Applied Mathematics, 2012, 3, 325-330
http://dx.doi.org/10.4236/am.2012.34049 Published Online April 2012 (http://www.SciRP.org/journal/am)

Blow-Up Phenomena for a Class of Parabolic Systems with Time Dependent Coefficients

Lawrence E. Payne[1], **Gérard A. Philippin**[2]
[1]Department of Mathematics, Cornell University, Ithaca, USA
[2]Département de Mathématiques et de Statistique, Université Laval, Québec City, Canada
Email: gphilip@mat.ulaval.ca

Received February 8, 2012; revised March 7, 2012; accepted March 14, 2012

ABSTRACT

Blow-up phenomena for solutions of some nonlinear parabolic systems with time dependent coefficients are investigated. Both lower and upper bounds for the blow-up time are derived when blow-up occurs.

Keywords: Parabolic Systems; Blow-Up; Sobolev Type Inequality

1. Introduction

It is well known that the solutions of parabolic problems may remain bounded for all time, or may blow-up in finite or infinite time. When blow-up occurs at time t^*, the evaluation of t^* is of great practical interest.

In a recent paper [1] Payne and Schaefer have investigated the blow-up phenomena of solutions in some parabolic systems of equations under homogeneous Dirichlet boundary conditions. The contribution of this note is to extend their investigations to a class of parabolic systems with time dependent coefficients. The case of a single parabolic equation was investigated recently in [2].

There is an abounding literature dealing with blow-up phenomena of solutions to parabolic partial differential equations. We refer the interested readers to [3-5]. A variety of physical, chemical, biological applications are discussed in [5,6]. Further references to the field are [1,7-19]. In this note we investigate the blow-up phenomena of the solution (u,v) of the following parabolic system

$$\begin{cases} u_t = \Delta u + k_1(t) f_1(v), x = (x_1,\cdots,x_N) \in \Omega, t \in (0,t^*) \\ v_t = \Delta v + k_2(t) f_2(u), x \in \Omega, t \in (0,t^*), \\ u(x,t) = v(x,t) = 0, \ x \in \partial\Omega, t \in (0,t^*), \\ u(x,0) = u_0(x) \ge 0, \ v(x,0) = v_0(x) \ge 0, x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N, N \ge 2$. The initial data (u_0,v_0) as well as the data $k_1(t), k_2(t)$, $f_1(t), f_2(t)$ are assumed nonnegative, so that the solution (u,v) of (1.1) will be nonnegative by the maximum principle. More specific assumptions on the data will be made later.

In Section 2 we derive conditions on the data of problem (1.1) sufficient to guarantee that blow-up will occur, and derive under these conditions some upper bound for t^*. In Section 3 we derive some lower bounds for the blow-up time t^* when blow-up occurs. However this section is limited to the case of Ω in $\mathbb{R}^2$ and in $\mathbb{R}^3$ respectively, because our technique makes use of some Sobolev type inequalities available in $\mathbb{R}^2$ and in $\mathbb{R}^3$ only. For convenience we include the proof of one of these inequalities in Section 4.

2. Conditions for Blow-Up in Finite Time t^*

Let λ_1 be the first eigenvalue and ϕ_1 be the associated eigenfunction of the Dirichlet-Laplace operator defined as

$$\Delta\phi_1 + \lambda_1\phi_1 = 0, \phi_1 > 0, x \in \Omega; \ \phi_1 = 0, x \in \partial\Omega, \tag{2.1}$$

$$\int_\Omega \phi_1 \mathrm{d}x = 1. \tag{2.2}$$

Let the auxiliary function $\theta(t)$ be defined in $(0,t^*)$ as

$$\theta(t) := \psi(t) + \chi(t), \tag{2.3}$$

with

$$\psi(t) := \int_\Omega u\phi_1 \mathrm{d}x, \quad \chi(t) := \int_\Omega v\phi_1 \mathrm{d}x, \tag{2.4}$$

where (u,v) is the solution of problem (1.1). We assume in this section that Ω is a bounded domain of $\mathbb{R}^N, N \ge 2$, and that

$$\begin{aligned} & f_1(s) \ge s^p, p = \text{constant} > 1, \\ & f_2(s) \ge s^q, q = \text{constant} > 1, s > 0, \end{aligned} \tag{2.5}$$

$$\min_{t>0}\{k_1(t), k_2(t)\} =: K > 0. \tag{2.6}$$

We then compute

$$\psi'(t) = \int_\Omega \left[\Delta u + k_1 f_1(v)\right]\phi_1 \mathrm{d}x \\ \ge -\lambda_1 \psi(t) + k_1(t)\int_\Omega v^p \phi_1 \mathrm{d}x \tag{2.7}$$

