

# General Solution of Two Generalized Form of Burgers Equation by Using the $\left(\frac{G'}{G}\right)$ -Expansion Method

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## ABSTRACT

In this work, the  $\left(\frac{G'}{G}\right)$ -expansion method is proposed for constructing more general exact solutions of two general form of Burgers type equation arising in fluid mechanics namely, Burgers-Korteweg-de Vries (Burgers-KdV) and Burger-Fisher equations. Our work is motivated by the fact that the  $\left(\frac{G'}{G}\right)$ -expansion method provides not only more general forms of solutions but also periodic and solitary waves. If we set the parameters in the obtained wider set of solutions as special values, then some previously known solutions can be recovered. The method appears to be easier and faster by means of a symbolic computation system.

**Keywords:**  $\left(\frac{G'}{G}\right)$ -Expansion Method; Generalized Burgers-KdV Equation; Generalized Burgers-Fisher Equation; Hyperbolic Function Solutions; Trigonometric Function Solutions

## 1. Introduction

Nonlinear evolution equations (NLEEs) have been the subject of study in various branches of mathematical-physical sciences such as physics, biology, chemistry, etc. The analytical solutions of such equations are of fundamental importance since a lot of mathematical-physical models are described by NLEEs. Among the possible solutions to NLEEs, certain special form solutions may depend only on a single combination of variables such as traveling wave variables. In the literature, there is a wide variety of approaches to nonlinear problems for constructing traveling wave solutions. Some of these approaches are the Jacobi elliptic function method [1], inverse scattering method [2], Hirota's bilinear method [3], homogeneous balance method [4], homotopy perturbation method [5], Weierstrass function method [6], symmetry method [7], Adomian decomposition method [8], sine/cosine method [9], tanh/coth method [10], the Exp-function method [11-16] and so on. But, most of the methods may sometimes fail or can only lead to a kind of special solution and the solution procedures become very com-

plex as the degree of nonlinearity increases.

Recently, the  $\left(\frac{G'}{G}\right)$ -expansion method, firstly introduced by Wang *et al.* [17], has become widely used to search for various exact solutions of NLEEs [17-27]. The  $\left(\frac{G'}{G}\right)$ -expansion method is based on the assumptions that the travelling wave solutions can be expressed by a polynomial in  $\left(\frac{G'}{G}\right)$ , and that  $G = G(\xi)$ , satisfies a second order linear ordinary differential equation

$$\frac{d^2 G(\xi)}{d\xi^2} + \lambda \frac{dG\xi}{d\xi} + \mu G\xi = 0 \quad (1)$$

where  $\lambda, \mu$  are arbitrary constants and  $\xi = kx + \omega t$ .

The degree of the polynomial can be determined by considering the homogeneous balance between the highest order derivative and nonlinear terms appearing in the given nonlinear evolution equations. The coefficients of the polynomial can be obtained by solving a set of algebraic equations resulted in from the process of using the

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method. The  $\left(\frac{G'}{G}\right)$ -expansion method is direct, concise, elementary and effective, and can be used for nonlinear partial differential equation involving higher-order nonlinear terms.

As we know, much work has been done on developing and extending the  $\left(\frac{G'}{G}\right)$ -expansion method for solving nonlinear evolution equations (see, for example [17-27] and their references). But in generalized cases, a small amount of work has been done (see, for example [28,29]). In this paper, we restrict our attention to the study of the following generalized forms of Burgers equation,

$$u_t + pu^n u_x + qu_{xx} + ru_{xxx} = 0, \tag{2}$$

g-Burgers-KdV

$$u_t + pu^n u_x + qu_{xx} + ru(1-u^n) = 0, \tag{3}$$

g-Burgers-Fisher

where  $n > 1$ , and  $p, q$  and  $r$  are constants. These equations are the generalized combined forms of Burgers equation

$$u_t + pu_x + qu_{xx} = 0 \tag{4}$$

Equation (4) is the lowest order approximation for the one-dimensional propagation of weak shock waves in a fluid [30,31]. It is also used in the description of the variation in vehicle density in highway traffic [32]. It is one of the fundamental model equations in fluid mechanics. The Burgers equation demonstrates the coupling between diffusion  $u_{xx}$  and the convection process  $uu_x$ . Burgers introduced this equation to capture some of the features of turbulent fluid in a channel caused by the interaction of the opposite effects of convection and diffusion [33]. It is also used to describe the structure of shock waves, traffic flow, and acoustic transmission [34].

The generalized Burgers-KdV Equation (2) are models for the propagation of waves on an elastic tube (see [28, 29,35,36] and their references). It can be regarded as a combination of the Burgers equation ( $p \neq 0, q \neq 0, r \neq 0$ ) and the KdV equation ( $p \neq 0, q = 0, r \neq 0$ ). Especially, Ismail Aslan in [26] applied the  $\left(\frac{G'}{G}\right)$ -expansion method for the Burgers-KdV equation ( $n = 1$ ) and in [36] Ahmet Bekir applied the tanh method to modified form of Burgers-KdV equation ( $n = 2$ ). In addition, the generalized Burgers-Fisher Equation (3), see [37], has a wide range of applications in plasma physics, fluid physics, capillary-gravity waves, nonlinear optics and chemical physics [38].

