\,\mathrm{d}\mathcal{H}^{n-1}=(\mu-\nu)(K).$$

In order to take into account the congestion effects, we suppose that the transportation cost $k(x)$ per resident at the point x depends on the traffic intensity at x, *i.e.* $k(x) = g(|\xi(x)|)$.

where $g:[0,+\infty] \to [0,+\infty]$ is an increasing function.

Then, the transportation cost moving μ to ν is:

$$C_g(\mu,\nu) = \inf\left\{\int_\Omega g(|\xi(x)|)|\xi(x)|\mathrm{d}x : \nabla\cdot\xi = \mu-\nu \text{ and } \xi\cdot n_\Omega = 0 \text{ on } \partial\Omega\right\}.$$

The problem (1) with the constraints (2) allows both to take into account the congestion effects by an appropriated choice of the functionals Ψ and to widen the choice of unhappiness function h and management cost function g of the first model to the local lower semi-continuous functionals on $\mathcal{M}_b(\bar{Q},\mathbb{R}^{d+1})$.

For more details, we refer the interested reader to the several recent papers on the subject (see for instance [4, 14-17]).

2.2. Network Problems Applied to the Urban Transportation

In the models of optimal design of an urban area we considered that

- the urban area Ω is a well known regular compact subset of $\mathbb{R}^d$;
- the density of residents μ and the density of services ν are two well known positives measures with equal mass.

The irrigation problem consists to find among all feasible structures (or feasible network) Σ those that minimize the transportation cost

$$I_\Sigma(\gamma) = \int_{\Omega\times\Omega} c(\mathrm{d}_\Sigma(x,y))\mathrm{d}\gamma(x,y);$$

The particular irrigation problem of the average distance consists to find an optimal network Σ_{opt} for which the average distance for a citizen to reach the mos $\mathbb{R}^d$ t nearby point of the network is minimal.

$$\min\left\{\int_\Omega dist(x,\Sigma)\mu(x)\mathrm{d}x : \sum\subset\Omega \text{ is a feasible network}\right\}.$$

Theorem 1 *For every* $L>0$ *there exists an optimal network* Σ_L *for the Optimization problem*

$$\min\left\{\int_\Omega dist(x,\Sigma)\, f(x)\mathrm{d}x : \sum \text{connected and } \mathcal{H}^1(\Sigma)\le L\right\}$$

where $\mathcal{H}^1$ is the Hausdorff measure defined on $\mathbb{R}^d$.

Notice that there are other proposition of functionals to be minimized.

For more details, we refer the interested reader to the several recent papers on the subject (see for instance [1-3, 18,19]).

Our aim is to concentrate our effort on the numerical questions to locate the optimal network in a given domain Ω, says for example a town.

3. Approximation and Numerical Simulations

3.1. First Steps for the Formulation of the Discrete Problem

In this subsection, we are going to propose a first approach of discretization. From this, we deduce a generalized formulation but not the own possible formulation.

Let

$\mathcal{A} = \left\{\text{applications on } \mathbb{R}^N \text{ measurable such that } T^\#\mu = \nu\right\}.$

For the simulation we are going to consider:

- $c(x,y) = \frac{1}{2}\|x-y\|^2$ and $\mathrm{supp}(\Omega)\subset\mathbb{R}^d$;
- $T^*\in\mathcal{A}$ such that $I(T^*) = \min I(T)$;
- a sequence of points $\{x_k\}_{k=1}^m \subset \mathrm{supp}(\mu)\subset\mathbb{R}^N$ and of balls $E_k = B(x_k,r_k)$ such that $\mu(E_1) = \cdots = \mu(E_m) = \varepsilon$, with $\mathrm{supp}(\mu) = \overline{\left\{x\in\mathbb{R}^N / \mu(x)\neq 0\right\}}$, and $x_k\in\mathbb{R}^N$.

and let $y_k = T^*(x_k)$ and $F_k = T^*(E_k)$. We build an other map $T\in\mathcal{A}$ switching round cyclically the images of balls $\{E_k\}_{k=1}^m$. Then T satisfies

$$\begin{cases} T(x_k) = y_{k+1}, T(E_k) = F_{k+1}, k = 1,\cdots,m \\ T = T^* \text{ on } \mathbb{R}^N\setminus\cup_{k=1}^m E_k \end{cases}$$

where $y_{m+1} = y_1$ and $F_{m+1} = F_1$.