Making use of Hölder's inequality, we have

$$\chi(t) = \int_\Omega v\phi_1 \mathrm{d}x \le \left(\int_\Omega v^p \phi_1 \mathrm{d}x\right)^{\frac{1}{p}}. \tag{2.8}$$

Combining (2.7) and (2.8), we obtain

$$\psi'(t) \ge -\lambda_1 \psi(t) + k_1(t)\left(\chi(t)\right)^p. \tag{2.9}$$

A similar computation leads to

$$\chi'(t) \ge -\lambda_1 \chi(t) + k_2(t)\left(\psi(t)\right)^q. \tag{2.10}$$

Adding (2.9) and (2.10), we obtain

$$\theta'(t) = \psi'(t) + \chi'(t) \ge -\lambda_1 \theta(t) + K\left(\psi^q + \chi^p\right), \tag{2.11}$$

where K is defined in (2.6). We first investigate the particular case $p = q$. Making use of Hölder's inequality, we have

$$\psi^q + \chi^q \ge 2^{1-q}\left(\psi + \chi\right)^q = 2^{1-q}\left(\theta(t)\right)^q. \tag{2.12}$$

Inserted in (2.11), we obtain the first order differential inequality

$$\theta'(t) \ge -\lambda_1 \theta + 2^{1-q} K \theta^q, t \in \left(0, t^*\right). \tag{2.13}$$

Integrating (2.13) from 0 to t, we obtain the inequality

$$\left(\theta(t)\right)^{1-q} \le e^{(q-1)\lambda_1 t}\left\{\left(\theta(0)\right)^{1-q} - \frac{2^{1-q}K}{\lambda_1}\right\} + \frac{2^{1-q}K}{\lambda_1} \\ =: \varepsilon(t). \tag{2.14}$$

Suppose that the data satisfy the condition

$$\theta(0) > 2\left(\frac{\lambda_1}{K}\right)^{1/(q-1)}. \tag{2.15}$$

Then $\varepsilon(t)$ vanishes at some time $t_0 > 0$, and $\theta(t)$ must blow up at some time $t^* \le t_0$. We obtain

$$t^* \le t_0 := -\frac{1}{(q-1)\lambda_1}\log\left\{1 - \frac{2^{q-1}\lambda_1}{K\left(\theta(0)\right)^{q-1}}\right\}. \tag{2.16}$$

In the general case, we suppose without loss of generality that $p > q$, and make use of the inequality

$$\chi^q = \left(c\chi^p\right)^{\frac{q}{p}}\left(c^{-\frac{q}{p-q}}\right)^{\frac{p-q}{p}} \le \frac{q}{p}c\chi^p + \frac{p-q}{p}c^{-\frac{q}{p-q}}, \tag{2.17}$$

valid for arbitrary $c > 0$. Choosing $c := \frac{p}{q}$, we obtain

$$\chi^q \le \chi^p + Q, \tag{2.18}$$

with

$$Q := \frac{p-q}{p}\left(\frac{q}{p}\right)^{\frac{q}{p-q}} > 0. \tag{2.19}$$

Inserted in (2.12), we obtain the first order differential inequality

$$\theta'(t) \ge 2^{1-q}K\theta^q - \lambda_1\theta - KQ =: \Theta(\theta). \tag{2.20}$$

Suppose that the initial data are so large that $\Theta\left(\theta(0)\right) > 0$. Then $\theta(t)$ is increasing for t small. Since $\Theta(\theta)$ is increasing in θ from its negative minimum, it follows then that $\Theta\left(\theta(t)\right)$ is increasing for $t > 0$. This shows that $\theta'(t)$ remains positive, so that $\theta(t)$ blows up at time t^*. Integrating (2.20) leads to the following upper bound for t^*

$$t^* = \int_0^{t^*}\mathrm{d}t \le \int_{\theta(0)}^{\infty}\frac{\mathrm{d}\theta}{\Theta(\theta)}. \tag{2.21}$$

These results are summarized in the following.

Theorem 1

1) *Assume* (2.5) *with* $p = q > 1$, (2.6), *and* (2.15). *Then* $\theta(t)$ *defined in* (2.3) *blows up at finite time* t^* *bounded above by* (2.16).