Our first interest in the present work is in implementing the  $\left(\frac{G'}{G}\right)$ -expansion method to stress its power in handling nonlinear equations, so that one can apply it to models of various types of nonlinearity. The next interest is in the determination of exact travelling wave solutions for the generalized form of Equations (2) and (3) especially the modified form of them  $\left(n = \frac{3}{2}\right)$ . Searching for exact solutions of nonlinear problems has attracted a considerable amount of research work where computer symbolic systems facilitate the computational work.

## 2. Description of the $\left(\frac{G'}{G}\right)$ -Expansion

### Method

The objective of this section is to outline the use of the  $\left(\frac{G'}{G}\right)$ -expansion method for solving certain nonlinear partial differential equations (PDEs). Suppose we have a nonlinear PDE for  $u(x,t)$ , in the form

$$P(u, u_x, u_t, u_{xx}, u_{xt}, \dots) = 0 \tag{5}$$

where  $P$  is a polynomial in its arguments, which includes nonlinear terms and the highest order derivatives. The transformation  $u(x,t) = U(\xi), \xi = kx + \omega t$ , reduces Equation (5) to the ordinary differential equation (ODE)

$$P(U, kU', \omega U', k^2 U'', k\omega U'', \dots) = 0 \tag{6}$$

where  $U = U(\xi)$ , and prime denotes derivative with respect to  $\xi$ . We assume that the solution of Equation (6) can be expressed by a polynomial in  $\left(\frac{G'}{G}\right)$  as follows:

$$U(\xi) = \sum_{i=1}^m \alpha_i \left(\frac{G'}{G}\right)^i + \alpha_0, \quad \alpha_m \neq 0 \tag{7}$$

where  $\alpha_0$ , and  $\alpha_i$  are constants to be determined later, and  $G(\xi)$  satisfies the second order linear ordinary differential Equation (1):

$$\frac{d^2 G(\xi)}{d\xi^2} + \lambda \frac{dG(\xi)}{d\xi} + \mu G(\xi) = 0 \tag{8}$$

where  $\lambda$  and  $\mu$  are arbitrary constants. It follows from (7) and (8), that

$$U' = -\sum_{i=1}^m i \alpha_i \left[ \left(\frac{G'}{G}\right)^{i+1} + \lambda \left(\frac{G'}{G}\right)^i + \mu \left(\frac{G'}{G}\right)^{i-1} \right] \tag{9}$$

$$U'' = \sum_{i=1}^m i \alpha_i \left[ (i+1) \left(\frac{G'}{G}\right)^{i+2} + (2i+1) \lambda \left(\frac{G'}{G}\right)^{i+1} + i(\lambda^2 + 2\mu) \left(\frac{G'}{G}\right)^i + (2i-1) \lambda \mu \left(\frac{G'}{G}\right)^{i-1} + (i-1) \mu^2 \left(\frac{G'}{G}\right)^{i-2} \right] \tag{10}$$

and so on, here the prime denotes the derivative with respect to  $\xi$ .

To determine  $u$  explicitly, we take the following three steps:

Step 1. Determine the integer  $m$  by substituting Equation (7) into Equation (6) and balancing the highest order nonlinear term(s) and the highest order derivative.

Step 2. Substitute Equation (7) along with Equation (8) into Equation (6) and collect all terms with the same order of  $\left(\frac{G'}{G}\right)$  together, the left-hand side of Equation

(6) is converted into a polynomial in  $\left(\frac{G'}{G}\right)$ . Then set

each coefficient of this polynomial to zero to derive a set of algebraic equations for  $k, \omega, \lambda, \mu, \alpha_0$  and  $\alpha_i$ .

Step 3. Solve the system of algebraic equations obtained in Step 2, for  $k, \omega, \lambda, \mu, \alpha_0$  and  $\alpha_i$  by use of Maple, along with the general solutions of Equation (8), we can have more travelling wave solutions of Equation (5).

### 3. Application

In this section, we will demonstrate the  $\left(\frac{G'}{G}\right)$ -expansion method on the generalized equations listed in (2) and (3).

#### 3.1. Generalized Burgers-KdV Equation

Considering the generalized Burgers-KdV equation with higher-order nonlinear terms

$$u_t + pu^n u_x + qu_{xx} + ru_{xxx} = 0 \tag{11}$$

Now, using the wave variable  $u(x, t) = U(\xi)$ ,  $\xi = kx + \omega t$ , in (11) and integrating the resulting equation and neglecting the constant of integration, we find

$$\omega U + \frac{pk}{n+1} U^{n+1} + qk^2 U' + rk^3 U'' = 0 \tag{12}$$

To achieve our goal, we use the transformation  $U(\xi) = V^{\frac{1}{n}}(\xi)$ , that will carry (12) into the ODE

$$\omega n^2 V^2 + \frac{pk n^2}{n+1} V^3 + qk^2 n V V' + rk^3 \left( n V V'' + (1-n)(V')^2 \right) = 0 \tag{13}$$

According to Step 1, we get  $3m = 2m + 2$ , hence  $m = 2$ . We then suppose that Equation (13) has the following formal solutions:

$$V = \alpha_2 \left(\frac{G'}{G}\right)^2 + \alpha_1 \left(\frac{G'}{G}\right) + \alpha_0, \quad \alpha_2 \neq 0 \tag{14}$$

where  $\alpha_2, \alpha_1$ , and  $\alpha_0$  are constants which are unknown to be determined later.