Then we have $I(T^*)\le I(T)$, *i.e.*

$$\sum_{k=1}^m\int_{E_k}\left|x-T^*(x)\right|^2\mathrm{d}\mu(x) \le \sum_{k=1}^m\int_{E_k}\left|x-T(x)\right|^2\mathrm{d}\mu(x).$$

Therefore when $\varepsilon\to 0$, we obtain

$$\sum_{k=1}^m\int_{E_k}\left\langle x, T(x)-T^*(x)\right\rangle\mathrm{d}\mu(x)\le 0 \tag{4}$$

Suppose that the map T^* and the measure μ are regular, the Equation (4) leads

$$\sum_{k=1}^m\left\langle x_k, y_{k+1}-y_k\right\rangle\le 0 \text{ with } y_k = T^*(x_k) \tag{5}$$

Then $\left\{(x,T^*(x))/x\in\mathrm{supp}(\mu)\right\}\subset\mathbb{R}^N\times\mathbb{R}^N$ is cyclically monotonous and $T^*\subset\partial\varphi^*$.

At first, in problem ($\mathcal{P}_1$) the objective is to find the points y_k which minimize

$\min\sum_{k=1}^m\|x_k-y_k\|^2$ subject to

$$\begin{cases} \sum_{k=1}^{m-1}\left\langle x_k, y_{k+1}-y_k\right\rangle\le 0 \\ y_{m+1} = y_1 \\ \sum_{k=1}^{m-1}\|y_k-y_{k+1}\|\le L \end{cases}$$

In the next section, we show that it is quite possible to give a more general approximation .

3.2. New Reformulation Using Permutations

In the literature, many formulations (see for example [1-4]) have not yet practical applications which deal with the permutation of points. In this paper we introduce a new reformulation of the problem by introducing permutations σ .

Let us take a permutation σ defined on $\{1,\cdots,m\}$ such that $\sigma(k)\neq k$, we set:

$$\begin{cases} T(x_k)=y_{\sigma(k)}, T(E_k)=F_{\sigma(k)}, k=1,\cdots,m \\ T=T^* \text{ on } \mathbb{R}^N \setminus \cup_{k=1}^{m} E_k \end{cases}$$

and we solve the two following problems: ($\mathcal{P}_2$)

$$\min \sum_{k=1}^{m} \left\| x_k - y_k \right\|^2$$

subject to

$$\left\{ \sum_{k=1}^{m-1} \left\| y_{\sigma(k)} - y_{\sigma(k+1)} \right\| \leq L \right.$$

and problem ($\mathcal{P}_3$)

$$\min \sum_{k=1}^{m} \left\| x_k - y_{\sigma(k)} \right\|^2$$

subject to

$$\left\{ \sum_{k=1}^{m-1} \left\| y_k - y_{k+1} \right\| \leq L \right.$$

with the norm $\|\cdot\|_2$ and

$$\mathcal{S}_m = \left\{ \sigma : \{1,\cdots,m\} \to \{1,\cdots,m\} \text{ permutation} \right\}.$$

This is a theoretical formulation. And our aim is to apply it to a practical urban transport network. As a first step, we decided to work on $\mathbb{R}^2$ with a reasonable number of points.

For a scenario in $\mathbb{R}^n$, if we consider m points: the number of programs to be solved becomes m^m. We leave the reader to verify that for:

- $m = 3$ points $\Rightarrow$ we solve 27 programs
- $m = 4$ points $\Rightarrow$ we solve 256 programs
- etc.

A scenario involving up to 18 points is used for problem ($\mathcal{P}_1$). Using this scenario with problems ($\mathcal{P}_2$) and ($\mathcal{P}_3$) requires to solve 18^{18} programs, it is the reason we consider only some of these points for permutations in ($\mathcal{P}_2$) and ($\mathcal{P}_3$).

3.3. Numerical Experiments

This section shows how the three models developed in the two previous Sections 3.1 and 3.2 are applied to real data of Dakar Dem Dikk (3D). Recall that 3D (see [20], [21]) is the main public urban transportation company in Dakar. This company manages a fleet of buses with different technical characteristics. Some of the buses can operate only in certain roads in the city center and the others can access in all over the network. Buses are parked overnight at Ouakam and Thiaroye terminals (see **Figure 1**).

To ensure network coverage, 3D manages its services by using 17 lines, with 11 from Ouakam terminal and 6 from Thiaroye terminal. Each line ensures a certain num-