2) *Assume* (2.5) *with* $p > q > 1$, (2.6), *and* $\Theta\left(\theta(0)\right) > 0$ *with* $\Theta(\theta)$ *defined in* (2.20). *Then* $\theta(t)$ *blows up at finite time* t^* *bounded above by* (2.21).

To conclude this section, we note that if the condition (2.6) is replaced by

$$\min_{t>\tau}\left\{k_1(t), k_2(t)\right\} =: K > 0, \tag{2.22}$$

then we have to replace the initial data $\theta(0)$ by $\theta(\tau)$ in Theorem 1. Clearly we may use a lower bound for $\theta(\tau)$. For instance we may integrate the differential inequality

$$\theta' \ge -\lambda_1\theta \tag{2.23}$$

that follows from (2.11), leading to the lower bound

$$\theta(\tau) \ge e^{-\lambda_1\tau}\theta(0). \tag{2.24}$$

3. Lower Bounds for t^*

In this section we assume that the data f_1, f_2, satisfy the conditions

$$0 \le f_1(s) \le s^p, p > 1; 0 \le f_2(s) \le s^q, q > 1, s > 0, \tag{3.1}$$

and that the data $k_1(t), k_2(t)$ are nonnegative for all $t > 0$. Moreover the solution is assumed to blow up in the sense that $\Phi(t) \to \infty$ as $t \to t^*$, where $\Phi(t)$ is defined as

$$\Phi(t) := M_1^{-1}U(t) + M_2^{-1}V(t), \tag{3.2}$$

with

$$U(t) := \int_\Omega u^{2q} \mathrm{d}x, \quad M_1 := \int_\Omega u_0^{2q} \mathrm{d}x, \tag{3.3}$$

$$V(t) := \int_\Omega v^{2p} \mathrm{d}x, \quad M_2 := \int_\Omega v_0^{2p} \mathrm{d}x. \tag{3.4}$$

Differentiating (3.3) and making use of (1.1), (3.1), we obtain

$$\begin{aligned} U'(t) &\le 2q\int_\Omega u^{2q-1}\left[\Delta u + k_1(t) v^p\right]\mathrm{d}x \\ &= 2qk_1(t)\int_\Omega u^{2q-1}v^p \mathrm{d}x - 2q(2q-1)J(t), \end{aligned} \tag{3.5}$$

with

$$J(t) := \int_\Omega u^{2(q-1)} |\nabla u|^2 \,\mathrm{d}x. \tag{3.6}$$

Making use of Schwarz and Hölder's inequalities we have

$$\begin{aligned} \int_\Omega u^{2q-1}v^p \mathrm{d}x &\le \left(\int_\Omega u^{2(2q-1)}\mathrm{d}x \int_\Omega v^{2p}\mathrm{d}x\right)^{1/2} \\ &\le \left(\int_\Omega u^{4q}\mathrm{d}x\right)^{\frac{q-1}{2q}} \left(\int_\Omega u^{2q}\mathrm{d}x\right)^{1/2q} \left(\int_\Omega v^{2p}\mathrm{d}x\right)^{1/2}. \end{aligned} \tag{3.7}$$

In $\mathbb{R}^2$ we make use of the following Sobolev type inequality

$$\int_\Omega u^{4q}\mathrm{d}x \le \frac{q^2}{2}\int_\Omega u^{2(q-1)}|\nabla u|^2 \,\mathrm{d}x \int_\Omega u^{2q}\mathrm{d}x, \tag{3.8}$$

derived in the last section of the paper. Combining (3.7) and (3.8), we obtain

$$\begin{aligned} &\int_\Omega u^{2q-1}v^p \mathrm{d}x \\ &\le \left(\frac{q^2}{2}\right)^{\frac{q-1}{2q}} \left(J(t)\right)^{\frac{q-1}{2q}} \left(\int_\Omega u^{2q}\mathrm{d}x\right)^{1/2} \left(\int_\Omega v^{2p}\mathrm{d}x\right)^{1/2} \\ &\le \frac{1}{2}\left(\frac{q^2}{2}\right)^{\frac{q-1}{2q}} \left(J(t)\right)^{\frac{q-1}{2q}} M_1^{1/2}M_2^{1/2}\Phi(t), \end{aligned} \tag{3.9}$$

where we have used the arithmetic-geometric mean inequality. Making use of the inequality

$$\begin{aligned} &a^r b^{1-r} \le ra + (1-r)b, \\ &r \in (0,1), a > 0, b > 0, \end{aligned} \tag{3.10}$$