Substituting Equation (14) into Equation (13) and collecting all terms with the same order of  $\left(\frac{G'}{G}\right)$ , together,

the left-hand sides of Equation (13) are converted into a polynomial in  $\left(\frac{G'}{G}\right)$ . Setting each coefficient of each

polynomial to zero, we derive a set of algebraic equations for  $k, \omega, \lambda, \mu, \alpha_0, \alpha_1$ , and  $\alpha_2$ , (collecting the coefficients of  $\left(\frac{G'}{G}\right)^i, i = 0, 1, \dots, 6$  and setting it to zero,

Appendix I) and solving them by use of Maple, we get the following general result:

$$\left\{ \begin{aligned} \lambda &= -\frac{n(pn^2\alpha_1 - 2qkn^2 + 4pn\alpha_1 - 6qkn - 4kq)}{2rk^2(n^3 + 7n^2 + 14n + 8)}, \\ \mu &= \frac{p(-4qkn^2 + pn^2\alpha_1 + 4pn\alpha_1 - 12qkn - 8kq)n^3\alpha_1}{16r^2k^4(n^5 + 10n^4 + 37n^3 + 64n^2 + 52n + 16)}, \\ \omega &= \frac{2kq^2(n+2)}{r(n+4)^2}, \quad \alpha_0 = -\frac{\alpha_1^2 pn^2}{8rk^2(2+3n+n^2)}, \\ \alpha_2 &= -\frac{2rk^2(2+3n+n^2)}{pn^2} \end{aligned} \right\} \tag{15}$$

Substitute the above general case in (10), we get

$$V = -\frac{2rk^2(2+3n+n^2)}{pn^2} \left(\frac{G'}{G}\right)^2 + \alpha_1 \left(\frac{G'}{G}\right) - \frac{\alpha_1^2 pn^2}{8rk^2(2+3n+n^2)} \tag{16}$$

then use the transformation  $U(\xi) = V^{\frac{1}{n}}(\xi)$ , the hyperbolic function solutions of Equation (11), becomes:

$$u_1(x,t) = \left[ \frac{-2rk^2(2+3n+n^2)}{pn^2} \left( \frac{\sqrt{\Delta} \left[ C_1 \sinh\left(\frac{1}{2}\sqrt{\Delta}\xi\right) + C_2 \cosh\left(\frac{1}{2}\sqrt{\Delta}\xi\right) \right]}{2 \left[ C_2 \sinh\left(\frac{1}{2}\sqrt{\Delta}\xi\right) + C_1 \cosh\left(\frac{1}{2}\sqrt{\Delta}\xi\right) \right]} - \frac{\lambda}{2} \right)^2 + \alpha_1 \left( \frac{\sqrt{\Delta} \left[ C_1 \sinh\left(\frac{1}{2}\sqrt{\Delta}\xi\right) + C_2 \cosh\left(\frac{1}{2}\sqrt{\Delta}\xi\right) \right]}{2 \left[ C_2 \sinh\left(\frac{1}{2}\sqrt{\Delta}\xi\right) + C_1 \cosh\left(\frac{1}{2}\sqrt{\Delta}\xi\right) \right]} - \frac{\lambda}{2} \right) - \frac{\alpha_1^2 pn^2}{8rk^2(2+3n+n^2)} \right]^{\frac{1}{n}} \tag{17}$$

and the trigonometric function solutions of Equation (11), will be:

$$u_2(x,t) = \left[ \frac{-2rk^2(2+3n+n^2)}{pn^2} \left( \frac{i\sqrt{\Delta} \left[ -C_1 \sin\left(\frac{i}{2}\sqrt{\Delta}\xi\right) + C_2 \cos\left(\frac{i}{2}\sqrt{\Delta}\xi\right) \right]}{2 \left[ C_2 \sin\left(\frac{i}{2}\sqrt{\Delta}\xi\right) + C_1 \cos\left(\frac{i}{2}\sqrt{\Delta}\xi\right) \right]} - \frac{\lambda}{2} \right)^2 + \alpha_1 \left( \frac{i\sqrt{\Delta} \left[ -C_1 \sin\left(\frac{i}{2}\sqrt{\Delta}\xi\right) + C_2 \cos\left(\frac{i}{2}\sqrt{\Delta}\xi\right) \right]}{2 \left[ C_2 \sin\left(\frac{i}{2}\sqrt{\Delta}\xi\right) + C_1 \cos\left(\frac{i}{2}\sqrt{\Delta}\xi\right) \right]} - \frac{\lambda}{2} \right) - \frac{\alpha_1^2 pn^2}{8rk^2(2+3n+n^2)} \right]^{\frac{1}{n}} \tag{18}$$

where

$$\xi = kx + \frac{2kq^2(n+2)}{r(n+4)^2}t, \\ \Delta = \frac{q^2 n^2}{k^2 r^2 (n+4)^2}, \\ \lambda = -\frac{n(pn^2\alpha_1 - 2qkn^2 + 4pn\alpha_1 - 6qkn - 4kq)}{2rk^2(n^3 + 7n^2 + 14n + 8)}$$

and  $C_1, C_2, k$ , and  $\alpha_1$  are arbitrary constants and  $i^2 = -1$ .