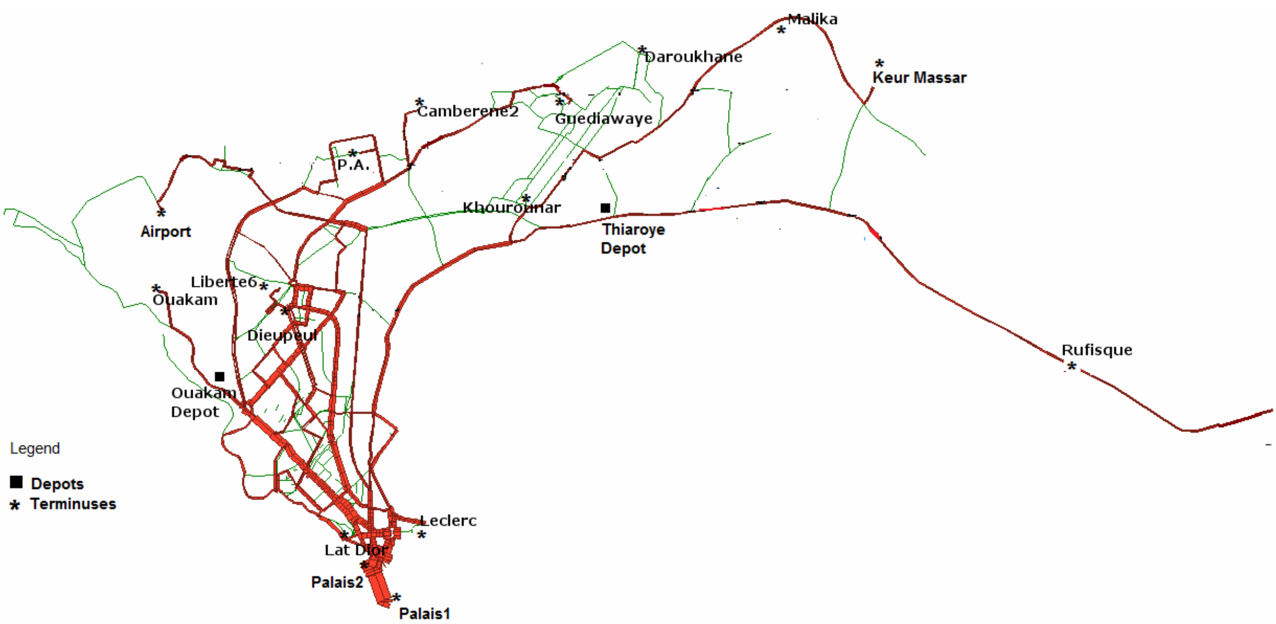

Figure 1. Urban transportation network of 3D.

ber of routes. At present, the total number of routes in the network is 289. First, the most important 18 sites of the network are identified. Thirty (30) permanent terminuses (terminals) and 810 bus stops are used (see **Figure 1**, where bus stops are not represented due to their size). The map in **Figure 1** is obtained by using the software EMME [22]. **Table 1** gives the 18 sites, their latitude and longitude.

The data are based on the scenario of 3D; and the input data needed to use the models are the:

- total length of the network $L = 5902.62$ kilometers;
- number of points x_k (terminals and bus stops) $m = 18$ (see **Figure 1**);
- latitude and longitude of points x_k representing the two terminals and bus stops.

The total distance covered by all the buses from terminals to starting points of routes and from end points back to their terminals represents the total length of the network; and we have L = 5902.62 kilometers (for the 18 sites).

The GPS (Global Positioning System) coordinates are calculated with Google map, and then transformed into coordinates on the plane with he formula: degree + (minute/60) + (second/3600). **Table 1** gives the coordinates of all points.

The numerical experiments are executed:

- on a computer: 2 × Intel(R) Core(TM)2 Duo CPU 2.00 GHz, 4.0 Gb of RAM, under UNIX system;
- and by the software IPOPT (Interior Point OPTimization) 3.9 stable [23,24], running with linear solver ma27.

For the objective function, we have:

$$\begin{aligned}\alpha &= \min \sum_{k=1}^{m} \left\| x_k - y_k \right\|^2 = \min \sum_{k=1}^{18} \left\| x_k - y_k \right\|^2 \\ &= \min \left(\left\| x_1 - y_1 \right\|^2 + \left\| x_2 - y_2 \right\|^2 + \cdots + \left\| x_{18} - y_{18} \right\|^2 \right) \\ &= \min \left(\alpha_1 + \alpha_2 + \cdots + \alpha_{18} \right)\end{aligned}$$

with

$$\alpha_i = \left\| x_i - y_i \right\|^2 = \left(x_i^1 - y_i^1 \right)^2 + \left(x_i^2 - y_i^2 \right)^2 \quad \forall i = 1, \cdots, 18.$$

Finally,

$$\alpha = \min \left[\sum_{i=1}^{18} \sum_{j=1}^{2} \left(y_i^j \right)^2 - 2 \sum_{i=1}^{18} \sum_{j=1}^{2} x_i^j y_i^j \right]$$

Table 1. Scenario of 3D.