we have

$$\begin{aligned} \left(J(t)\right)^{\frac{q-1}{2q}}\Phi &= \left(c^{-1}J\right)^{\frac{q-1}{2q}} \left(c^{\frac{q-1}{q+1}}\Phi^{\frac{2q}{q+1}}\right)^{\frac{q+1}{2q}} \\ &\le \frac{q-1}{2q}c^{-1}J(t) + \frac{q+1}{2q}c^{\frac{q-1}{q+1}}\Phi^{\frac{2q}{q+1}}, \end{aligned} \tag{3.11}$$

valid for arbitrary $c > 0$ to be chosen later. Inserted in (3.9) and (3.5), we obtain

$$\begin{aligned} &U'(t) \\ &\le \left\{\frac{(q-1)k_1(t)}{2}\left(\frac{q^2}{2}\right)^{\frac{q-1}{2q}} M_1^{1/2}M_2^{1/2}c^{-1} - 2q(2q-1)\right\}J(t) \\ &\quad + \frac{(q+1)k_1(t)}{2}\left(\frac{q^2}{2}\right)^{\frac{q-1}{2q}} M_1^{1/2}M_2^{1/2}c^{\frac{q-1}{q+1}}\Phi^{\frac{2q}{q+1}}. \end{aligned} \tag{3.12}$$

We now select

$$c := \frac{(q-1)k_1(t)}{4q(2q-1)}M_1^{1/2}M_2^{1/2}\left(\frac{q^2}{2}\right)^{\frac{q-1}{2q}} \tag{3.13}$$

in order to have $\{\} = 0$ in (3.12), arriving at

$$M_1^{-1}U'(t) \le F(q)M_1^{-\frac{1}{q+1}}M_2^{\frac{q}{q+1}}\left(k_1(t)\right)^{\frac{2q}{q+1}}\left(\Phi(t)\right)^{\frac{2q}{q+1}}, \tag{3.14}$$

with

$$F(q) := 2^{-\frac{2(2q-1)}{q+1}}(q+1)\left(\frac{q(q-1)}{2q-1}\right)^{\frac{q-1}{q+1}}. \tag{3.15}$$

A similar computation leads to

$$\begin{aligned} &M_2^{-1}V'(t) \\ &\le F(p)M_1^{\frac{p}{p+1}}M_2^{-\frac{1}{p+1}}\left(k_2(t)\right)^{\frac{2p}{p+1}}\left(\Phi(t)\right)^{\frac{2p}{p+1}}, \end{aligned} \tag{3.16}$$

where $V(t)$ is defined in (3.4). In $\mathbb{R}^3$, we replace (3.7) by

$$\begin{aligned} \int_\Omega u^{2q-1}v^p \mathrm{d}x &\le \left(\int_\Omega u^{2(2q-1)}\mathrm{d}x\int_\Omega v^{2p}\mathrm{d}x\right)^{1/2} \\ &\le \left(\int_\Omega u^{6q}\mathrm{d}x\right)^{\frac{q-1}{4q}} \left(\int_\Omega u^{2q}\mathrm{d}x\right)^{\frac{q+1}{4q}} \left(\int_\Omega v^{2p}\mathrm{d}x\right)^{1/2}. \end{aligned} \tag{3.17}$$

and make use of the Sobolev type inequality

$$\begin{aligned} \left(\int_\Omega u^{6q}\mathrm{d}x\right)^{1/6} &\le \gamma q\left(\int_\Omega u^{2(q-1)}|\nabla u|^2 \,\mathrm{d}x\right)^{1/2} \\ &= \gamma q\left(J(t)\right)^{1/2}, \end{aligned} \tag{3.18}$$

derived by Talenti in [20] with $\gamma := 4^{1/3}3^{-1/2}\pi^{-2/3}$. Inserted in (3.17), we obtain

$$\begin{aligned} &\int_\Omega u^{2q-1}v^p \mathrm{d}x \\ &\le C(q)\left(J(t)\right)^{\frac{3(q-1)}{4q}}\left(M_1^{-1}U(t)\right)^{\frac{q+1}{4q}}\left(M_2^{-1}V(t)\right)^{1/2}M_1^{\frac{q+1}{4q}}M_2^{1/2}, \end{aligned} \tag{3.19}$$