known results in the literature can be rediscovered, for instance, setting  $C_2 = 0$ , then the general solutions (17)-(18) reduces, respectively,

If  $C_1$  and  $C_2$  take up special values, the various

$$u_{1,1}(x,t) = \left[ \frac{-2rk^2(2+3n+n^2)}{pn^2} \left( \frac{\sqrt{\frac{q^2 n^2}{k^2 r^2 (n+4)^2}}}{2} \tanh\left(\frac{1}{2}\sqrt{\frac{q^2 n^2}{k^2 r^2 (n+4)^2}}\xi\right) - \frac{\lambda}{2} \right)^2 + \alpha_1 \left( \frac{\sqrt{\frac{q^2 n^2}{k^2 r^2 (n+4)^2}}}{2} \tanh\left(\frac{1}{2}\sqrt{\frac{q^2 n^2}{k^2 r^2 (n+4)^2}}\xi\right) - \frac{\lambda}{2} \right) - \frac{\alpha_1^2 pn^2}{8rk^2(2+3n+n^2)} \right]^{\frac{1}{n}} \tag{19}$$

$$u_{2,1}(x,t) = \left[ \frac{-2rk^2(2+3n+n^2)}{pn^2} \left( i \frac{\sqrt{\frac{q^2 n^2}{k^2 r^2 (n+4)^2}}}{2} \tan \left( \frac{i}{2} \sqrt{\frac{q^2 n^2}{k^2 r^2 (n+4)^2}} \xi \right) + \frac{\lambda}{2} \right)^2 - \alpha_1 \left( i \frac{\sqrt{\frac{q^2 n^2}{k^2 r^2 (n+4)^2}}}{2} \tan \left( \frac{i}{2} \sqrt{\frac{q^2 n^2}{k^2 r^2 (n+4)^2}} \xi \right) + \frac{\lambda}{2} \right) - \frac{\alpha_1^2 pn^2}{8rk^2(2+3n+n^2)} \right]^{\frac{1}{n}} \tag{20}$$

and when  $C_1 = 0$ , then we deduce from general solutions (17)-(18) that, respectively

$$u_{1,2}(x,t) = \left[ \frac{-2rk^2(2+3n+n^2)}{pn^2} \left( \frac{\sqrt{\frac{q^2 n^2}{k^2 r^2 (n+4)^2}}}{2} \coth \left( \frac{1}{2} \sqrt{\frac{q^2 n^2}{k^2 r^2 (n+4)^2}} \xi \right) - \frac{\lambda}{2} \right)^2 + \alpha_1 \left( \frac{\sqrt{\frac{q^2 n^2}{k^2 r^2 (n+4)^2}}}{2} \coth \left( \frac{1}{2} \sqrt{\frac{q^2 n^2}{k^2 r^2 (n+4)^2}} \xi \right) - \frac{\lambda}{2} \right) - \frac{\alpha_1^2 pn^2}{8rk^2(2+3n+n^2)} \right]^{\frac{1}{n}} \tag{21}$$

$$u_{2,2}(x,t) = \left[ \frac{2rk^2(2+3n+n^2)}{pn^2} \left( i \frac{\sqrt{\frac{q^2 n^2}{k^2 r^2 (n+4)^2}}}{2} \cot \left( \frac{i}{2} \sqrt{\frac{q^2 n^2}{k^2 r^2 (n+4)^2}} \xi \right) - \frac{\lambda}{2} \right)^2 + \alpha_1 \left( i \frac{\sqrt{\frac{q^2 n^2}{k^2 r^2 (n+4)^2}}}{2} \cot \left( \frac{i}{2} \sqrt{\frac{q^2 n^2}{k^2 r^2 (n+4)^2}} \xi \right) - \frac{\lambda}{2} \right) - \frac{\alpha_1^2 pn^2}{8rk^2(2+3n+n^2)} \right]^{\frac{1}{n}} \tag{22}$$

where

$$\xi = kx + \frac{2kq^2(n+2)}{r(n+4)^2}t, \quad \lambda = -\frac{n(pn^2\alpha_1 - 2qkn^2 + 4pn\alpha_1 - 6qkn - 4kq)}{2rk^2(n^3 + 7n^2 + 14n + 8)}$$

and  $k$ , and  $\alpha_1$  are arbitrary constants and  $i^2 = -1$ .

In particulars cases, if  $n = \frac{3}{2}$ , and  $\alpha_1 = 0$  then

$$u_1(x,t) = \left[ -\frac{70rk^2}{9p} \left( \frac{3}{22} \sqrt{\frac{q^2}{k^2 r^2}} \tanh \left( \frac{3}{22} \sqrt{\frac{q^2}{k^2 r^2}} \left( kx + \frac{28kq^2t}{121r} \right) \right) - \frac{3q}{22rk} \right)^2 \right]^{\frac{2}{3}}$$

$$u_2(x,t) = \left[ -\frac{70rk^2}{9p} \left( \frac{3}{22} \sqrt{\frac{q^2}{k^2 r^2}} \coth \left( \frac{3}{22} \sqrt{\frac{q^2}{k^2 r^2}} \left( kx + \frac{28kq^2t}{121r} \right) \right) - \frac{3q}{22rk} \right)^2 \right]^{\frac{2}{3}}$$