	GPS coordinates		Coordinates on the plane	
x_k	**latitude**	**longitude**	**latitude**	**longitude**
Ouakam terminal	14˚42'25.79"N	17˚28'43.52"O	14.7071638888889	17.4787555555556
Thiaroye terminal	14˚44'48.49"N	17˚22'48.41"O	14.7468027777778	17.3801138888889
Camberene 2	14˚43'54.80"N	17˚26'38.99"O	14.7318888888889	17.4441638888889
PA	14˚45'37.04"N	17˚26'16.79"O	14.7602888888889	17.4379972222222
Keur Massar	14˚47'7.72"N	17˚18'33.33"O	14.7854777777778	17.3092583333333
Lat Dior	14˚40'8.59"N	17˚26'28.51"O	14.6690527777778	17.4412527777778
Palais2	14˚39'10.63"N	17˚25'59.86"O	14.6529527777778	17.4332944444444
Guediawaye	14˚46'21.40"N	17˚23'20.18"O	14.7726111111111	17.3889388888889
Daroukhane	14˚46'53.08"N	17˚22'19.47"O	14.7814111111111	17.3720750000000
Dieuppeul	14˚43'23.54"N	17˚27'31.21"O	14.7232055555556	17.4586694444444
Leclerc	14˚40'16.54"N	17˚25'53.06"O	14.6712611111111	17.4314055555556
Liberte 6	14˚43'42.11"N	17˚27'36.32"O	14.7283638888889	17.4600888888889
Aeroport	14˚44'44.02"N	17˚29'21.65"O	14.7455611111111	17.4893472222222
Khourounar	14˚44'59.60"N	17˚24'25.10"O	14.7498888888889	17.4069722222222
Palais1	14˚39'10.63"N	17˚25'59.86"O	14.6529527777778	17.4332944444444
Malika	14˚47'37.78"N	17˚20'11.42"O	14.7938277777778	17.3365055555556
Rufisque	14˚42'44.30"N	17˚16'1.66"O	14.7123055555556	17.2671277777778
Ouakam	14˚43'56.98"N	17˚29'35.02"O	14.7324944444444	17.4930611111111

and $\beta = \sum_{i=1}^{18}\sum_{j=1}^{2}\left(x_i^j\right)^2$ is added to the value of the objective function.

Constraint $\sum_{k=1}^{m-1}\left\langle x_k, y_{k+1} - y_k\right\rangle \le 0$ gives

$$\sum_{k=1}^{2}\left\langle x_k, y_{k+1} - y_k\right\rangle = \left\langle x_1, y_2 - y_1\right\rangle + \left\langle x_2, y_3 - y_2\right\rangle \le 0 .$$

Urban transportation network of 3D

Thus, $\left\langle -x_1, y_1\right\rangle + \left\langle x_1 - x_2, y_2\right\rangle + \left\langle x_2, y_3\right\rangle \le 0$, *i.e.*:

$$\begin{aligned}&-x_1^1 y_1^1 - x_1^2 y_1^2 + \left(x_1^1 - x_2^1\right) y_2^1 \\ &+ \left(x_1^2 - x_2^2\right) y_2^2 + x_2^1 y_3^1 + x_2^2 y_3^2 \le 0.\end{aligned}$$

And $\sum_{k=1}^{m-1}\left\| y_k - y_{k+1}\right\| \le L$ gives

$$\sum_{k=1}^{2}\left\| y_k - y_{k+1}\right\| = \left\| y_1 - y_2\right\| + \left\| y_2 - y_3\right\| \le L ,$$

i.e.:

$$\sqrt{\left(y_1^1 - y_2^1\right)^2 + \left(y_1^2 - y_2^2\right)^2} + \sqrt{\left(y_2^1 - y_3^1\right)^2 + \left(y_2^2 - y_3^2\right)^2} \le L$$

with $y_4^1 = y_1^1$ and $y_4^2 = y_1^2$.

We simply formulate the problem in AMPL [25] syntax, and solve the problem through the AMPL environment; with a total number of 38 variables for problem $\left(\mathcal{P}_1\right)$. The solution obtained is an optimal one (for each case) wherein the priority is assigned to the minimization of the distance between x_k and y_k. The IPOPT found an optimal point within desired tolerances; and we obtain the following results: the total number of iterations is 16 and for the optimal network Σ_{opt} we have $y_k = x_k \forall k \in \{2, \cdots, m-1\}$. The others obtained solutions y_1, y_m and y_{m+1} are given in **Table 2** with the GPS coordinates. The points y_1 and y_m are represented in the network (see **Figure 2**).

Permutations make the resolution more complicated but can give better results. The number of sub-problems to solve depends on the number of points in the network. Thus, we choose $m = 3$ points in 3D's network.

Now, let us take the permutation which is the main idea of this work. For problem $\left(\mathcal{P}_2\right)$, $\forall i = 1,2,3$ such that $\sigma(i) = \sigma(i+1)$, we have $\left\| y_{\sigma(i)} - y_{\sigma(i+1)}\right\| = 0$. Therefore some constraints are similar and we have 9 sub-problems to solve. An illustration:

$$\begin{aligned}\sum_{k=1}^{m-1}\left\| y_{\sigma(k)} - y_{\sigma(k+1)}\right\| &= \sum_{k=1}^{2}\left\| y_{\sigma(k)} - y_{\sigma(k+1)}\right\| \\ &= \left\| y_{\sigma(1)} - y_{\sigma(2)}\right\| + \left\| y_{\sigma(2)} - y_{\sigma(3)}\right\|\end{aligned}$$

The three unconstrained sub-problems are obtained for $(\sigma(1), \sigma(2), \sigma(3))$ with $\sigma(1) = \sigma(2) = \sigma(3)$, *i.e.* in the three permutations (1,1,1), (2,2,2) and (3,3,3).