with

$$C(q) := (\gamma q)^{\frac{3(q-1)}{2q}}. \tag{3.20}$$

Moreover we make use of (3.10) to write

$$\left(M_1^{-1}U(t)\right)^{\frac{q+1}{4q}} J^{\frac{3(q-1)}{4q}} = \left(c^{-1}J\right)^{\frac{3(q-1)}{4q}} \left[c^{\frac{3(q-1)}{q+3}} \left(M_1^{-1}U\right)^{\frac{q+1}{q+3}} \right]^{\frac{q+3}{4q}} \le \frac{3(q-1)}{4q} c^{-1} J + \frac{q+3}{4q} c^{\frac{3(q-1)}{4q}} \left(M_1^{-1}U\right)^{\frac{q+1}{q+3}}, \quad (3.21)$$

with arbitrary $c > 0$ to be chosen later. Combining (3.5), (3.19) and (3.21), we obtain

$$U'(t) \le \left\{ \frac{3(q-1)}{2} C(q) M_1^{\frac{q+1}{4q}} M_2^{1/2} \left(M_2^{-1}V\right)^{1/2} k_1(t) c^{-1} - 2q(2q-1) \right\} J(t) + \frac{q+3}{2} C(q) M_1^{\frac{q+1}{4q}} M_2^{1/2} \left(M_2^{-1}V\right)^{1/2} c^{\frac{3(q-1)}{q+3}} \left(M_1^{-1}U\right)^{\frac{q+1}{q+3}} k_1(t). \quad (3.22)$$

We now select c such that the quantity $\{\}$ in (3.22) vanishes. We are then led to the inequality

$$U'(t) \le A(q) M_1^{\frac{q+1}{q+3}} M_2^{\frac{2q}{q+3}} \left(k_1(t)\right)^{\frac{4q}{q+3}} \left(M_1^{-1}U\right)^{\frac{q+1}{q+3}} \left(M_2^{-1}V\right)^{\frac{2q}{q+3}}, \quad (3.23)$$

with

$$A(q) := \frac{q+3}{2} \left(C(q)\right)^{\frac{4q}{q+3}} \left(\frac{3(q-1)}{4q(2q-1)} \right)^{\frac{3(q-1)}{q+3}}. \quad (3.24)$$

Finally we make use of (3.10) to write

$$\left(M_1^{-1}U\right)^{\frac{q+1}{q+3}} \left(M_2^{-1}V\right)^{\frac{2q}{q+3}} = \left(\left(M_1^{-1}U\right)^{\frac{q+1}{3q+1}} \left(M_2^{-1}V\right)^{\frac{2q}{3q+1}} \right)^{\frac{3q+1}{q+3}} \le \left\{ \frac{q+1}{3q+1} c\left(M_1^{-1}U\right) + \frac{2q}{3q+1} c^{-\frac{q+1}{2q}} \left(M_2^{-1}V\right) \right\}^{\frac{3q+1}{q+3}}, \quad (3.25)$$

and select c to satisfy $(q+1)c = 2qc^{-(q+1)/2q}$, leading to

$$\left(M_1^{-1}U\right)^{\frac{q+1}{q+3}} \left(M_2^{-1}V\right)^{\frac{2q}{q+3}} \le \left(\frac{q+1}{3q+1} \right)^{\frac{3q+1}{q+3}} \left(\frac{2q}{q+1} \right)^{\frac{2q}{q+3}} \Phi^{\frac{3q+1}{q+3}}. \quad (3.26)$$

Inserted in (3.23), we obtain

$$M_1^{-1}U'(t) \le \Gamma(q) M_1^{-\frac{2}{q+3}} M_2^{\frac{2q}{q+3}} \left(k_1(t)\right)^{\frac{4q}{q+3}} \Phi^{\frac{3q+1}{q+3}}, \quad (3.27)$$

with

$$\Gamma(q) := \frac{q+3}{2} \left(\frac{3(q-1)}{4q(2q-1)} \right)^{\frac{3(q-1)}{q+3}} \left(\frac{q+1}{3q+1} \right)^{\frac{3q+1}{q+3}} \times \left(\frac{2q}{q+1} \right)^{\frac{2q}{q+3}} \left(C(q)\right)^{\frac{4q}{q+3}}. \quad (3.28)$$

A similar computation leads to

$$M_2^{-1}V'(t) \le \Gamma(p) M_1^{\frac{2p}{p+3}} M_2^{-\frac{2}{p+3}} \left(k_2(t)\right)^{\frac{4p}{p+3}} \Phi^{\frac{3p+1}{p+3}}. \quad (3.29)$$