and

$$u_3(x,t) = \left[ -\frac{70}{9} \frac{rk^2}{p} \left( \frac{-3}{22} \sqrt{\frac{q^2}{k^2 r^2}} \tan \left( \frac{3i}{22} \sqrt{\frac{q^2}{k^2 r^2}} \left( kx + \frac{28}{121} \frac{kq^2 t}{r} \right) - \frac{3}{22} \frac{q}{rk} \right) \right)^2 \right]^{\frac{2}{3}}$$

$$u_4(x,t) = \left[ -\frac{70}{9} \frac{rk^2}{p} \left( \frac{-3}{22} \sqrt{\frac{q^2}{k^2 r^2}} \cot \left( \frac{3i}{22} \sqrt{\frac{q^2}{k^2 r^2}} \left( kx + \frac{28}{121} \frac{kq^2 t}{r} \right) - \frac{3}{22} \frac{q}{rk} \right) \right)^2 \right]^{\frac{2}{3}}$$

are the four solitary wave solutions of the particular Burgers-KdV equation

$$u_t + pu^{\frac{3}{2}}u_x + qu_{xx} + ru_{xxx} = 0 \tag{23}$$

where  $k$  is arbitrary constant and  $i^2 = -1$ .

Different real solutions of the general Burgers-KdV Equation (2) were obtained by Ahmet Bekir in [36] using the extended tanh method.

### 3.2. Generalized Burgers-Fisher Equation

Now considering the generalized Burgers-Fisher equation with higher-order nonlinear terms

$$u_t + pu^n u_x + qu_{xx} + ru(1-u^n) = 0 \tag{24}$$

Using the wave variable  $u(x,t) = U(\xi), \xi = kx + \omega t$ , in (24), we find

$$\omega U' + pkU^n U' + qk^2 U'' + rU(1-U^n) = 0 \tag{25}$$

To achieve our goal, we use the transformation

$U(\xi) = V^{\frac{1}{n}}(\xi)$ , that will carry (25) into the ODE

$$\left\{ \mu = \frac{\lambda^2}{4} - \frac{n^2 p^2}{4k^2 q^2 (n+1)^2}, \omega = -\frac{k(qr(n+1)^2 + p^2)}{p(n+1)}, \alpha_0 = \frac{1}{2} + \frac{qk\lambda(n+1)}{2pn}, \alpha_1 = \frac{qk(1+n)}{pn} \right\} \tag{28}$$

Substitute the above general case in (25), we get

$$V = \frac{bk(1+n)}{an} \left[ \left( \frac{G'}{G} \right) + \frac{\lambda}{2} \right] + \frac{1}{2} \tag{29}$$

$$rn^2(V^2 - V^3) + n\omega VV' + pknV^2V' + qk^2((1-n)(V')^2 + nVV'') = 0 \tag{26}$$

According to Step 1, we get  $3m+1=2m+2$ , hence  $m=1$ . We then suppose that Equation (26) has the following formal solutions:

$$V = \alpha_1 \left( \frac{G'}{G} \right) + \alpha_0, \quad \alpha_1 \neq 0 \tag{27}$$

where  $\alpha_1$  and  $\alpha_0$  are constants which are unknown to be determined later.

Similar on previous section, substituting Equation (29) into Equation (26) and collecting all terms with the same order of  $\left( \frac{G'}{G} \right)$ , together, we derive a set of algebraic equations for  $k, \omega, \lambda, \mu, \alpha_0$ , and  $\alpha_1$ , (collecting the coefficients of  $\left( \frac{G'}{G} \right)^i, i = 0, 1, \dots, 4$  and setting it to zero,

Appendix II) and solving them by use of Maple, we get the following general result:

then use the transformation  $U(\xi) = V^{\frac{1}{n}}(\xi)$ , the hyperbolic and the trigonometric function solutions of Equation (11), will be:

$$U_1(\xi) = \left[ \frac{\frac{qk}{2p} \sqrt{\frac{p^2}{k^2 q^2}} \left( C_1 \sinh \left( \frac{n}{2(1+n)} \sqrt{\frac{p^2}{k^2 q^2}} \xi \right) + C_2 \cosh \left( \frac{n}{2(1+n)} \sqrt{\frac{p^2}{k^2 q^2}} \xi \right) \right)}{C_2 \sinh \left( \frac{n}{2(1+n)} \sqrt{\frac{p^2}{k^2 q^2}} \xi \right) + C_1 \cosh \left( \frac{n}{2(1+n)} \sqrt{\frac{p^2}{k^2 q^2}} \xi \right)} + \frac{1}{2} \right]^{\frac{1}{n}} \tag{30}$$

$$U_2(\xi) = \left[ \frac{\frac{qki}{2p} \sqrt{\frac{p^2}{k^2 q^2}} \left( -C_1 \sin \left( \frac{ni}{2(1+n)} \sqrt{\frac{p^2}{k^2 q^2}} \xi \right) + C_2 \cos \left( \frac{ni}{2(1+n)} \sqrt{\frac{p^2}{k^2 q^2}} \xi \right) \right)}{C_2 \sin \left( \frac{ni}{2(1+n)} \sqrt{\frac{p^2}{k^2 q^2}} \xi \right) + C_1 \cos \left( \frac{ni}{2(1+n)} \sqrt{\frac{p^2}{k^2 q^2}} \xi \right)} + \frac{1}{2} \right]^{\frac{1}{n}} \tag{31}$$

where  $\xi = kx - \frac{k(qr(n+1)^2 + p^2)}{p(n+1)}t$ , and  $C_1, C_2, k$  are arbitrary constants and  $i^2 = -1$ .