All sub-problems constraints are reported in **Table 3**.

Table 2. y_1 and y_m for optimal network Σ_{opt}.

	GPS coordinates	
y_k	**latitude**	**longitude**
y_1	14˚42'52.20"N	17˚29'15.00"O
y_m	14˚43'30.36"N	17˚29'3.84"O
$y_{m+1} = y_1$	14˚42'52.20"N	17˚29'15.00"O

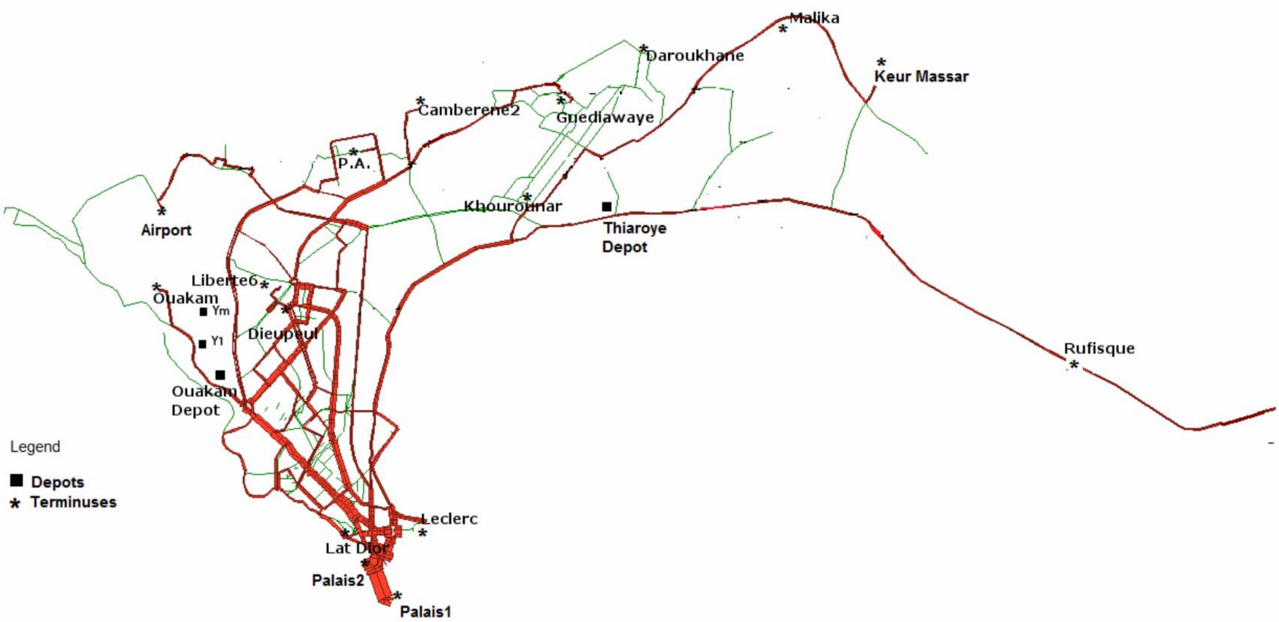

Figure 2. Optimal network of 3D without permutation.

Table 3. All constraints with permutations.

P_i	$(\sigma(1),\sigma(2),\sigma(3))$	**constraints**
1	(1,1,1)	
2	**(1,1,2)**	$\|y_1-y_2\|$
3	**(1,1,3)**	$\|y_1-y_3\|$
4	**(1,2,1)**	$2\|y_1-y_2\|$
5	(1,2,2)	$\|y_1-y_2\|$
6	**(1,2,3)**	$\|y_1-y_2\|+\|y_2-y_3\|$
7	**(1,3,1)**	$2\|y_1-y_3\|$
8	**(1,3,2)**	$\|y_1-y_3\|+\|y_2-y_3\|$
9	(1,3,3)	$\|y_1-y_3\|$
10	(2,1,1)	$\|y_1-y_2\|$
11	(2,1,2)	$2\|y_1-y_2\|$
12	**(2,1,3)**	$\|y_1-y_2\|+\|y_2-y_3\|$
13	(2,2,1)	$\|y_1-y_2\|$
14	(2,2,2)	
15	**(2,2,3)**	$\|y_2-y_3\|$
16	(2,3,1)	$\|y_1-y_3\|+\|y_2-y_3\|$
17	**(2,3,2)**	$2\|y_2-y_3\|$
18	(2,3,3)	$\|y_2-y_3\|$
19	(3,1,1)	$\|y_1-y_3\|$
20	(3,1,2)	$\|y_1-y_2\|+\|y_1-y_3\|$
21	(3,1,3)	$2\|y_1-y_3\|$
22	(3,2,1)	$\|y_1-y_2\|+\|y_2-y_3\|$
23	(3,2,2)	$\|y_2-y_3\|$
24	(3,2,3)	$2\|y_2-y_3\|$
25	(3,3,1)	$\|y_1-y_3\|$
26	(3,3,2)	$\|y_2-y_3\|$
27	(3,3,3)	