If we suppose that

$$\Phi(t) \to \infty \ \ as\ t \to t^*, \quad (3.30)$$

then there exists $t_1 \ge 0$ such that $\Phi(t) \ge 1\ \forall t \ge t_1$ and we have

$$\Phi'(t) = M_1^{-1}U' + M_2^{-1}V' \le \begin{cases} k(t)\Phi^{2\sigma/(\sigma+1)} & \text{if } \Omega \subset \mathbb{R}^2 \\ \tilde{k}(t)\Phi^{3\sigma/(\sigma+3)} & \text{if } \Omega \subset \mathbb{R}^3 \end{cases} \quad (3.31)$$

valid for $t \ge t_1$, with

$$\sigma := \max\{p, q\}, \quad (3.32)$$

$$k(t) := F(q) M_1^{-\frac{1}{q+1}} M_2^{\frac{q}{q+1}} \left(k_1(t)\right)^{\frac{2q}{q+1}} + F(p) M_1^{\frac{p}{p+1}} M_2^{-\frac{1}{p+1}} \left(k_2(t)\right)^{\frac{2p}{p+1}}, \quad (3.33)$$

$$\tilde{k}(t) := \Gamma(q) M_1^{-\frac{2}{q+3}} M_2^{\frac{2q}{q+3}} \left(k_1(t)\right)^{\frac{4q}{q+3}} + \Gamma(p) M_1^{\frac{2p}{p+3}} M_2^{-\frac{2}{p+3}} \left(k_2(t)\right)^{\frac{4p}{p+3}}, \quad (3.34)$$

Integrating (3.31), we obtain in the two-dimensional case

$$\frac{\sigma+1}{\sigma-1} = \int_1^\infty \Phi^{-2\sigma/(\sigma+1)} \mathrm{d}\Phi \le \int_{t_1}^{t^*} k(t)\mathrm{d}t \le \int_0^{t^*} k(t)\mathrm{d}t =: K(t^*), \quad (3.35)$$

from which we obtain a lower bound for t^* of the form

$$t^* \ge K^{-1}\left(\frac{\sigma+1}{\sigma-1} \right), \quad (3.36)$$

where K^{-1} is the inverse function of K. In the three-dimensional case, we obtain

$$\frac{\sigma+3}{2(\sigma-1)} \le \int_{t_1}^{t^*} \tilde{k}(t)\mathrm{d}t \le \int_0^{t^*} \tilde{k}(t)\mathrm{d}t =: \tilde{K}(t^*), \quad (3.37)$$

from which we obtain a lower bound for t^* of the form

$$\tilde{t}^* \ge \tilde{K}^{-1} \frac{\sigma+3}{2(\sigma-1)}. \quad (3.38)$$

These results are summarized in the following

Theorem 2

Under the assumption (3.30), *a lower bound for the blow-up time* t^* *of the solution* (u,v) *of* (1.1) *is given by* (3.36) *in the two-dimensional case and by* (3.38) *in the three-dimensional case.*

In the particular case in which $k_1(t)$ and $k_2(t)$ are constant, we have

$$t^* \geq \frac{\sigma+1}{\sigma-1}\left\{F(q)M_1^{-\frac{1}{q+1}}M_2^{\frac{q}{q+1}}k_1^{\frac{2q}{q+1}} + F(p)M_1^{\frac{p}{p+1}}M_2^{-\frac{1}{p+1}}k_2^{\frac{2p}{p+1}}\right\}^{-1} \quad (3.39)$$

in the two-dimensional case and

$$t^* \geq \frac{\sigma+3}{2(\sigma-1)}\left\{\Gamma(q)M_1^{-\frac{2}{q+3}}M_2^{\frac{2q}{q+3}}k_1^{\frac{4q}{q+3}} + \Gamma(p)M_1^{\frac{2p}{p+3}}M_2^{-\frac{2}{p+3}}k_2^{\frac{4p}{p+3}}\right\}^{-1} \quad (3.40)$$

in the three-dimensional case.

Theorem 2 could easily be extended to systems of n parabolic equations of the form

$$\frac{\partial u_i}{\partial t} = \Delta u_i + k_i(t)f_i(u_j), j \neq i = 1,\cdots,n. \quad (3.41)$$

4. Sobolev Type Inequality in $\mathbb{R}^2$

The Sobolev type inequality (3.8) in $\mathbb{R}^2$ may be known, but for the convenience of the reader we present a proof here.