If  $C_1$  and  $C_2$  take up special values, the various known results in the literature can be rediscovered, for instance, setting  $C_2 = 0$ , then the general solutions (30)-(31) reduces, respectively,

$$u_{1,1}(x,t) = \left[ \frac{qk}{2p} \sqrt{\frac{p^2}{k^2q^2}} \tanh \left( \frac{n}{2(1+n)} \sqrt{\frac{p^2}{k^2q^2}} \left( kx - \frac{k(qr(n+1)^2 + p^2)}{p(n+1)}t \right) \right) + \frac{1}{2} \right]^{\frac{1}{n}} \tag{32}$$

$$u_{2,1}(x,t) = \left[ -\frac{qki}{2p} \sqrt{\frac{p^2}{k^2q^2}} \tan \left( \frac{ni}{2(1+n)} \sqrt{\frac{p^2}{k^2q^2}} \left( kx - \frac{k(qr(n+1)^2 + p^2)}{p(n+1)}t \right) \right) + \frac{1}{2} \right]^{\frac{1}{n}} \tag{33}$$

and when  $C_1 = 0$ , then we deduce from general solutions (30)-(31) that, respectively

$$u_{1,1}(x,t) = \left[ \frac{qk}{2p} \sqrt{\frac{p^2}{k^2q^2}} \coth \left( \frac{n}{2(1+n)} \sqrt{\frac{p^2}{k^2q^2}} \left( kx - \frac{k(qr(n+1)^2 + p^2)}{p(n+1)}t \right) \right) + \frac{1}{2} \right] \tag{34}$$

$$u_{2,1}(x,t) = \left[ \frac{qki}{2p} \sqrt{\frac{p^2}{k^2q^2}} \cot \left( \frac{ni}{2(1+n)} \sqrt{\frac{p^2}{k^2q^2}} \left( kx - \frac{k(qr(n+1)^2 + p^2)}{p(n+1)}t \right) \right) + \frac{1}{2} \right] \tag{35}$$

where  $\xi = kx - \frac{k(qr(n+1)^2 + p^2)}{p(n+1)}t$ ,  $k$ , is arbitrary constant and  $i^2 = -1$ .

In particular, if  $n = \frac{3}{2}$ , then

$$u_1(x,t) = \left[ \frac{qk}{2p} \sqrt{\frac{p^2}{k^2q^2}} \tanh \left( \frac{3}{10} \sqrt{\frac{p^2}{k^2q^2}} \left( kx - \frac{2}{5} \frac{k \left( \frac{25}{4} qr + p^2 \right)}{p} t \right) \right) + \frac{1}{2} \right]^{\frac{2}{3}}$$

$$u_2(x,t) = \left[ -\frac{qki}{2p} \sqrt{\frac{p^2}{k^2q^2}} \tan \left( \frac{3i}{10} \sqrt{\frac{p^2}{k^2q^2}} \left( kx - \frac{2}{5} \frac{k \left( \frac{25}{4} qr + p^2 \right)}{p} t \right) \right) + \frac{1}{2} \right]^{\frac{2}{3}}$$

and

$$u_3(x,t) = \left[ \frac{qk}{2p} \sqrt{\frac{p^2}{k^2q^2}} \coth \left( \frac{3}{10} \sqrt{\frac{p^2}{k^2q^2}} \left( kx - \frac{2}{5} \frac{k \left( \frac{25}{4} qr + p^2 \right)}{p} t \right) \right) + \frac{1}{2} \right]^{\frac{2}{3}}$$

$$u_4(x,t) = \left[ \frac{qki}{2p} \sqrt{\frac{p^2}{k^2q^2}} \cot \left( \frac{3i}{10} \sqrt{\frac{p^2}{k^2q^2}} \left( kx - \frac{2}{5} \frac{k \left( \frac{25}{4} qr + p^2 \right)}{p} t \right) \right) + \frac{1}{2} \right]^{\frac{2}{3}}$$

are the four solitary wave solutions of the particular Burgers-Fisher equation

$$u_t + pu^2u_x + qu_{xx} + ru\left(1-u^2\right) = 0 \quad (36)$$

$k$ , is arbitrary constant and  $i^2 = -1$ .

Different real solutions of the general Burgers-Fisher Equation (3) were obtained by Wazwaz in [39] using the tanh method and by El-Wakil in [38] using a modified tanh-function method and recently by Luwai Wazzan in [28] using a modified tanh-coth method.

#### 4. Conclusions

This study shows that the  $\left(\frac{G'}{G}\right)$ -expansion method is quite efficient and practically well suited for use in finding exact solutions for the generalized form of Burgers-Korteweg-de Vries (Burgers-KdV) and Burger-Fisher equations. The results show that this method is a powerful Mathematical tool for obtaining exact solutions for the general B-Fisher and B-KdV equations. With the aid of Maple, we have assured the correctness of the obtained solutions by putting them back into the original equation. It is also a promising method to solve other nonlinear partial differential equations arising in engineering sciences.