It is sufficient to solve $P_1,P_2,P_3,P_4,P_5,P_6,P_7,P_{11},P_{14}$ and P_{16}; since we have $P_{27}=P_{14}=P_1$, $P_{13}=P_{10}=P_5=P_2$, $P_{25}=P_{19}=P_9=P_3$, $P_{11}=P_4$, $P_{22}=P_6$, $P_{21}=P_7$, $P_{16}=P_8$, $P_{20}=P_{12}$, $P_{24}=P_{17}$ and $P_{26}=P_{23}=P_{18}=P_{15}$.

We only compute the quantities $\|y_1-y_2\|$, $\|y_1-y_3\|$ and $\|y_2-y_3\|$. For all sub-problems $P_i\,(i=1,\cdots,27)$, the objective function is the same as problem $(\mathcal{P}_1)$ with $m=3$.

$$\alpha=\min\sum_{k=1}^{m}\|x_k-y_k\|^2=\min\sum_{k=1}^{3}\|x_k-y_k\|^2$$
$$=\min\left(\|x_1-y_1\|^2+\|x_2-y_2\|^2+\|x_3-y_3\|^2\right)$$
$$=\min\left(\alpha_1+\alpha_2+\alpha_3\right)$$

with

$$\alpha_1=\left(x_1^1-y_1^1\right)^2+\left(x_1^2-y_1^2\right)^2,$$
$$\alpha_2=\left(x_2^1-y_2^1\right)^2+\left(x_2^2-y_2^2\right)^2$$

and

$$\alpha_3=\left(x_3^1-y_3^1\right)^2+\left(x_3^2-y_3^2\right)^2.$$

Finally,

$$\alpha=\min\Big[\left(y_1^1\right)^2+\left(y_1^2\right)^2+\left(y_2^1\right)^2+\left(y_2^2\right)^2+\left(y_3^1\right)^2+\left(y_3^2\right)^2$$
$$-2x_1^1y_1^1-2x_1^2y_1^2-2x_2^1y_2^1-2x_2^2y_2^2-2x_3^1y_3^1-2x_3^2y_3^2\Big]$$

and $\beta=\sum_{i=1}^{3}\sum_{j=1}^{2}\left(x_i^j\right)^2$ is added to the value of the objective function.

From computational results, we have obtained the same value for all 10 sub-problems. Thus, permutations do not influence the distance constraint on the curve of Σ_k. The optimal value is $\alpha=1170.332935387$, for all P_i with $i=1,\cdots,27$.

For problem $(\mathcal{P}_3)$, we consider $\sigma(k)$ and the number of possible permutation is $\sigma(k)\in\{1,2,3\}$. Recall that the number of sub-problems to solve depends on the number of points in the network. Also, we choose $m=3$ points in 3D's network. Thus, for the considered permutation $(\sigma(1),\sigma(2),\sigma(3))$ we have obtained a total of $3^3=27$ sub-problems $P_i\,(i=1,\cdots,27)$ to solve, see **Table 4**.

In order not to overload explanations, we develop only the sub-problem P_1, the rest are left to the reader as an exercise.

In $\mathbb{R}^2$, the sub-problem P_1 is obtained for

$$(\sigma(1),\sigma(2),\sigma(3))=(1,1,1),$$

with vectors $x_i=\left(x_i^1,x_i^2\right)$ and $y_i=\left(y_i^1,y_i^2\right)$ $\forall i=1,2,3$. For the objective function, we have

$$\alpha_1=\min\sum_{k=1}^{m}\|x_k-y_{\sigma(k)}\|^2=\min\sum_{k=1}^{3}\|x_k-y_{\sigma(k)}\|^2$$
$$=\min\left(\|x_1-y_{\sigma(1)}\|^2+\|x_2-y_{\sigma(2)}\|^2+\|x_3-y_{\sigma(3)}\|^2\right)$$

$$\alpha_1=\min\left(\|x_1-y_1\|^2+\|x_2-y_1\|^2+\|x_3-y_1\|^2\right)$$
$$=\min\left(\alpha_1^1+\alpha_2^1+\alpha_3^1\right)$$

Table 4. Optimal values of α_i with different permutations.