Lemma 1

Let $u(x,y)$ *be a nonnegative piecewise* C^1*-function defined in a bounded domain* Ω *that vanishes on the boundary* $\partial\Omega$. *Let* q *be any constant* ≥ 1. *Then we have the following Sobolev type inequality*

$$\iint_\Omega u^{4q}\mathrm{d}x\mathrm{d}y \leq \frac{q^2}{2}\iint_\Omega u^{2(q-1)}|\nabla u|^2\mathrm{d}x\mathrm{d}y\iint_\Omega u^{2q}\mathrm{d}x\mathrm{d}y, \quad (4.1)$$

valid for $\Omega \subset \mathbb{R}^2$.

For the proof of (4.1), we follow the argument of Payne in [21]. We note that (4.1) is equivalent to

$$\iint_{\tilde\Omega} \tilde u^{4q}\mathrm{d}x\mathrm{d}y \leq \frac{q^2}{2}\iint_{\tilde\Omega} \tilde u^{2(q-1)}|\nabla \tilde u|^2\mathrm{d}x\mathrm{d}y\iint_{\tilde\Omega} \tilde u^{2q}\mathrm{d}x\mathrm{d}y, \quad (4.2)$$

where $\tilde\Omega$ is the convex hull of Ω, and $\tilde u := u,(x,y)\in\Omega,\ \tilde u = 0,(x,y)\in\tilde\Omega\setminus\Omega.$ It is therefore sufficient to establish (4.1) for Ω convex. For the proof, let $P := (\bar x,\bar y)$ be an arbitrary point in $\Omega\subset\mathbb{R}^2$. Let $P_k := (x_k,\bar y)\in\partial\Omega, Q_k := (\bar x, y_k)\in\partial\Omega, k=1,2$ be two pairs of boundary points associated to P with $x_1 \leq x_2$, $y_1 \leq y_2$. Since u vanishes on $\partial\Omega$, we have for any constant $q \geq 1$

$$u^{2q}(P) = 2q\int_{P_1}^{P} u^{2q-1}u_x\mathrm{d}x = -2q\int_{P_2}^{P} u^{2q-1}u_x\mathrm{d}x, \quad (4.3)$$

from which we obtain

$$u^{2q}(P) \leq q\int_{P_1}^{P_2} u^{2q-1}|u_x|\mathrm{d}x. \quad (4.4)$$

Similarly we have

$$u^{2q}(P) \leq q\int_{Q_1}^{Q_2} u^{2q-1}|u_y|\mathrm{d}y. \quad (4.5)$$

Multiplying (4.4) by (4.5) and integrating over Ω leads to

$$\begin{aligned}\iint_\Omega u^{4q}\mathrm{d}x\mathrm{d}y &\leq q^2\iint_\Omega u^{2q-1}|u_x|\mathrm{d}x\mathrm{d}y\iint_\Omega u^{2q-1}|u_y|\mathrm{d}x\mathrm{d}y\\ &\leq q^2\left(\iint_\Omega u^{2(q-1)}u_x^2\mathrm{d}x\mathrm{d}y\iint_\Omega u^{2(q-1)}u_y^2\mathrm{d}x\mathrm{d}y\right)^{1/2}\iint_\Omega u^{2q}\mathrm{d}x\mathrm{d}y\\ &\leq \frac{1}{2}q^2\iint_\Omega u^{2(q-1)}|\nabla u|^2\mathrm{d}x\mathrm{d}y\iint_\Omega u^{2q}\mathrm{d}x\mathrm{d}y, \end{aligned} \quad (4.6)$$

which is the desired inequality (4.1). We note that we have used the Schwarz and the arithmetic-geometric mean inequalities in the two last steps of (4.6).

REFERENCES

[1] L. E. Payne and P. W. Schaefer, "Blow-Up Phenomena for Some Nonlinear Parabolic Systems," *International Journal of Pure and Applied Mathematics*, Vol. 48, No. 2, 2008, pp. 193-202.

[2] L. E. Payne and G. A. Philippin, "On Blow-Up Phenomena for Solutions of a Class of Nonlinear Parabolic Problems with Time Dependent Coefficients under Dirichlet Boundary Conditions," *Proceedings of the American Mathematical Sociery*, accepted.