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### Appendix I

$$\begin{aligned} \left(\frac{G'}{G}\right)^0 &: \omega n^3 \alpha_0^2 + \omega n^2 \alpha_0^2 + pkn^2 \alpha_0^3 + rk^3 \alpha_1^2 \mu^2 - qk^2 n^2 \alpha_0 \alpha_1 \mu - qk^2 n \alpha_0 \alpha_1 \mu - rk^3 n^2 \alpha_1^2 \mu^2 + 2rk^3 n \alpha_0 \alpha_2 \mu^2 \\ &+ rk^3 n \alpha_0 \alpha_1 \lambda \mu + 2rk^3 n^2 \alpha_0 \alpha_2 \mu^2 + rk^3 n^2 \alpha_0 \alpha_1 \lambda \mu = 0, \\ \left(\frac{G'}{G}\right)^1 &: 2\omega n^3 \alpha_1 \alpha_0 + 2\omega n^2 \alpha_1 \alpha_0 + 3pkn^2 \alpha_1 \alpha_0^2 - 2qk^2 n \alpha_0 \alpha_2 \mu - qk^2 n \alpha_0 \alpha_1 \lambda - qk^2 n^2 \alpha_1^2 \mu - qk^2 n \alpha_1^2 \mu \\ &- 2qk^2 n^2 \alpha_0 \alpha_2 \mu - qk^2 n^2 \alpha_0 \alpha_1 \lambda + 4rk^3 \alpha_1 \alpha_2 \mu^2 + 2rk^3 \alpha_1^2 \lambda \mu + 2rk^3 n \alpha_1 \alpha_2 \mu^2 + rk^3 n \alpha_1^2 \lambda \mu \\ &+ 2rk^3 n \alpha_0 \alpha_1 \mu + rk^3 n \alpha_0 \alpha_1 \lambda^2 + 6rk^3 n \alpha_0 \alpha_2 \lambda \mu - 2rk^3 n^2 \alpha_1 \alpha_2 \mu^2 - rk^3 n^2 \alpha_1^2 \lambda \mu + 2rk^3 n^2 \alpha_0 \alpha_1 \mu \\ &+ rk^3 n^2 \alpha_0 \alpha_1 \lambda^2 + 6rk^3 n^2 \alpha_0 \alpha_2 \lambda \mu = 0, \\ \left(\frac{G'}{G}\right)^2 &: \omega n^3 \alpha_1^2 + \omega n^2 \alpha_1^2 + 2\omega n^3 \alpha_2 \alpha_0 + 2\omega n^2 \alpha_2 \alpha_0 + 4rk^3 \alpha_2^2 \mu^2 + 2rk^3 \alpha_1^2 \mu + rk^3 \alpha_1^2 \lambda^2 + 3pkn^2 \alpha_2 \alpha_0^2 \\ &+ 3pkn^2 \alpha_1^2 \alpha_0 - 3qk^2 n \alpha_2 \alpha_1 \mu - 2qk^2 n \alpha_0 \alpha_2 \lambda - qk^2 n^2 \alpha_1^2 \lambda - qk^2 n^2 \alpha_0 \alpha_1 - qk^2 n \alpha_1^2 \lambda - qk^2 n \alpha_0 \alpha_1 \\ &- 3qk^2 n^2 \alpha_2 \alpha_1 \mu - 2qk^2 n^2 \alpha_0 \alpha_2 \lambda + rk^3 n \alpha_1^2 \lambda^2 + 2rk^3 n \alpha_1^2 \mu + 2rk^3 n \alpha_2^2 \mu^2 + 8rk^3 \alpha_2 \alpha_1 \lambda \mu \\ &+ 8rk^3 n \alpha_0 \alpha_2 \mu + 4rk^3 n \alpha_0 \alpha_2 \lambda^2 + 3rk^3 n \alpha_0 \alpha_1 \lambda + 7rk^3 n \alpha_2 \alpha_1 \lambda \mu - 2rk^3 n^2 \alpha_2^2 \mu^2 + 8rk^3 n^2 \alpha_0 \alpha_2 \mu \\ &+ 4rk^3 n^2 \alpha_0 \alpha_2 \lambda^2 + 3rk^3 n^2 \alpha_0 \alpha_1 \lambda - rk^3 n^2 \alpha_2 \alpha_1 \lambda \mu = 0, \\ \left(\frac{G'}{G}\right)^3 &: 2\omega n^3 \alpha_2 \alpha_1 + 2\omega n^2 \alpha_2 \alpha_1 + pkn^2 \alpha_1^3 - qk^2 n^2 \alpha_1^2 - qk^2 n \alpha_1^2 + 2rk^3 \alpha_1^2 \lambda + 6pkn^2 \alpha_2 \alpha_1 \alpha_0 - 3qk^2 n \alpha_2 \alpha_1 \lambda \\ &- 2qk^2 n \alpha_0 \alpha_2 - 2qk^2 n^2 \alpha_2^2 \mu - 2qk^2 n^2 \alpha_0 \alpha_2 - 2qk^2 n \alpha_2^2 \mu - 3qk^2 n^2 \alpha_2 \alpha_1 \lambda + 3rk^3 n \alpha_1^2 \lambda + 8rk^3 \alpha_2^2 \lambda \mu \\ &+ 8rk^3 \alpha_2 \alpha_1 \mu + 4rk^3 \alpha_2 \alpha_1 \lambda^2 + 2rk^3 n \alpha_0 \alpha_1 + 6rk^3 n \alpha_2^2 \lambda \mu + 10rk^3 n \alpha_2 \alpha_1 \mu + 5rk^3 n \alpha_2 \alpha_1 \lambda^2 \\ &+ 10rk^3 n \alpha_0 \alpha_2 \lambda + rk^3 n^2 \alpha_1^2 \lambda + 2rk^3 n^2 \alpha_0 \alpha_1 - 2rk^3 n^2 \alpha_2^2 \lambda \mu + 2rk^3 n^2 \alpha_2 \alpha_1 \mu + rk^3 n^2 \alpha_2 \alpha_1 \lambda^2 \\ &+ 10rk^3 n^2 \alpha_0 \alpha_2 \lambda = 0, \\ \left(\frac{G'}{G}\right)^4 &: \omega n^3 \alpha_2^2 + \omega n^2 \alpha_2^2 + rk^3 \alpha_1^2 + 3pkn^2 \alpha_2^2 \alpha_0 + 3pkn^2 \alpha_2 \alpha_1^2 - 2qk^2 n^2 \alpha_2^2 \lambda - 3qk^2 n^2 \alpha_2 \alpha_1 - 2qk^2 n \alpha_2^2 \lambda \\ &- 3qk^2 n \alpha_2 \alpha_1 + 6rk^3 n \alpha_0 \alpha_2 + 4rk^3 n \alpha_2^2 \lambda^2 + 8rk^3 n \alpha_2^2 \mu + 8rk^3 \alpha_2 \alpha_1 \lambda + rk^3 n^2 \alpha_1^2 + 8rk^3 \alpha_2^2 \mu \\ &+ 4rk^3 \alpha_2^2 \lambda^2 + 2rk^3 n \alpha_1^2 + 13rk^3 n \alpha_2 \alpha_1 \lambda + 6rk^3 n^2 \alpha_0 \alpha_2 + 5rk^3 n^2 \alpha_2 \alpha_1 \lambda = 0, \\ \left(\frac{G'}{G}\right)^5 &: 3pkn^2 \alpha_2^2 \alpha_1 - 2qk^2 n \alpha_2^2 - 2qk^2 n^2 \alpha_2^2 + 8rk^3 n \alpha_2 \alpha_1 + 10rk^3 n \alpha_2^2 \lambda + 8rk^3 \alpha_2^2 \lambda + 4rk^3 \alpha_2 \alpha_1 \\ &+ 4rk^3 n^2 \alpha_2 \alpha_1 + 2rk^3 n^2 \alpha_2^2 \lambda = 0, \\ \left(\frac{G'}{G}\right)^6 &: 4rk^3 \alpha_2^2 + pkn^2 \alpha_2^3 + 2rk^3 n^2 \alpha_2^2 + 6rk^3 n \alpha_2^2 = 0. \end{aligned}$$