P_i	$(\sigma(1),\sigma(2),\sigma(3))$	α_i
1	(1,1,1)	0.007723332000
2	(1,1,2)	0.005650763000
3	(1,1,3)	0.005650763000
4	(1,2,1)	0.001765520000
5	(1,2,2)	0.004168715000
6	**(1,2,3)**	**0.000000000000**
7	(1,3,1)	0.438482963000
8	**(1,3,2)**	**0.000000000000**
9	(1,3,3)	0.004168715000
10	(2,1,1)	0.004168715000
11	(2,1,2)	0.001765520000
12	**(2,1,3)**	**0.000000000000**
13	(2,2,1)	0.005650763000
14	(2,2,2)	0.007723332000
15	(2,2,3)	0.005650763000
16	**(2,3,1)**	**0.000000000000**
17	(2,3,2)	0.001765520000
18	(2,3,3)	0.004168715000
19	(3,1,1)	0.004168715000
20	**(3,1,2)**	**0.000000000000**
21	(3,1,3)	0.001765520000
22	**(3,2,1)**	**0.000000000000**
23	(3,2,2)	0.004168715000
24	(3,2,3)	0.001765520000
25	(3,3,1)	0.005650763000
26	(3,3,2)	0.005650763000
27	(3,3,3)	0.007723332000

with

$$\alpha_1^1 = \left(x_1^1 - y_1^1\right)^2 + \left(x_1^2 - y_1^2\right)^2,$$

$$\alpha_2^1 = \left(x_2^1 - y_1^1\right)^2 + \left(x_2^2 - y_1^2\right)^2$$

and

$$\alpha_3^1 = \left(x_3^1 - y_1^1\right)^2 + \left(x_3^2 - y_1^2\right)^2$$

Finally,

$$\alpha_1 = \min\Big[3\left(y_1^1\right)^2 + 3\left(y_1^2\right)^2 - 2\left(x_1^1 + x_2^1 + x_3^1\right)y_1^1 - 2\left(x_1^2 + x_2^2 + x_3^2\right)y_1^2\Big]$$

and $\beta = \sum_{i=1}^{3}\sum_{j=1}^{2}\left(x_i^j\right)^2$ is added to the value of the objective function for all sub-problems $P_i, \forall i = 1,\cdots,27$.

The constraint

$$\sum_{k=1}^{m-1}\left\|y_k - y_{k+1}\right\| \le L$$

gives

$$\sum_{k=1}^{2}\left\|y_k - y_{k+1}\right\| = \left\|y_1 - y_2\right\| + \left\|y_2 - y_3\right\| \le L,$$

i.e.:

$$\sqrt{\left(y_1^1 - y_2^1\right)^2 + \left(y_1^2 - y_2^2\right)^2} + \sqrt{\left(y_2^1 - y_3^1\right)^2 + \left(y_2^2 - y_3^2\right)^2} \le L$$

For the scenario of problems $(\mathcal{P}_2)$ and $(\mathcal{P}_3)$, we choose x_1 = Ouakam terminal, x_2 = Thiaroye terminal and x_3 = Leclerc (see **Table 5**).

We denote by α_i and $y_{k,i}^*$ the optimal value and the optimal solution of sub-problem P_i $\forall k = 1,2,3$, respectively.

The results show that the following six sub-problems: P_6, P_8, P_{12}, P_{16}, P_{20}, P_{22} give the best value $\left(\alpha = \alpha_i = 0.0, \forall i = 6,8,12,16,20 \text{ and } 22\right)$. The six solutions are different, *i.e.*:

$$y_{k,6}^* \ne y_{k,8}^* \ne y_{k,12}^* \ne y_{k,16}^* \ne y_{k,20}^* \ne y_{k,22}^*,$$

with only $y_{k,6}^* = x_k$.

The curve Σ_{opt} can be described by one of the points $y_{k,i}^* \in T(x_k)$; see **Figure 3** where

$$\sigma = \left(\sigma(1),\sigma(2),\sigma(3)\right).$$

The points $y_{k,i}^*$ which define Σ_{opt} (with coordinates $y_{k,i}^{*j}$ $\forall j = 1,2$) are given in **Table 6**, with $y_{k,6} = x_k$.

The solutions giving the best possible permutations (optimum) are illustrated in **Table 6** and include all permutations $\sigma(i) \ne \sigma(j)$ $\forall i, j$.

According to the simulations, we determine a set of optimal policy that can describe the optimal network Σ_{opt}. Finally: after comparison of the simulated models, we can deduce that the model for problem $(\mathcal{P}_3)$ is better. It provides the best curve describing the optimal value Σ_{opt}, obtained with the permutations introduced in the objective function.

4. Conclusions

In this paper, we describe applications of mass transportation theory and develop how to optimize the curve design of urban network problems. Using the discrete formulations, we give three nonlinear programming problems with continue variables, and have described urban transportation problem of 3D applied to these three models. The results have shown that the optimal network is obtained with permutations including $\sigma(i) \ne \sigma(j)\ \forall i, j$.