[3] V. A. Galaktionov and J. L. Vazquez, "The Problem of Blow-Up in Nonlinear Parabolic Equations," *Journal of Dynamical and Control System*, Vol. 8, No. 3, 2002, pp. 399-433. doi:10.1023/A:1016334621818

[4] A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov and A. P. Mikhailov, "Blow-Up in Quasilinear Parabolic Equations," Walter de Gruyter & Co., Berlin, 1995. doi:10.1515/9783110889864

[5] B. Straughan, "Explosive Instabilities in Mechanics," Springer, Berlin, 1998. doi:10.1007/978-3-642-58807-5

[6] C. Bandle and H. Brunner, "Blow-Up in Diffusion Equations: A Survey," *Journal of Computational and Applied Mathematics*, Vol. 97, No. 1-2, 1998, pp. 3-22. doi:10.1016/S0377-0427(98)00100-9

[7] L. E. Payne, G. A. Philippin and P. W. Schaefer, "Bounds for Blow-Up Time in Nonlinear Parabolic Problems," *Journal of Mathematical Analysis and Applications*, Vol. 338, No. 1, 2008, pp. 438-447. doi:10.1016/j.jmaa.2007.05.022

[8] L. E. Payne, G. A. Philippin and P. W. Schaefer, "Blow-Up Phenomena for Some Nonlinear Parabolic Problems," *Nonlinear Analysis*: *Theory*, *Methods & Applications*, Vol. 69, No. 10, 2008, pp. 3495-3502. doi:10.1016/j.na.2007.09.035

[9] L. E. Payne, G. A. Philippin and S. Vernier-Piro, "Blow-Up Phenomena for a Semilinear Heat Equation with Nonlinear Boundary Condition, I," *Zeitschrift für Angewandte Mathematik und Physik*, Vol. 61, No. 6, 2010, pp. 999-1007. doi:10.1007/s00033-010-0071-6

[10] L. E. Payne, G. A. Philippin and S. Vernier-Piro, "Blow-Up Phenomena for a semilinear Heat Equation with Nonlinear Boundary Condition, II," *Nonlinear Analysis*: *Theory*, *Methods & Applications*, Vol. 73, No. 4, 2010, pp. 971-978. doi:10.1016/j.na.2010.04.023

[11] L. E. Payne and P. W. Schaefer, "Lower Bound for Blow-Up Time in Parabolic Problems under Neumann Conditions," *Applicable Analysis*, Vol. 85, No. 10, 2006, pp. 1301-1311. doi:10.1080/00036810600915730

[12] L. E. Payne and P. W. Schaefer, "Lower Bound for Blow -Up Time in Parabolic Problems under Dirichlet Conditions," *Journal of Mathematical Analysis and Applications*, Vol. 328, No. 2, 2007, pp. 1196-1205. doi:10.1016/j.jmaa.2006.06.015

[13] L. E. Payne and P. W. Schaefer, "Bounds for the Blow-Up Time for the Heat Equation under Nonlinear Boundary Conditions," *Proceedings of the Royal Society of Edinburgh*, Vol. 139, No. 6, 2009, pp. 1289-1296.

[14] L. E. Payne and J. C. Song, "Lower Bounds for the Blow-Up Time in a Temperature Dependent Navier-Stokes Flow," *Journal of Mathematical Analysis and Applications*, Vol. 335, No. 1, 2007, pp. 371-376. doi:10.1016/j.jmaa.2007.01.083

[15] P. Quittner, "On Global Existence and Stationary Solutions of Two Classes of Semilinear Parabolic Equations," *Commemtationes Mathematicae Universitatis Carolinae*, Vol. 34, No. 1, 1993, pp. 105-124.

[16] P. Quittner and P. Souplet, "Superlinear Parabolic Problems. Blow-Up, Global Existence and Steady States," Birkhäuser, Basel, 2007.

[17] J. L. Vazquez, "The Problem of Blow-Up for Nonlinear Heat Equations. Complete Blow-Up and Avalanche Formation," *Rendiconti Lincei Matematica e Applicazioni*, Vol. 15, No. 34, 2004, pp. 281-300.

[18] F. B. Weissler, "Local Existence and Nonexistence for Semilinear Parabolic Equations in L^P," *Indiana University Mathematics Journal*, Vol. 29, No. 1, 1980, pp. 79-102. doi:10.1512/iumj.1980.29.29007

[19] F. B. Weissler, "Existence and Nonexistence of Global Solutions for a Heat Equation," *Israël Journal of Mathematics*, Vol. 38, No. 1-2, 1981, pp. 29-40.

[20] G. Talenti, "Best Constant in Sobolev Inequality," *Annali di Matematica Pura ed Applicata*, Vol. 110, No. 1, 1976, pp. 353-372.

[21] L. E. Payne, "Uniqueness Criteria for Steady State Solutions of the Navier-Stokes Equations," In: *Atti del Simposio Internazionale Sulle Applicazioni Dell'Analisi Alla Fisica Matematica*, Cagliari-Sassari, 1964, pp. 130-153.