### Appendix II

$$\left(\frac{G'}{G}\right)^0 : qk^2\alpha_1^2\mu^2 - n\omega\alpha_1\alpha_0\mu - pkn\alpha_1\alpha_0^2\mu + rn^2\alpha_0^2 - rn^2\alpha_0^3 + qk^2n\alpha_1\alpha_0\lambda\mu - qk^2n\alpha_1^2\mu^2 = 0,$$

$$\left(\frac{G'}{G}\right)^1 : -n\omega\alpha_1^2\mu - n\omega\alpha_1\alpha_0\lambda - 2pkn\alpha_1^2\alpha_0\mu - pkn\alpha_1\alpha_0^2\lambda + 2qk^2\alpha_1^2\lambda\mu - qk^2n\alpha_1^2\lambda\mu \\ + 2qk^2n\alpha_1\alpha_0\mu + qk^2n\alpha_1\alpha_0\lambda^2 + 2rn^2\alpha_1\alpha_0 - 3rn^2\alpha_1\alpha_0^2 = 0,$$

$$\left(\frac{G'}{G}\right)^2 : rn^2\alpha_1^2 - n\omega\alpha_1^2\lambda - n\omega\alpha_1\alpha_0 - pkn\alpha_1^3\mu - 2pkn\alpha_1^2\alpha_0\lambda - pkn\alpha_1\alpha_0^2 + 2qk^2\alpha_1^2\mu \\ + qk^2\alpha_1^2\lambda^2 + 3qk^2n\alpha_1\alpha_0\lambda - 3rn^2\alpha_1^2\alpha_0 = 0,$$

$$\left(\frac{G'}{G}\right)^3 : -n\omega\alpha_1^2 - rn^2\alpha_1^3 - pkn\alpha_1^3\lambda - 2pkn\alpha_1^2\alpha_0 + 2qk^2\alpha_1^2\lambda + qk^2n\alpha_1^2\lambda + 2qk^2n\alpha_1\alpha_0 = 0,$$

$$\left(\frac{G'}{G}\right)^4 : qk^2\alpha_1^2 - pkn\alpha_1^3 + qk^2n\alpha_1^2 = 0.$$