In future works, we will study an application in $\mathbb{R}^3$ and make a reformulation that solves a unique program,

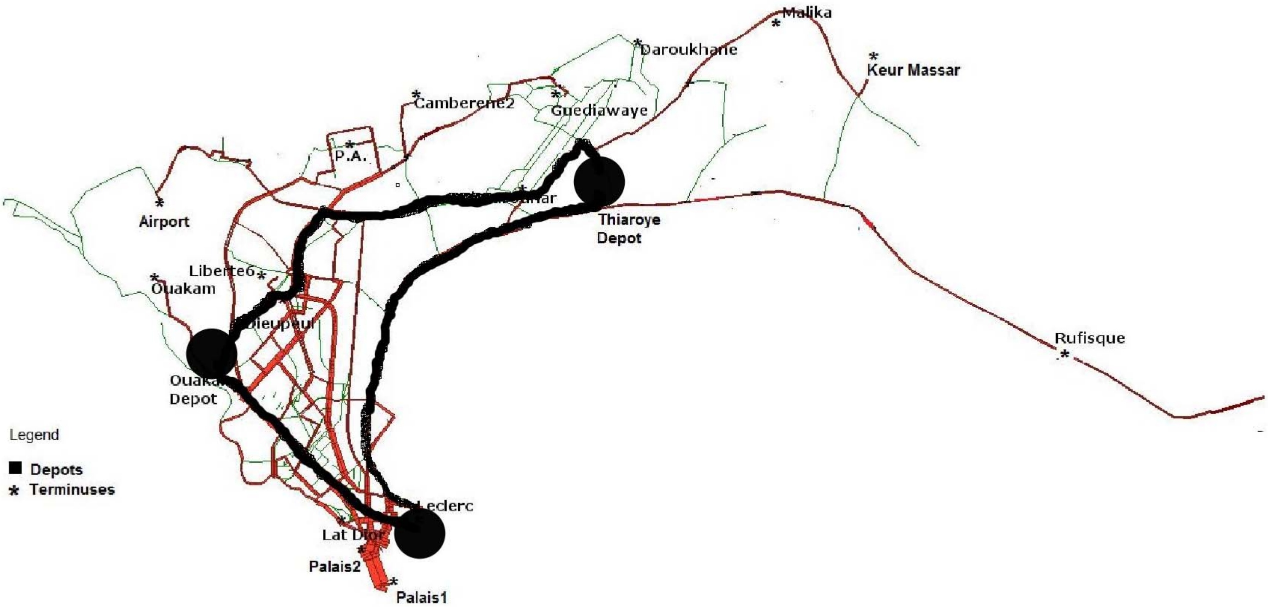

Figure 3. Optimal network Σ_{opt}.

Table 5. The network Σ_{opt} with permutations.

	GPS coordinates	
x_k	**latitude**	**longitude**
x_1	14˚42'25.79"N	17˚28'43.52"O
x_2	14˚44'48.49"N	17˚22'48.41"O
x_3	14˚40'16.54"N	17˚25'53.06"O

Table 6. The optimal solutions $y^*_{k,i}$ with permutations.

	GPS coordinates		σ
y_k	**latitude** (j = 1)	**longitude** (j = 2)	
$y_{1,6}$	14˚42'25.79"N	17˚28'43.52"O	
$y_{2,6}$	14˚44'48.49"N	17˚22'48.41"O	(1,2,3)
$y_{3,6}$	14˚40'16.54"N	17˚25'53.06"O	
$y_{1,8}$	14˚42'25.79"N	17˚28'43.52"O	
$y_{2,8}$	14˚40'16.54"N	17˚25'53.06"O	(1,3,2)
$y_{3,8}$	14˚44'48.49"N	17˚22'48.41"O	
$y_{1,12}$	14˚44'48.49"N	17˚22'48.41"O	
$y_{2,12}$	14˚42'25.79"N	17˚28'43.52"O	(2,1,3)
$y_{3,12}$	14˚40'16.54"N	17˚25'53.06"O	
$y_{1,16}$	14˚40'16.54"N	17˚25'53.06"O	
$y_{2,16}$	14˚42'25.79"N	17˚28'43.52"O	(2,3,1)
$y_{3,16}$	14˚44'48.49"N	17˚22'48.41"O	
$y_{1,20}$	14˚44'48.49"N	17˚22'48.41"O	
$y_{2,20}$	14˚40'16.54"N	17˚25'53.06"O	(3,1,2)
$y_{3,20}$	14˚42'25.79"N	17˚28'43.52"O	
$y_{1,22}$	14˚40'16.54"N	17˚25'53.06"O	
$y_{2,22}$	14˚44'48.49"N	17˚22'48.41"O	(3,2,1)
$y_{3,22}$	14˚42'25.79"N	17˚28'43.52"O	

instead of solving m^m problems.

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